On a Theorem of Tignol for Defectless extensions and its converse

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Abstract. Let \((K, v)\) be a Henselian valued field of arbitrary rank. In 1990, Tignol proved that if \((K', v')/(K, v)\) is a finite separable defectless extension of degree a prime number, then the set \(A_{K'/K} = \{v(Tr_{K'/K}(\alpha)) - v'(\alpha) \mid \alpha \in K', \alpha \neq 0\}\) has a minimum element. This paper extends Tignol’s result to all finite separable extensions. Moreover a characterization of finite separable defectless extensions is given by showing that \((K', v')/(K, v)\) is a defectless extension if and only if the set \(A_{K'/K}\) has a minimum element. Our proof also leads to a new proof of the well known result that each finite extension of a formally \(\varphi\)-adic field (or more generally of a finitely ramified valued field) is defectless.

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1. Introduction

Throughout this paper, \( v \) is a Henselian valuation of arbitrary rank of a field \( K \) with residue field \( R(K) \) and \( \bar{v} \) is the unique prolongation of \( v \) to a fixed algebraic closure \( \overline{K} \) of \( K \). A finite extension \( (K', v')/(K, v) \) (or briefly \( K'/K \)) will be called defectless if \([K': K] = ef\) where \( e \) and \( f \) are respectively the index of ramification and the residual degree of \( v'/v \). This extension will be referred to as tame if (a) it is defectless; (b) the residue field of \( v' \) is a separable extension of the residue field of \( v \); (c) the ramification index of \( v'/v \) is not divisible by the characteristic of the residue field of \( v \).

Let \((K', v') \subseteq (\overline{K}, \bar{v})\) be a finite extension of \((K, v)\). Since \((K, v)\) is Henselian, for any \( \alpha \) in \( K' \) and \( \sigma \) in \( Gal(\overline{K}/K) \), \( \bar{v} \circ \sigma(\alpha) = \bar{v}(\alpha) \) and consequently \( v(T r_{K'/K}(\alpha)) \geq v'(\alpha) \); here and elsewhere \( T r \) stands for the trace. In 1990, Tignol proved that if \((K', v')/(K, v)\) is a finite separable extension of degree any prime number, then the set \( A_{K'/K} \) defined by

\[
A_{K'/K} = \{ v(T r_{K'/K}(\alpha)) - v'(\alpha) | \alpha \in K', \alpha \neq 0 \}
\]

has a minimum element provided \((K', v')/(K, v)\) is a defectless extension (cf.[4, Prop. 2.5] or [5, Lemma 1.1]). He also proved that the smallest element of \( A_{K'/K} \) is zero in case \((K', v')/(K, v)\) is a tame extension. In 2000, Khanduja [2] proved that the above result of Tignol in fact holds for all finite tame extensions and showed that a finite separable extension \((K', v')\) of a Henselian valued field \((K, v)\) is tame if and only if zero is the minimum element of \( A_{K'/K} \). We have observed that if \((K', v')/(K, v)\) is any finite separable defectless extension, then the set \( A_{K'/K} \) has a minimum element (see Lemma 2.2). This gives rise to the following natural question.

Let \((K', v')/(K, v)\) be a finite separable extension for which the set \( A_{K'/K} \) has a minimum element. Is it true that \((K', v')\) is a defectless extension of \((K, v)\)?

In this paper, we prove that the answer to the above question is in the affirmative. In other words, it is proved that a finite separable extension \((K', v')\) of \((K, v)\) is de-
fectless if and only if the set $A_{K'/K}$ has a minimum element. It will be shown that this characterization of defectless extensions quickly implies that every finite extension of a finitely ramified valued field is defectless, thereby providing a new proof of this well known result. Recall that a valued field $(K,v)$ is said to be finitely ramified if the value group of $v$ admits a least positive element $\lambda$ and there is a prime number $p$ and a natural number $e$ such that $v(p) = e\lambda$; such a valued field has characteristic 0 and $p$ is the characteristic of its residue field.

In the course of proof we use the notion of valuation basis. A set $\{x_1, ..., x_n\}$ of elements of an $n$-dimensional extension $(K',v')$ of $(K,v)$ is said to be a valuation basis of $(K',v')/(K,v)$ if for every choice of elements $a_i \in K$, we have $v'(\sum_{i=1}^{n} a_i x_i) = \min_{i=1}^{n} \{v'(a_i x_i)\}$. Note that a valuation basis of $(K',v')/(K,v)$ is linearly independent over $K$ and hence is a basis of $K'/K$.

The main result of the present paper is the following:

**Theorem 1.1.** Let $v$ be a Henselian valuation of arbitrary rank of a field $K$. Let $K'/K$ be a finite separable extension and $v'$ be the prolongation of $v$ to $K'$. Then the following statements are equivalent.

(i) $(K',v')$ is a defectless extension of $(K,v)$.

(ii) $(K',v')/(K,v)$ has a valuation basis.

(iii) The set $A_{K'/K} = \{v(Tr_{K'/K} (\beta)) - v'(\beta) | \beta \in K', \beta \neq 0\}$ has a minimum element.

The following corollary will be deduced from the above theorem.

**Corollary 1.2.** Each finite extension of a finitely ramified Henselian valued field is defectless.

**2. Some preliminary results**

Let $(K,v)$ and $(\overline{K}, \overline{v})$ be as in the preceding section. For any $\xi$ in the valuation ring
of $\tilde{v}$, $\xi^*$ will denote its $\tilde{v}$-residue, i.e., the image of $\xi$ under the canonical homomorphism from the valuation ring of $\tilde{v}$ onto its residue field.

The result of the following lemma is well known. For the sake of completeness, we give its proof here.

**Lemma 2.1.** Let $(K', v')$ be a finite defectless extension of a Henselian valued field $(K, v)$. Then it has a valuation basis.

**Proof.** Let $G \subseteq G'$ and $R(K) \subseteq R(K')$ denote respectively the value groups and the residue fields of $v$ and $v'$. Let $e$ and $f$ stand respectively for the index of $G$ in $G'$ and the degree of the extension $R(K')/R(K)$. Choose elements $x_1, \ldots, x_e$ in $K'$ for which the cosets $G + v'(x_1), \ldots, G + v'(x_e)$ are all distinct. Choose $y_1, \ldots, y_f$ in the valuation ring of $v'$ such that their $v'$-residues $y^*_1, \ldots, y^*_f$ are linearly independent over $R(K)$. Observe that the extension $(K', v')/(K, v)$ being defectless, has degree $ef$. Claim is that the set

$$\{ x_i y_j, \ 1 \leq i \leq e, \ 1 \leq j \leq f \}$$

is a valuation basis of $(K', v')/(K, v)$. Suppose that the claim is false. Then there exists an element $x = \sum_{j=1}^f \sum_{i=1}^e a_{ij} x_i y_j$ in $K'$ with $a_{ij}$ in $K$ for which $v'(x) > \min_{i,j} \{v'(a_{ij} x_i y_j)\}$. If necessary after renaming, we may assume that $\min_{i,j} \{v'(a_{ij} x_i y_j)\} = v'(a_{11} x_1 y_1)$. The elements $y^*_1, \ldots, y^*_f$ being linearly independent over $R(K)$ are non-zero and hence $v'(y_j) = 0, \ 1 \leq j \leq f$. Thus we have

$$v'(\sum_{i=1}^e \sum_{j=1}^f a_{ij} x_i y_j) > \min_{i,j} \{v'(a_{ij} x_i y_j)\} = v'(a_{11} x_1). \quad (2)$$

Since $G + v'(x_1)$ is different from $G + v'(x_i)$ when $2 \leq i \leq e$, it follows from the equality in (2) that $v'(a_{ij} x_i y_j) > v'(a_{11} x_1)$ for $2 \leq i \leq e, \ 1 \leq j \leq f$; consequently

$$v'(\sum_{i=2}^e \sum_{j=1}^f a_{ij} x_i y_j) > v'(a_{11} x_1).$$

Therefore (2) implies that

$$v'(\sum_{j=1}^f a_{1j} x_1 y_j) > v'(a_{11} x_1).$$

The above inequality shows that $\sum_{j=1}^f \left( \frac{a_{1j}}{a_{11}} \right) y^*_j = 0^*$ which contradicts the linear indepen-
dence of $y^*_1, \ldots, y^*_n$ over $R(K)$. This contradiction proves the lemma.

**Lemma 2.2.** Suppose that a finite separable extension $(K', v')$ of a Henselian valued field $(K, v)$ has a valuation basis $w_1, \ldots, w_n$. Then the set $A_{K'/K}$ defined by (1) has smallest element equal to $\min_{1 \leq i \leq n} \{v(Tr_{K'/K}(w_i)) - v'(w_i)\}$.

**Proof.** Let $\beta = \sum_{i=1}^{n} a_i w_i$ be any non-zero element of $K'$, $a_i \in K$. Then

$$v'(\beta) = \min_{i} v'(a_i w_i) = v'(a_k w_k) \text{ (say).} \quad (3)$$

Using the triangle law, we have

$$v(Tr_{K'/K}(\beta)) \geq \min_{i} \{v(a_i Tr_{K'/K}(w_i))\} = v(a_j) + v(Tr_{K'/K}(w_j)) \text{ (say).} \quad (4)$$

It follows from (3) and (4) that

$$v(Tr_{K'/K}(\beta)) - v'(\beta) \geq v(a_j) + v(Tr_{K'/K}(w_j)) - v'(a_k w_k)$$

$$\geq v(a_j) + v(Tr_{K'/K}(w_j)) - v'(a_j w_j)$$

$$= v(Tr_{K'/K}(w_j)) - v'(w_j).$$

Thus we have shown that for any $\beta \neq 0$ in $K'$, the inequality

$$v(Tr_{K'/K}(\beta)) - v'(\beta) \geq \min_{1 \leq i \leq n} \{v(Tr_{K'/K}(w_i)) - v'(w_i)\}$$

holds as desired.

As usual, an extension $(K', v')/(K, v)$ (or briefly $K'/K$ when the underlying valuations are clear) will be called an immediate extension if $v'$ and $v$ have the same value group and the same residue field.

**Lemma 2.3.** Let $(K', v')$ be a finite separable extension of a Henselian valued field $(K, v)$. Let $L$ be an intermediate field such that $K'/L$ is an immediate extension of
degree strictly greater than one. Then the set $A_{K'/K}$ defined by (1) does not have any minimum element.

Proof. To prove the lemma, it is clearly enough to show that for any given non-zero element $\xi$ in $K'$, there exists an element $\eta$ in $K'$ satisfying the following two conditions

$$v'(\eta) > v'(\xi), \quad Tr_{K'/K}(\eta) = Tr_{K'/K}(\xi).$$  \hspace{1cm} (5)

We split the proof in two cases.

Case (i). Char $K = 0$. In this case there exists a generator $\theta$ of the extension $K'/L$ with $Tr_{K'/L}(\theta) = 0$. Since $K'/L$ is an immediate extension, on replacing $\theta$ by $\theta/\alpha$ for a suitable element $\alpha \in L$, we can assume that

$$v'(\theta) = 0 \text{ and } \theta^* = 1^*.$$ \hspace{1cm} (6)

Let $\xi$ be any non-zero element of $K'$. Using the fact that $K'/L$ is an immediate extension, we can choose an element $c$ belonging to $L$ satisfying

$$(\xi/c)^* = -1^*.$$ \hspace{1cm} (7)

We verify that (5) holds for the element $\eta$ defined by $\eta = \xi + c\theta$. It follows from (6) and (7) that

$$(\eta/\xi)^* = 1^* + (c/\xi)^* \theta^* = 0^*.$$ 

Therefore $v'(\eta) > v'(\xi)$. Since $Tr_{K'/L}(\theta) = 0$, we have

$$Tr_{K'/K}(\eta) = Tr_{K'/K}(\xi) + Tr_{L/K}(cTr_{K'/L}(\theta)) = Tr_{K'/K}(\xi)$$

as desired.

Case (ii). Char $K = p > 0$. Let $\xi$ be any non-zero element of $K'$. Fix an element $c$ of $L$ satisfying (7). Define an element $\eta$ of $K'$ by $\eta = \xi + c$. Then clearly

$$(\eta/\xi)^* = 1 + (c/\xi)^* = 0^*.$$
Since \( \text{char } K = p > 0 \), and \( K'/L \) is an extension of degree \( p^r > 1 \), we have \( \text{Tr}_{K'/L}(c) = p^r c = 0 \). Therefore \( \eta \) satisfies (5).

**Lemma 2.4.** Let \((K', v')(K, v)\) be a finite separable extension of Henselian valued fields. Let \( L \) be an intermediate field such that \( K'/L \) is a defectless extension with respect to the valuation obtained by restricting \( v' \) to \( L \). Suppose that \( A_{K'/K} \) has a minimum element, then \( A_{L/K} \) has a minimum element.

**Proof.** As \( K'/L \) is a defectless extension, it has a valuation basis \( \theta_1, \ldots, \theta_m \) by virtue of Lemma 2.1. We denote \( \min A_{K'/K} \) by \( \lambda \) and set

\[
t_i = \text{Tr}_{K'/L} (\theta_i), \quad 1 \leq i \leq m.
\]

Let \( \beta = \sum_{i=1}^{m} a_i \theta_i, a_i \in L \), be an element of \( K' \) such that \( \lambda = v(\text{Tr}_{K'/K}(\beta)) - v'(\beta) \), i.e.,

\[
\lambda = v\left( \sum_i T_{L/K}(a_i t_i) \right) - v'(\sum_i a_i \theta_i).
\]

If an index \( s \) is defined so as

\[
\min_i \{ v(T_{L/K}(a_i t_i)) \} = v(T_{L/K}(a_s t_s)),
\]

then we are going to show that \( a_s t_s \neq 0 \) and

\[
\lambda = v(T_{K'/K}(a_s \theta_s)) - v'(a_s \theta_s);
\]

this will be used to prove that

\[
\min A_{L/K} = v(T_{L/K}(a_s t_s)) - v'(a_s t_s)
\]

which will complete the proof of the lemma.

Observe that \( a_s t_s \neq 0 \), for otherwise \( T_{L/K}(a_i t_i) = 0 \) for \( 1 \leq i \leq m \) by virtue of (9); this would imply that \( T_{K'/K}(\beta) = \sum_i T_{K'/K}(a_i \theta_i) = \sum_i T_{L/K}(a_i t_i) = 0 \) leading to \( \lambda = \infty \), which is impossible as \( K'/K \) is a separable extension. Using (8) and (9) and the fact that \( \theta_1, \ldots, \theta_m \) is a valuation basis of \( K'/L \), we see that

\[
\lambda \geq \min_i \{ v(T_{L/K}(a_i t_i)) \} - \min_i \{ v'(a_i \theta_i) \}
\]
\[ \geq v(Tr_{L/K}(a_st_s)) - v'(a_s\theta_s) \]

\[ = v(Tr_{K'/'K}(a_s\theta_s)) - v'(a_s\theta_s). \]

Indeed the inequality \( \lambda \geq v(Tr_{K'/'K}(a_s\theta_s)) - v'(a_s\theta_s) \) just proved must be an equality by virtue of the fact that \( \lambda \) is minimum of \( A_{K'/'K} \). This proves (10).

Suppose to the contrary that (11) is false. Then there exists a non-zero element \( c \) of \( L \) such that

\[ v(Tr_{L/K}(c)) - v'(c) < v(Tr_{L/K}(a_st_s)) - v'(a_st_s). \quad (12) \]

As \( t_s \neq 0 \), we can write \( c \) as \( bt_s, b \in L \). Consider the element \( b\theta_s \) of \( K' \). Keeping in mind (12) and the equality \( Tr_{K'/'L}(\theta_s) = t_s \), a simple calculation shows that

\[ v(Tr_{K'/'K}(b\theta_s)) - v'(b\theta_s) = v(Tr_{L/K}(bt_s)) - v'(bt_s) \]

\[ < v(Tr_{L/K}(a_st_s)) - v'(a_st_s) + v'(bt_s) - v'(b\theta_s) \]

\[ = v(Tr_{K'/'K}(a_s\theta_s)) - v'(a_s\theta_s). \]

Therefore it now follows from (10) that

\[ v(Tr_{K'/'K}(b\theta_s)) - v'(b\theta_s) < \lambda \]

which is impossible as \( \lambda \) is the minimum element of the set \( A_{K'/'K} \). This contradiction proves (11) and hence the lemma.

We shall use the following already known theorem. Its proof is omitted (see [2]).

**Theorem 2A.** A finite separable extension \((K', v')\) of a Henselian valued field \((K, v)\) is tame if and only if there exists \( \alpha \neq 0 \) in \( K' \) satisfying \( v(Tr_{K'/'K}(\alpha)) = v'(\alpha) \).

We now prove a theorem which will be used to prove Theorem 1.1; it is of independent interest as well.
Theorem 2.5. Let \((K, v) \subseteq (K', v') \subseteq (K'', v'')\) be a tower of finite separable extensions. Suppose that \(A_{K''/K'}\) and \(A_{K'/K}\) have minimum elements. Then \(A_{K''/K}\) has a minimum element which equals \(\min A_{K''/K'} + \min A_{K'/K}\).

**Proof.** Let \(\alpha\) be any non-zero element of \(K''\). We can write

\[
v(T_{K''/K}(\alpha)) - v''(\alpha) = v(T_{K'/K}(T_{K''/K'}(\alpha))) - v'(T_{K''/K'}(\alpha)) + \]
\[
v'(T_{K''/K'}(\alpha)) - v''(\alpha).
\]

This shows that \(A_{K''/K} \subseteq A_{K''/K'} + A_{K'/K}\); hence \(A_{K''/K}\) is bounded from below by \(\min A_{K''/K'} + \min A_{K'/K}\). On the other hand, if \(a' \in K'\) and \(\gamma \in K''\) satisfy

\[
v(T_{K'/K}(a')) - v'(a') = \min A_{K'/K}
\]

and

\[
v'(T_{K''/K'}(\gamma)) - v''(\gamma) = \min A_{K''/K'},
\]

then one can quickly verify that \(b = \gamma a'T_{K''/K'}(\gamma)^{-1}\) satisfies

\[
v(T_{K''/K}(b)) - v''(b) = \min A_{K''/K'} + \min A_{K'/K};
\]

hence \(\min A_{K''/K'} + \min A_{K'/K} \in A_{K''/K}\). The theorem follows.

The corollary stated below is an immediate consequence of the above theorem and Theorem 2.A.

**Corollary.** Let \((K, v) \subseteq (K', v') \subseteq (K'', v'')\) be a tower of finite separable extensions such that \(K''/K'\) is a tame extension. Suppose that \(A_{K'/K}\) has a minimum element. Then \(A_{K''/K}\) has a minimum element which equals \(\min A_{K'/K}\).

The following theorem which will be used in the sequel is essentially proved in [3, Lemma 3.15]. For the sake of readers’ convenience and ready reference, we give its proof here.
**Theorem 2.6.** Let \( v \) be a Henselian valuation of a field \( K \) whose residue field is of characteristic \( p > 0 \). Let \( w \) be its prolongation to the separable closure \( K^{\text{sep}} \) of \( K \). Let \( K' \subseteq K^{\text{sep}} \) be a finite extension of \( K \) which is not tame. Then there exists a finite tame extension \( T \) of \( K \) such that \( TK'/T \) is a tower of extensions of degree \( p \) each.

**Proof.** Let \( K^V \) denote the maximal tame extension of \((K,v)\) contained in \((K^{\text{sep}},w)\). By ramification theory, \( K^V \) is the ramification field of the extension \((K^{\text{sep}},w)/(K,v)\) and \( K^{\text{sep}}/K^V \) is a \( p \)-extension (cf. [1, 22.7, 20.18]). Write \( K' = K(\alpha) \). Let \( K^V(\alpha_1,\ldots,\alpha_s) \) be the smallest Galois extension of \( K^V \) containing \( \alpha \). Consider the groups

\[
H_o = \text{Gal}(K^V(\alpha_1,\ldots,\alpha_s)/K^V), \quad H = \text{Gal}(K^V(\alpha_1,\ldots,\alpha_s)/K^V(\alpha)).
\]

Since \( K'/K \) is not a tame extension, \( \alpha \) does not belong to \( K^V \). Therefore \( |H_o| > 1 \); in fact by what has been said in the above paragraph, the order of \( H_o \) must be a power of \( p \). So there exists a descending chain of subgroups

\[
H_o \supset H_1 \supset \ldots \supset H_t = H \supset H_{t+1} \supset \ldots \supset \{e\}
\]

such that each \( H_i \) is a normal subgroup of \( H_{i-1} \) of index \( p \). Let \( K^V(\beta_1), K^V(\beta_1,\beta_2), \ldots, K^V(\beta_1,\ldots,\beta_t) = K^V(\alpha) \) denote respectively the fixed fields of \( H_1,\ldots,H_t = H \). It is clear that

\[
K^V \subset K^V(\beta_1) \subset K^V(\beta_1,\beta_2) \subset \ldots \subset K^V(\beta_1,\ldots,\beta_t) = K^V(\alpha)
\]

is a tower of extensions of degree \( p \) each. Assume without loss of generality that \( \beta_t = \alpha \).

Let \( X^p + a_{11}X^{p-1} + \ldots + a_{1p} \) be the minimal polynomial of \( \beta_1 \) over \( K^V \). Let \( K_1 \) denote the field obtained by adjoining to \( K \) the coefficients \( a_{11},\ldots,a_{1p} \). Let \( X^p + b_{21}X^{p-1} + \ldots + b_{2p} \) be the minimal polynomial of \( \beta_2 \) over \( K^V[\beta_1] \). We can write \( b_{2i} \) as

\[
b_{2i} = \sum_{j=0}^{p-1} a_{2ij}\beta_1^j, \quad a_{2ij} \in K^V.
\]

Let \( K_2 \) denote the field obtained by adjoining to \( K_1 \) the \( p^2 \) elements \( \{a_{2ij}, 1 \leq i \leq p, 0 \leq j \leq p - 1\} \). Repeating this process \( t \) times, we obtain a subfield \( K_t \) of \( K^V \) which
is a finite tame extension of $K$. Denote $K_t$ by $T$. Clearly

$$T \subset T(\beta_1) \subset T(\beta_1, \beta_2) \subset \ldots \subset T(\beta_1, \ldots, \beta_t)$$

(14)

is a tower of extensions of degree $p$ each. Since $T(\beta_1, \ldots, \beta_t)$ contains $\beta_t = \alpha$ and $\alpha$ is algebraic over $K^V$ of degree $p^t$ by virtue of (13), it now follows from (14) that $T(\beta_1, \ldots, \beta_t) = T(\alpha) = TK'$. This completes the proof of the theorem.

3. Proof of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. The assertions $(i) \implies (ii)$ and $(ii) \implies (iii)$ hold in view of Lemma 2.1 and Lemma 2.2 respectively. We now prove $(iii) \implies (i)$. Since every finite tame extension is defectless, it may be assumed that $K'/K$ is not a tame extension. Let the prime number $p$ denote the characteristic of the residue field of $v$. Applying Theorem 2.6, we see that there exists a tame extension $T$ of $K$ such that $TK'/T$ is a tower of extensions $T \subset T_1 \subset \ldots \subset T_s = TK'$ of degree $p$ each. Since tameness is preserved under composition [1, 20.15(b)], $TK'/K'$ is a tame extension. By hypothesis $A_{K'/K}$ has a minimum element. Therefore by the corollary following Theorem 2.5, $\min A_{TK'/K}$ exists. It now follows from Lemma 2.3 that the extension $T_s = TK'$ of $T_{s-1}$ having degree $p$ is defectless. Now applying Lemma 2.4 to the tower of extensions $K \subset T_{s-1} \subset T_s$, we see that $\min A_{T_{s-1}/K}$ exists. Repetition of the above argument (with $T_s$ replaced by $T_{s-1}$) yields that $T_{s-2}/T_{s-1}$ is defectless and $\min A_{T_{s-2}/K}$ exists. Continuing this process $s$ times, we conclude that $T_s = TK'$ is a defectless extension of $T$. Also $T/K$ being tame is defectless. Consequently $TK'/K$ is a defectless extension and so is $K'/K$.

Proof of Corollary 1.2. Let $(K', v')$ be an extension of a finitely ramified Henselian valued field $(K, v)$ of degree $\nu$. Let $p$ be the characteristic of the residue field of $v$ and $v(p)/e$ be the least positive element of the value group $G$ of $v$. Let $r$ be the largest positive integer such that $v(p)/er$ belongs to the value group $G'$ of $v'$. We indeed verify that the smallest convex subgroup $C$ of $G'$ containing $v(p)$ is the cyclic group generated by

$$\ldots$$
$v(p)/er$. Note that an element $g'$ of $G'$ belongs to $C$ if and only if $\max\{g', -g'\} \leq sv(p)$ for some positive integer $s$. Let $h$ be any positive element of $C$. There exists a non-negative integer $m$ such that $mv(p)/e \leq nh < (m + 1)v(p)/e$. As $v(p)/e$ is the least positive element of $G$ and $nh - mv(p)/e$ belongs to $G$, it follows that $nh = mv(p)/e$. So we can write $h = av(p)/ber$ where $a$ and $b$ are coprime positive integers. If $a', b'$ are integers satisfying $aa' + bb' = 1$, then it is clear that $v(p)/ber = a'h + (b'v(p)/er)$ is an element of $G'$. Since $r$ is the largest integer such that $v(p)/er$ belongs to $G'$, we conclude that $b = 1$ and hence $h = av(p)/er$ is in the cyclic group generated by $v(p)/er$ as desired.

To prove that $(K', v')/(K, v)$ is defectless, in view of Theorem 1.1, it is enough to show that the set $A_{K'/K}$ has a minimum element. Observe that $v(Tr_{K'/K}(1)) - v(1) = v(n)$ belongs to $A_{K'/K} \cap C$. Since $C$ is the cyclic group generated by $v(p)/er$, it follows that $\min A_{K'/K} = qv(p)/er$ where $q$ is the least non-negative integer such that $qv(p)/er$ belongs to $A_{K'/K}$.

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References


