Coordinate Geometry & Lines

It is essential that the student be able to automatically apply the very basic formulas of elementary Analytic Geometry:

**Distance Formula**

The distance between the points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) is

\[
|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

In our diagram, we take \( P_1 = (x_1, y_1) \) to be \((-1, 1)\) and \( P_2 = (x_2, y_2) \) to be \((3, 4)\).

Then

\[
|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3 - (-1))^2 + (4 - 1)^2} = \sqrt{16 + 9} = \sqrt{25} = 5
\]

**Slope**

The slope of the line passing through the points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) is

\[
m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}
\]

so in our example \( m = \frac{4 - 1}{3 - (-1)} = \frac{3}{4} \)
Equations of Lines

These come in many useful forms:

**Point-Slope Form**
The equation of the line passing through the point \( P_1 = (x_1, y_1) \) with slope \( m \) is \( y - y_1 = m(x - x_1) \)

Thus given the point \( P_1 = (1, 2) \) and the slope \( m = -\frac{1}{3} \) the equation of the line is \( y - 2 = -\frac{1}{3}(x - 1) \)

**Point-Point Form**
The equation of the line passing through the points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \) is

\[
y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)
\]

This just comes from putting the two previous formulas together.

**Slope-Intercept Form**
The equation of the line passing through the \( y \)-axis at the point \((0, b)\) with slope \( m \) is

\( y = mx + b \). For example, if \( m = -\frac{1}{2} \) and \( b = 2 \), the equation is \( y = -\frac{1}{2}x + 2 \)

**Intercept-Intercept Form**
The equation of the line passing through the intercepts \((a, 0)\) and \((0, b)\) is

\[
\frac{x}{a} + \frac{y}{b} = 1
\]

For example, the equation of the line through \((4, 0)\) and \((0, 3)\) is \( \frac{x}{4} + \frac{y}{3} = 1 \).
**General Form**

Every line has infinitely many equations of the form

\[ Ax + By + C = 0. \]

For any fixed line, they are non-zero multiples of each other.

---

**Parallel & Perpendicular Lines**

Two lines with slopes \( m_1 \) and \( m_2 \) are parallel if \( m_1 = m_2 \).

perpendicular if \( m_1 m_2 = -1 \).

**Example:** Find the equation of the line through the point \((2, 1)\) which is perpendicular to the line \( y = -\frac{1}{2}x + 2 \).

**Solution:** The slope of the perpendicular line is \( -\frac{1}{-\frac{1}{2}} = 2 \), so the equation of the perpendicular line is, using the Point-Slope Form:

\[ y - 1 = 2(x - 2) \]
Distance from a Point to a Line

The distance from the point \( P_0 = (x_0, y_0) \) to a line \( \ell \) with equation \( Ax + By + C = 0 \) is

\[
d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}
\]

Example: Find the distance from the point \((3, 4)\) to the line with equation \( y = -\frac{1}{2}x + 2 \).

Solution: We must rewrite the equation of the line in General form:

\[
y = -\frac{1}{2}x + 2 \quad \text{becomes} \quad 2y = -x + 2 \quad \text{or} \quad x + 2y - 2 = 0,
\]

so we apply the Distance Formula with \( x_0 = 3, \; y_0 = 4, \; A = 1, \; B = 2, \) and \( C = -2 \):

\[
d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} = \frac{|(1)(3) + (2)(4) + (-2)|}{\sqrt{(1)^2 + (2)^2}} = \frac{|3 + 8 - 2|}{\sqrt{1 + 4}} = \frac{9}{\sqrt{5}}
\]