If \( f(x) \geq g(x) \) on the interval \([a, b]\), then to find the area \( A \) between the graphs of \( y = f(x) \) and \( y = g(x) \) from \( a \) to \( b \) we simply evaluate

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx
\]
In practice, difficulties arise from the form or statement of a problem. For example, the problem “Find the area between the curves $y = x^2$ and $y = 1 - x^2$”, if interpreted strictly, would have answer $\infty$. Yet many people would state such a problem believing that they are asking the question:

“What is the area of the region of the area consisting of points which both lie above the curve $y = x^2$ and below the curve $y = 1 - x^2$?”
To solve this problem, we need to find the points of intersection of the two curves:

\[ x^2 = 1 - x^2 \text{ if } 2x^2 = 1 \text{ or } x^2 = \frac{1}{2}, \]
so the curves intersect when \( x = -\frac{\sqrt{2}}{2} \) and \( x = \frac{\sqrt{2}}{2} \), so in our area integral we take \( a = -\frac{\sqrt{2}}{2} \) and \( b = \frac{\sqrt{2}}{2} \):

\[
A = \int_{a}^{b} [f(x) - g(x)] \, dx = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} [(1 - x^2) - x^2] \, dx = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 1 - 2x^2 \, dx = \\
\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} 1 - 2x^2 \, dx = x - \frac{2}{3} x^3 \bigg|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} = \\
\left[ \left( \frac{\sqrt{2}}{2} \right) - \frac{2}{3} \left( \frac{\sqrt{2}}{2} \right)^3 \right] - \left[ \left( -\frac{\sqrt{2}}{2} \right) - \frac{2}{3} \left( -\frac{\sqrt{2}}{2} \right)^3 \right] = \\
\left[ \frac{\sqrt{2}}{2} - \frac{2 \sqrt{2}}{3} \right] - \left[ -\frac{\sqrt{2}}{2} - \frac{2}{3} \left( -\frac{2 \sqrt{2}}{8} \right) \right] = \frac{\sqrt{2}}{2} \left[ 1 - \frac{1}{3} \right] + \frac{\sqrt{2}}{2} \left[ 1 - \frac{1}{3} \right] = \\
\frac{2 \sqrt{2}}{3}\]
Note that we can simplify the calculation by making use of the fact that we have symmetry about the $y$-axis:

\[
A = 2 \int_{0}^{\frac{\sqrt{2}}{2}} 1 - 2x^2 \, dx = 2 \left( x - \frac{2}{3}x^3 \right) \bigg|_{0}^{\frac{\sqrt{2}}{2}} =
\]

\[
2 \left( \frac{\sqrt{2}}{2} - \frac{2}{3} \left( \frac{\sqrt{2}}{2} \right)^3 \right) = \sqrt{2} \left( 1 - \frac{1}{3} \right) = \frac{2\sqrt{2}}{3}
\]

**Problem:** Find the area of the simple regions lying between the intersections of the curves $y = \sin x$ and $y = \cos x$
We have to be very careful to make sure that the function we take for $f$ lies above the function $g$ on the interval $[a, b]$. We let $a = \frac{\pi}{4}$, $b = \frac{5\pi}{4}$, $f(x) = \sin x$, and $g(x) = \cos x$, so that

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} [\sin x - \cos x] \, dx = (- \sin x - \cos x) \bigg|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} =$$

$$\left( - \sin \frac{5\pi}{4} - \cos \frac{5\pi}{4} \right) - \left( - \sin \frac{\pi}{4} - \cos \frac{\pi}{4} \right) =$$

$$\left( - \left( -\frac{\sqrt{2}}{2} \right) - \left( -\frac{\sqrt{2}}{2} \right) \right) - \left( -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = 2\sqrt{2}$$
Suppose we have two functions of $y$ like $f(y) = |y|$ and $g(y) = y^2$ which intersect at $c$ and $d$, (-1 and 1 in this example) and wish to find the area between them.

We use the formula

$$A = \int_c^d [f(y) - g(y)] \, dy$$

In our example we have

$$A = \int_{-1}^{1} [|y| - y^2] \, dy = 2 \int_0^{1} [|y| - y^2] \, dy = 2 \int_0^{1} [y - y^2] \, dy =$$

$$2 \left( \frac{y^2}{2} - \frac{y^3}{3} \right) \bigg|_0^1 = 2 \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{3}$$
Example: Find the area of the region bounded by the given curves by two methods:
(a) integrating with respect to $x$, (b) integrating with respect to $y$, if:

$$4x + y^2 = 0, \; y = 2x + 4$$

Solution: (a) The upper boundary of the region is the graph of the somewhat complicated function

$$f(x) = \begin{cases} 
2x + 4 & \text{if } -4 \leq x \leq -1 \\
\sqrt{-4x} & \text{if } -1 \leq x \leq 0 
\end{cases}$$
while the lower part is the graph of \( y = -\sqrt{-4x} \), \(-4 \leq x \leq 0\).

The area is \( A = \int_{-4}^{0} [f(x) - g(x)] \, dx = \)

\[
\int_{-4}^{-1} [f(x) - g(x)] \, dx + \int_{-1}^{0} [f(x) - g(x)] \, dx = \\
\int_{-4}^{-1} \left[ 2x + 4 - (-\sqrt{-4x}) \right] \, dx + \int_{-1}^{0} \left[ \sqrt{-4x} - (-\sqrt{-4x}) \right] \, dx = \\
\int_{-4}^{-1} 2x + 4 + 2(-x)^{\frac{1}{2}} \, dx + 2 \int_{-1}^{0} 2(-x)^{\frac{1}{2}} \, dx = \\
x^2 + 4x \bigg|_{-4}^{-1} + 2 \int_{-4}^{-1} (-x)^{\frac{1}{2}} \, dx + 4 \int_{-1}^{0} (-x)^{\frac{1}{2}} \, dx =
\]

**Sidetrack:** We need to find \( \int (-x)^{\frac{1}{2}} \, dx \) by making the substitution \( u = -x \), \( dx = -du \):

\[
\int (-x)^{\frac{1}{2}} \, dx = \int u^{\frac{1}{2}} (-du) = - \int u^{\frac{1}{2}} \, du = - \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = -2 \frac{2}{3} (-x)^{\frac{3}{2}} + C
\]

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Thus we get

\[ A = x^2 + 4x \bigg|_{-4}^{-1} + 2 \int_{-4}^{-1} (-x)^{\frac{1}{2}} \, dx + 4 \int_{-1}^{0} (-x)^{\frac{1}{2}} \, dx = \]

\[ x^2 + 4x \bigg|_{-4}^{-1} + 2 \left( -\frac{2}{3} (-x)^{\frac{3}{2}} \right) \bigg|_{-4}^{-1} + 4 \left( -\frac{2}{3} (-x)^{\frac{3}{2}} \right) \bigg|_{-1}^{0} = \]

\[ \left( (-1)^2 + 4(-1) \right) - \left( (-4)^2 + 4(-4) \right) + \left[ -\frac{4}{3} (-(-1))^{\frac{3}{2}} - \left( -\frac{4}{3} (-(-4))^{\frac{3}{2}} \right) \right] + \]

\[ \left[ 4\frac{2}{3} (-0)^{\frac{3}{2}} - 4\frac{2}{3} (-(-1))^{\frac{3}{2}} \right] = \]

\[ (1 - 4) - (16 - 16) + \left[ -\frac{4}{3} + \frac{4}{3} (4)^{\frac{3}{2}} \right] + \left[ 0 - \frac{8}{3} \right] = \]

\[-3 + \left[ -\frac{4}{3} + \frac{4}{3} \cdot 8 \right] + \frac{8}{3} = 9\]

(b) We first solve the two equations \( 4x + y^2 = 0, \) and \( y = 2x + 4 \) for \( x \) as a function of \( y \) and get

\[ x = -\frac{y^2}{4} \text{ and } x = \frac{y - 4}{2} \]
Thus we have

\[ A = \int_{-4}^{2} \left[ -\frac{y^2}{4} - \frac{y - 4}{2} \right] \, dy = \int_{-4}^{2} -\frac{y^2}{4} - \frac{y}{2} + 2 \, dy = \]

\[ -\frac{y^3}{12} - \frac{y^2}{4} + 2y \bigg|_{-4}^{2} = \left( -\frac{2^3}{12} - \frac{2^2}{4} + 2(2) \right) - \left( -\frac{(-4)^3}{12} - \frac{(-4)^2}{4} + 2(-4) \right) \]

\[ \left( -\frac{8}{12} - \frac{4}{4} + 4 \right) - \left( -\frac{64}{12} - \frac{16}{4} - 8 \right) = \left( -\frac{2}{3} - 1 + 4 \right) - \left( -\frac{16}{3} - 4 - 8 \right) = \]

\[ -\frac{2}{3} + 3 - \frac{16}{3} + 12 = 15 - \frac{18}{3} = 9 \]
**Example:** Find the area of the region bounded by the given curves by two methods:
(a) integrating with respect to \( x \), (b) integrating with respect to \( y \), if:
\[ x + 1 = 2(y - 2)^2, \quad x + 6y = 7 \]

**Solution:** (a) The two curves intersect at the points \( (1, 1) \) and \( (7, 0) \), so we have
\[ A = \int_{1}^{7} \left[ \frac{7-x}{6} - \left(2 - \sqrt{\frac{x+1}{2}}\right) \right] \, dx = \int_{1}^{7} -\frac{5}{6} - \frac{x}{6} + \sqrt{\frac{x+1}{2}} \, dx = \]
\[ = \left. \left(-\frac{5}{6}x - \frac{x^2}{12} + \frac{2}{3\sqrt{2}} (x + 1)^{\frac{3}{2}} \right) \right|_{1}^{7} \]
\[ = \left( -\frac{5}{6} \cdot 7 - \frac{7^2}{12} + \frac{2}{3\sqrt{2}} (7 + 1)^{\frac{3}{2}} \right) - \left( -\frac{5}{6} \cdot 1 - \frac{1^2}{12} + \frac{2}{3\sqrt{2}} (1 + 1)^{\frac{3}{2}} \right) = \]
\[ = \left( -\frac{35}{6} - \frac{49}{12} + \frac{2}{3\sqrt{2}} (8)^{\frac{3}{2}} \right) - \left( -\frac{5}{6} - \frac{1}{12} + \frac{2}{3\sqrt{2}} (2)^{\frac{3}{2}} \right) = \]
\[ = \left( -\frac{119}{12} + \frac{2}{3\sqrt{2}} \cdot 8 \cdot 8 \right) - \left( -\frac{11}{12} + \frac{2}{3\sqrt{2}} \cdot 2 \cdot 2 \right) = \]
\[ = \left( -\frac{108}{12} + \frac{2}{3\sqrt{2}} \cdot 8 \cdot 2 \right) - \left( \frac{4}{3} \right) = -9 + \frac{32}{3} - \frac{4}{3} = \frac{1}{3} \]

(b) \[ A = \int_{0}^{1} \left[ (7 - 6y) - (2(y^2 - 2)^2 - 1) \right] \, dy = \]
\[ = \int_{0}^{1} 7 - 6y - (2(y^2 - 4y + 4) - 1) \, dy = \]

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\[ \int_{0}^{1} \left( 7 - 6y - (2y^2 - 8y + 8 - 1) \right) dy = \]
\[ \int_{0}^{1} -2y^2 + 2y \, dy = -2 \frac{y^3}{3} + 2 \frac{y^2}{2} \bigg|_{0}^{1} = -\frac{2}{3} + 1 = \frac{1}{3} \]

Two strategies become clear from looking at these two examples:

**First:** if possible, avoid functions whose definitions must involve different formulas on different intervals.

**Second:** choose the integral that will have the simplest expression.

In both of the examples just looked at, it was best to integrate with respect to \( y \). It is easy to find examples where it is better to integrate with respect to \( x \): just rotate the above examples by 90 degrees!