On algebraically maximal valued fields and defectless extensions*

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Abstract. Let v be a henselian Krull valuation of a field K. In this paper, the authors give some necessary and sufficient conditions for a finite simple extension of (K, v) to be defectless. Various characterizations of algebraically maximal valued fields are also given which lead to a new proof of a result proved by Yu. L. Ershov.

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1. Basic definitions and statement of results

Throughout this paper, by a valuation v of a field K we mean a Krull valuation, i.e., v is a mapping from K onto $G \cup \{\infty\}$, where G is a totally ordered additively written abelian group, such that for all $x,\ y$ in K, the following properties are satisfied:

(i) $v(x) = \infty$ if and only if $x = 0$;

(ii) $v(xy) = v(x) + v(y)$;

(iii) $v(x+y) \geqslant \min\{v(x), v(y)\}$.

The pair (K, v) is called a valued field and G the value group of v. The subring $\mathcal{O}_v = \{x \in K |\ v(x) \geqslant 0\}$ of K with unique maximal ideal $\mathcal{M}_v = \{x \in K |\ v(x) > 0\}$ is called the valuation ring of v, and $\mathcal{O}_v/\mathcal{M}_v$ its residue field. A valuation v' is said to be an extension or prolongation of v to an overfield K' of K if v' coincides with v on K, in which case (K', v') is said to be an extension of (K, v). For a valued field extension $(K', v')/(K, v)$, if $G \subseteq G'$ and $\mathcal{O}_v/\mathcal{M}_v \subseteq \mathcal{O}_{v'}/\mathcal{M}_{v'}$, denote respectively the value groups and the residue fields of v, v', then the index $[G' : G]$ and the degree of the field extension $\mathcal{O}_{v'}/\mathcal{M}_{v'}$ over $\mathcal{O}_v/\mathcal{M}_v$ are called respectively the index of ramification and the residual degree of v'/v. An extension (K', v') of (K, v) is said to be immediate if the value groups and the residue fields of v' and v coincide, i.e., the index of ramification and the residual degree of v'/v are both one. A valued field (K, v) is said to be henselian if v has a unique prolongation to the algebraic closure of K. Henselian valued fields form an important class of valued fields and have several equivalent characterizations (cf. [3], [4, Theorem 4.1.3], [8]). In this paper, we characterize some special types of henselian valued fields.

In what follows, v is a henselian valuation of a field K and $\tilde{v}$ is the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K. In this paper, we prove that a valued field (K, v) is algebraically maximal, i.e., it has no proper immediate algebraic extension if and only if the set $\{\tilde{v}(\theta - a) \mid a \in K\}$ has a maximum element for every θ in $\widetilde{K} \setminus K$. It is shown that the above characterization quickly yields a result proved by Yu. L. Ershov which states that (K, v) is algebraically maximal if and only if the set $\{v(f(a)) \mid a \in K\}$ has a maximum element for every polynomial $f(x)$ belonging to $K[x]$ (cf. [5, Prop. 1.5.4, p.54, p.259]). Further it is also shown that for any fixed θ in $\widetilde{K}$ which is algebraic over K of degree $n > 1$, each of the sets $M_j(\theta)$, $1 \leqslant j \leqslant n-1$, defined by

$$M_j(\theta) = \{\tilde{v}(\theta - \beta) \mid \beta \in \widetilde{K},\ [K(\beta) : K] \leqslant j\} \tag{1}$$

has a maximum if and only if $K(\theta)$ is a defectless extension of (K, v). Recall that a finite extension (K', v') of a henselian valued field (K, v) (or briefly K'/K) is said to be defectless if $[K' : K] = ef$ where e, f are the index of ramification and the residual degree of v'/v. Precisely stated, we prove

Theorem 1.1. *Let v be a henselian valuation of a field K and $\tilde{v}$ be the unique prolongation of v to the algebraic closure $\widetilde{K}$ of K. The following statements are equivalent:*
(i) (K, v) is algebraically maximal.
(ii) For every θ in $\widetilde{K} \setminus K$, the set $\{\tilde{v}(\theta - a) \mid a \in K\}$ has a maximum.
(iii) For each monic irreducible polynomial $f(x) \in K[x]$, there exists an element a_f belonging to K such that $v(f(a_f)) \geq v(f(a))$ for every a in K.
(iv) For each polynomial $f(x)$ belonging to $K[x]$, there exists a_f belonging to K such that $v(f(a_f)) \geq v(f(a))$ for every a in K.

The equivalence of (i) and (ii) above will be deduced from a slightly more general result to be proved as Theorem 1.2.
Theorem 1.2. *Let (K, v), $(\widetilde{K}, \tilde{v})$ be as in the above theorem and θ be an element of $\widetilde{K} \setminus K$. Then the set $\{\tilde{v}(\theta - a) \mid a \in K\}$ has no maximum if and only if there exists γ belonging to $\widetilde{K} \setminus K$ with $[K(\gamma) : K] \leqslant [K(\theta) : K]$ such that $K(\gamma)/K$ is an immediate extension and $\tilde{v}(\gamma - a) = \tilde{v}(\theta - a)$ for every a in K.*

As regards defectless extensions, we prove
Theorem 1.3. *Let $(K, v), (\widetilde{K}, \tilde{v})$ be as in Theorem 1.1 and θ be an element of $\widetilde{K} \setminus K$ with the minimal polynomial $g(x)$ over K of degree n. The following statements are equivalent:*
(i) $K(\theta)/K$ is a defectless extension.
(ii) The set $M_j(\theta) = \{\tilde{v}(\theta - \beta) \mid \beta \in \widetilde{K},\ [K(\beta) : K] \leq j\}$ has a maximum element for each number j not exceeding $n - 1$.
(iii) For $1 \leq j \leq n-1$, the set $N_j(g) = \{\tilde{v}(g(\beta)) \mid \beta \in \widetilde{K},\ [K(\beta) : K] \leq j\}$ has a maximum element.

Our proof in fact specifies the elements β_j with $[K(\beta_j) : K] \leqslant j$ such that $\max N_j(g) = \tilde{v}(g(\beta_j))$, $1 \leqslant j \leqslant n-1$ (see Remark 3.6).

It may be pointed out that some other characterizations of finite separable defectless extensions are given in [2] and [6].

2. Proof of Theorems 1.1, 1.2.

In what follows, (K, v) and $(\widetilde{K}, \tilde{v})$ are as in Theorem 1.1. By the degree of an element α in $\widetilde{K}$, we shall mean the degree of the extension $K(\alpha)/K$ and shall denote it by *deg* α. For an element ξ in the valuation ring of $\tilde{v}$, $\bar{\xi}$ will stand for its $\tilde{v}$-residue, i.e., the image of ξ under the canonical homomorphism from the valuation ring of $\tilde{v}$ onto its residue field. When there is no chance of confusion, we shall write $\tilde{v}(\alpha)$ as $v(\alpha)$ for α belonging to $\widetilde{K}$.

Proposition 2.1. *Suppose that the set* $M_1 = \{\tilde{v}(\alpha - a) \mid a \in K\}$ *does not have a maximum element for some* α *belonging to* $\widetilde{K} \setminus K$. *Then either* $K(\alpha)$ *is an immediate extension of* (K, v) *or there exists* β *belonging to* $\widetilde{K} \setminus K$ *with* *deg* $\beta <$ *deg* α *such that* $\tilde{v}(\alpha - a) = \tilde{v}(\beta - a)$ *for each* a *in* K.

Proof. Let M denote the set $\{\tilde{v}(\alpha - \beta) \mid \beta \in \widetilde{K},\ deg\ \beta < deg\ \alpha\}$ containing M_1 and $\sup M$ its supremum. The proof is split into two cases.

Case I. $\sup M_1 < \sup M$. Then there exists β belonging to $\widetilde{K}$ with *deg* $\beta <$ *deg* α such that $\tilde{v}(\alpha - \beta) \geq \sup M_1$. Since M_1 does not have a maximum element, the above inequality shows that $\tilde{v}(\alpha - \beta) > \tilde{v}(\alpha - a)$ for every a in K. Therefore by the strong triangle law, for any element a of K, we have

$$\tilde{v}(\beta - a) = \min\{\tilde{v}(\beta - \alpha), \tilde{v}(\alpha - a)\} = \tilde{v}(\alpha - a).$$

Case II. $\sup M_1 = \sup M$. Then M_1 is a cofinal subset of M. In this case we show that $K(\alpha)/K$ is an immediate extension. For this it is enough to prove that for any polynomial $h(x)$ belonging to $K[x]$ with *deg* $h(x) <$ *deg* α, there exists $c \in K$ such that

$$\tilde{v}\left(\frac{h(\alpha)}{h(c)} - 1\right) > 0. \tag{2}$$

Write $h(x) = a\prod_{j=1}^{t}(x - \gamma_j)$. Since *deg* $\gamma_j \leqslant$ *deg* $h(x) <$ *deg* α and $\tilde{v}(\alpha - \gamma_j) \in M$, there exists an element $\tilde{v}(\alpha - a_s)$ of M_1 such that $\tilde{v}(\alpha - \gamma_j) < \tilde{v}(\alpha - a_s)$ for $1 \leq j \leq t$; consequently by the strong triangle law, we have

$$\tilde{v}(a_s - \gamma_j) = \min\{\tilde{v}(a_s - \alpha), \tilde{v}(\alpha - \gamma_j)\} = \tilde{v}(\alpha - \gamma_j) < \tilde{v}(\alpha - a_s).$$

On writing $\frac{h(\alpha)}{h(a_s)} = \prod_{j=1}^{t} \left(\dfrac{\alpha - \gamma_j}{a_s - \gamma_j} \right)$ as $\prod_{j=1}^{t} \left(1 + \dfrac{\alpha - a_s}{a_s - \gamma_j} \right)$ and using the above inequality, we see that $\tilde{v} \left(\frac{h(\alpha)}{h(a_s)} - 1 \right) > 0$ which proves (2) with $c = a_s$.

Proof of Theorem 1.2. Suppose first that $\{\tilde{v}(\theta - a) \mid a \in K\}$ does not have a maximum element. Then by Proposition 2.1, either $K(\theta)/K$ is an immediate extension or there exists η belonging to $\widetilde{K} \setminus K$ with $deg\ \eta < deg\ \theta$ such that $v(\theta - a) = v(\eta - a)$ for every a in K. If $K(\theta)/K$ is an immediate extension, then we take $\gamma = \theta$, otherwise applying Proposition 2.1 to η, we see that there exists β belonging to $\widetilde{K} \setminus K$ with $deg\ \beta < deg\ \eta$ such that either $K(\beta)/K$ is an immediate extension or $v(\beta - a) = v(\eta - a) = v(\theta - a)$ for every a in K. The above process must terminate after a finite number of steps giving us an element γ belonging to $\widetilde{K} \setminus K$ with $deg\ \gamma \leqslant deg\ \theta$ such that $K(\gamma)/K$ is an immediate extension and $v(\gamma - a) = v(\theta - a)$ for every a belonging to K.

Conversely suppose that there exists γ belonging to $\widetilde{K} \setminus K$ such that $K(\gamma)/K$ is an immediate extension and $\tilde{v}(\gamma - a) = \tilde{v}(\theta - a)$ for every a in K. We now show that the set $S = \{\tilde{v}(\gamma - a) \mid a \in K\}$ has no maximum element. Suppose to the contrary that $\tilde{v}(\gamma - c)$, $c \in K$ is the maximum element of S. Since $K(\gamma)/K$ is an immediate extension, there exists b in K such that $\tilde{v}(\gamma - c) = v(b)$; also we can find $d \in K$ such that the $\tilde{v}$-residue of $\frac{\gamma - c}{b}$ equals the $\tilde{v}$-residue of d, i.e., $\tilde{v}(\frac{\gamma - c}{b} - d) > 0$, which implies that $\tilde{v}(\gamma - c - bd) > v(b) = \tilde{v}(\gamma - c)$. This contradicts the choice of $\tilde{v}(\gamma - c)$.

Proof of Theorem 1.1. The equivalence of (i) and (ii) follows immediately from Theorem 1.2.

(ii)$\Rightarrow$(iii). Let $f(x) = \prod_{i=1}^{n} (x - \alpha^{(i)})$ be any monic irreducible polynomial over K having a root α in $\widetilde{K}$. There exists c belonging to K such that $v(\alpha - c) = \max\{v(\alpha - a) \mid a \in K\}$. Since (K, v) is henselian for any a in K, we have

$$v(f(a)) = nv(\alpha - a) \leq nv(\alpha - c) = v(f(c)).$$

(iii)$\Rightarrow$(iv). Let $f(x)$ be any polynomial belonging to $K[x]$ with the factorization $bf_1(x)^{m_1} f_2(x)^{m_2} \ldots f_r(x)^{m_r}$ into powers of distinct monic irreducible polynomials over K. Let n_i denote the degree of $f_i(x)$ and θ_i be a root of $f_i(x)$. By (iii), there exist c_i belonging to K for $1 \leq i \leq r$ such that $v(f_i(c_i)) = \max\{\ v(f_i(a)) \mid a \in K\}$, i.e., $v(\theta_i - c_i) = \max\{v(\theta_i - a) \mid a \in K\}$. It will be proved that for each d belonging to K, we have

$$v(f(d)) \leq \max_{1\leq i\leq r}\{v(f(c_i))\}. \tag{3}$$

Fix any d in K. Choose an index $j \geq 1$ such that

$$v(c_j - d) = \max_{1\leq i\leq r}\{v(c_i - d)\}. \tag{4}$$

We are going to prove that $v(f(d)) \leqslant v(f(c_j))$, which is the same as showing

$$\sum_{i=1}^{r} m_i n_i v(d-\theta_i) \leqslant \sum_{i=1}^{r} m_i n_i v(c_j - \theta_i).$$

This would follow as soon as it is shown that

$$v(d-\theta_i) \leqslant v(c_j-\theta_i), \quad 1 \leqslant i \leqslant r. \tag{5}$$

Note that $v(c_i - d) \geqslant \min\{v(c_i-\theta_i), v(\theta_i - d)\} = v(\theta_i - d)$. In view of (4) and the above inequality, we have

$$v(c_j - d) \geqslant v(c_i - d) \geqslant v(\theta_i - d), \quad 1 \leqslant i \leqslant r,$$

which gives $v(c_j-\theta_i) \geqslant \min\{v(c_j-d), v(d-\theta_i)\} = v(d-\theta_i)$. Thus (5) and hence (3) is proved.

(iv)⇒(ii). Let θ be an element of $\widetilde{K}\setminus K$ and $f(x)$ be its minimal polynomial over K of degree n. By hypothesis, there exists an element a_f belonging to K such that $v(f(a_f)) \geqslant v(f(a))$ for each a in K. Since (K,v) is henselian, the above inequality is equivalent to saying that $v(\theta - a_f) = \max\{v(\theta - a) \mid a \in K\}$.

3. Proof of Theorem 1.3.

We retain the notations introduced in the opening lines of the second section. For a subfield L of $\widetilde{K}$, let v_L denote the valuation of L obtained by restricting $\tilde{v}$. As usual, *def*(L/K) will stand for the defect of a finite extension L of (K,v) defined by

$$\mathit{def}(L/K) = [L:K]/ef$$

where e, f are the index of ramification and the residual degree of v_L/v.

As in [7], a pair (θ, α) of elements of $\widetilde{K}$ is called a distinguished pair (more precisely a (K,v)-distinguished pair) if the following three conditions are satisfied:

(i) $deg\ \theta > deg\ \alpha$, (ii) $\tilde{v}(\theta-\beta) \leqslant \tilde{v}(\theta-\alpha)$ for every β in $\widetilde{K}$ with $deg\ \beta < deg\ \theta$, (iii) whenever $\gamma \in \widetilde{K}$ with $deg\ \gamma < deg\ \alpha$, then $\tilde{v}(\theta-\gamma) < \tilde{v}(\theta-\alpha)$.

Remark 3.1. If (θ,α) is a distinguished pair and $deg\ \theta = n$, then the set $M_{n-1}(\theta)$ defined by (1) has a maximum element, viz. $\tilde{v}(\theta-\alpha)$. Conversely if α is an element of smallest degree over K for which $\tilde{v}(\theta-\alpha)$ is the maximum of $M_{n-1}(\theta)$, then clearly (θ,α) is a distinguished pair.

The following already known result will be used in the sequel; its proof is omitted (cf. [1, Section 3, p.223], [2, Theorem 1.1(iii)]).

Theorem A. *Let (θ,α) be a (K,v)-distinguished pair. Then def$(K(\theta)/K)$ = def$(K(\alpha)/K)$.*

We now prove

Lemma 3.2. *Let (θ,α) be a (K,v)-distinguished pair with $deg\ \alpha = n_1$. Then $M_j(\theta) = M_j(\alpha)$ for $1 \leqslant j \leqslant n_1 - 1$.*

Proof. Let γ be any element of $\widetilde{K}$ with $deg\ \gamma \leqslant j \leqslant n_1 - 1$. Then by the definition of a distinguished pair $\tilde{v}(\theta-\gamma) < \tilde{v}(\theta-\alpha)$; consequently by the strong triangle law $\tilde{v}(\alpha-\gamma) = \min\{\tilde{v}(\alpha-\theta), \tilde{v}(\theta-\gamma)\} = \tilde{v}(\theta-\gamma)$ which proves the lemma.

The result stated below is proved implicitly in the fourth section of [1] and explicitly in [2, Theorem 2.4]. Its proof is omitted.

Lemma 3.3. *Suppose that $K(\theta)/K$ is a defectless extension of degree $n > 1$. Then the set $M_{n-1}(\theta)$ has a maximum element.*

Lemma 3.4. *Let (θ,α) be a (K,v)-distinguished pair. Let $f(x)$, $g(x)$ be the minimal polynomials over K of α, θ respectively of degree n_1 and n. Then for any γ belonging to $\widetilde{K}$ with $deg\ \gamma \leqslant n_1 - 1$, one has $\tilde{v}(g(\gamma)) = \frac{n}{n_1}\tilde{v}(f(\gamma))$.*

Proof. Let $h(x)$ belonging to $K[x]$ be the minimal polynomial of γ of degree m. Write $g(x) = \prod_{j=1}^{n}(x-\theta^{(j)})$, $h(x) = \prod_{i=1}^{m}(x-\gamma^{(i)})$. Since $g(x)$, $h(x)$ are irreducible over the henselian valued field (K,v), it follows that

$$\tilde{v}(g(\gamma^{(i)})) = \tilde{v}(g(\gamma)), \quad \tilde{v}(h(\theta^{(j)})) = \tilde{v}(h(\theta)), \quad 1 \leqslant i \leqslant m, \quad 1 \leqslant j \leqslant n. \qquad (6)$$

Keeping in view (6) and the equality $\prod_{i=1}^{m} g(\gamma^{(i)}) = \pm \prod_{j=1}^{n} h(\theta^{(j)})$, it follows that

$m\tilde{v}(g(\gamma)) = n\tilde{v}(h(\theta))$, i.e.,

$$\tilde{v}(g(\gamma)) = \frac{n}{m}\tilde{v}(h(\theta)). \tag{7}$$

Writing $f(x) = \prod_{k=1}^{n_1}(x - \alpha^{(k)})$ and arguing as above, it can be seen that

$$\tilde{v}(f(\gamma)) = \frac{n_1}{m}\tilde{v}(h(\alpha)). \tag{8}$$

Since $deg\ \gamma \leqslant n_1 - 1$, it follows from the definition of a distinguished pair that $\tilde{v}(\theta - \gamma^{(i)}) < \tilde{v}(\theta - \alpha)$; consequently by the strong triangle law

$$\tilde{v}(\alpha - \gamma^{(i)}) = \min\{\tilde{v}(\alpha - \theta), \tilde{v}(\theta - \gamma^{(i)})\} = \tilde{v}(\theta - \gamma^{(i)}).$$

On summing over i, we see that $\tilde{v}(h(\alpha)) = \tilde{v}(h(\theta))$, which combined with (7) and (8) proves the lemma.

The following lemma needed for the proof of Theorem 1.3 is also of independent interest as pointed out in Remark 3.6.

Lemma 3.5. *Let (θ, α) be a (K, v)-distinguished pair and $g(x)$ be the minimal polynomial of θ over K of degree n. For any β belonging to $\widetilde{K}$ with deg $\beta \leqslant n-1$, one has $\tilde{v}(g(\beta)) \leqslant \tilde{v}(g(\alpha))$.*

Proof. Let β be as above. Since $\tilde{v}(g(\beta)) = \tilde{v}(g(\beta'))$ for every K-conjugate β' of β, it may be assumed without loss of generality that

$$\tilde{v}(\theta - \beta) = \max\{\tilde{v}(\theta - \beta') \mid \beta' \text{ runs over all } K\text{-conjugates of } \beta\}. \tag{9}$$

Write $g(x) = \prod_{i=1}^{n}(x - \theta^{(i)})$. It will be shown that for $1 \leqslant i \leqslant n$,

$$\tilde{v}(\beta - \theta^{(i)}) \leqslant \tilde{v}(\alpha - \theta^{(i)}). \tag{10}$$

Since (θ, α) is a distinguished pair and $deg\ \beta \leqslant n - 1$, we have

$$\tilde{v}(\alpha - \beta) \geqslant \min\{\tilde{v}(\alpha - \theta), \tilde{v}(\theta - \beta)\} = \tilde{v}(\theta - \beta). \tag{11}$$

Fix any i, $1 \leqslant i \leqslant n$. Since (K, v) is henselian, $\tilde{v}(\theta^{(i)} - \beta) = \tilde{v}(\theta - \beta')$ for some K-conjugate β' of β. Therefore using (11) and (9), we obtain

$$\tilde{v}(\alpha - \beta) \geqslant \tilde{v}(\theta - \beta) \geqslant \tilde{v}(\theta - \beta') = \tilde{v}(\theta^{(i)} - \beta). \tag{12}$$

It follows from (12) and the triangle law that

$$\tilde{v}(\alpha - \theta^{(i)}) \geqslant \min\{\tilde{v}(\alpha - \beta), \tilde{v}(\beta - \theta^{(i)})\} = \tilde{v}(\beta - \theta^{(i)}),$$

which proves (10) and hence the lemma.

Proof of Theorem 1.3. We prove the equivalence of (i) and (ii) and then of (ii) and (iii) by induction on n.
(i)$\Rightarrow$(ii). If $K(\theta)/K$ is a defectless extension of degree 2, then the set $M_1(\theta) = \{v(\theta - a) \mid a \in K\}$ has a maximum element in view of Proposition 2.1. Assume that the result holds for all elements of degree not exceeding $n-1$ and that $K(\theta)/K$ is a defectless extension of degree $n \geqslant 3$. Now by Lemma 3.3 and Remark 3.1, there exists an element θ_1 belonging to $\widetilde{K}$ such that (θ, θ_1) is a distinguished pair. Let n_1 denote the degree of θ_1. Applying Theorem A, we see that $K(\theta_1)/K$ is a defectless extension. By Lemma 3.2, $M_j(\theta) = M_j(\theta_1)$ for $1 \leqslant j \leqslant n_1 - 1$. Therefore by induction hypothesis, $M_j(\theta_1)$ and hence $M_j(\theta)$ has a maximum element for $1 \leqslant j \leqslant n_1 - 1$. Also it is clear from the definition of a distinguished pair that $v(\theta - \theta_1)$ is the maximum element of $M_j(\theta)$ when $n_1 \leqslant j \leqslant n-1$ which completes the proof of (i) implies (ii).
(ii)$\Rightarrow$(i). When $n = 2$, then using the hypothesis that the set $\{v(\theta - a) \mid a \in K\}$ has a maximum element and arguing as in the last lines of the proof of Theorem 1.2, we conclude that $K(\theta)/K$ is not an immediate extension and hence it is a defectless extension of degree 2. Suppose that θ has degree n and the result is true for all elements of degree $\leqslant n-1$. Since $M_{n-1}(\theta)$ has a maximum element, there exists an element θ_1 of degree n_1 (say) such that (θ, θ_1) is a distinguished pair. By Lemma 3.2, $M_j(\theta) = M_j(\theta_1)$ for $1 \leqslant j \leqslant n_1 - 1$ and hence $M_j(\theta_1)$ has a maximum element. Therefore by induction hypothesis, $K(\theta_1)/K$ is a defectless extension and hence so is $K(\theta)/K$ in view of Theorem A.
(ii)$\Rightarrow$(iii). Let c be an element of K such that $v(\theta - c) = \max\{v(\theta - a) \mid a \in K\}$. Then in the case $n = 2$, the set $N_1(g) = \{v(g(a)) \mid a \in K\}$ has $2v(\theta - c)$ as maximum. Suppose that θ has degree n and the result is true for all elements of smaller degree. In view of the hypothesis, there exists an element θ_1 belonging to $\widetilde{K}$ such that (θ, θ_1) is a distinguished pair with *deg* $\theta_1 = n_1$ (say). Then by Lemma 3.2, $M_j(\theta_1) = M_j(\theta)$ for $1 \leqslant j \leqslant n_1 - 1$. Therefore by induction hypothesis, if $f(x)$ is the minimal polynomial of θ_1 over K, then the set $N_j(f) = \{v(f(\beta)) \mid \beta \in \widetilde{K},\ deg\ \beta \leqslant j\}$ will have a maximum element for $1 \leqslant j \leqslant n_1 - 1$. Recall that by virtue of Lemma 3.4, for β belonging to $\widetilde{K}$ with *deg* $\beta \leqslant n_1 - 1$,

$v(g(\beta)) = \frac{n}{n_1} v(f(\beta))$. So it follows that the sets $N_j(g)$ will also have a maximum element for $1 \leqslant j \leqslant n_1 - 1$. Further by Lemma 3.5, $v(g(\theta_1))$ is the maximum of $N_{n-1}(g)$ and hence it is also the maximum of $N_j(g)$ when $n_1 \leqslant j \leqslant n-1$ which completes the proof of the desired assertion.

(iii)⇒(ii). For $n = 2$, the set $N_1(g) = \{2v(\theta - a) | a \in K\}$ has a maximum by (iii) and hence $M_1(\theta)$ has a maximum element. Suppose that $deg\ \theta = n$ and the result holds for elements of lower degree. Let α be an element of degree not exceeding $n-1$ such that $v(g(\alpha))$ is the maximum of the set $N_{n-1}(g)$. Replacing α by its K-conjugate, we can assume that

$$v(\theta - \alpha) = \max\{v(\theta - \alpha')|\ \alpha' \text{ runs over all } K\text{-conjugates of } \alpha\}. \tag{13}$$

Claim is that $M_{n-1}(\theta)$ has $v(\theta - \alpha)$ as maximum element. Suppose to the contrary that there exists an element γ belonging to $\widetilde{K}$ of degree $\leqslant n-1$ such that

$$v(\theta - \alpha) < v(\theta - \gamma). \tag{14}$$

We shall obtain the desired contradiction by showing that

$$v(g(\gamma)) > v(g(\alpha)). \tag{15}$$

To verify (15), note that in view of (14) and the strong triangle law, we have

$$v(\gamma - \alpha) = \min\{v(\gamma - \theta), v(\theta - \alpha)\} = v(\theta - \alpha). \tag{16}$$

Let $\theta^{(i)}$ be any K-conjugate of θ. Keeping in mind (16), (13) and the fact that $v(\alpha - \theta^{(i)}) = v(\alpha' - \theta) \leqslant v(\alpha - \theta)$, we have

$$v(\gamma - \theta^{(i)}) \geqslant \min\{v(\gamma - \alpha), v(\alpha - \theta^{(i)})\} = v(\alpha - \theta^{(i)}).$$

Summing over all K-conjugates $\theta^{(i)}$ of θ and using (14), we obtain (15). Hence the claim is proved. So there exists θ_1 in $\widetilde{K}$ such that (θ, θ_1) is a distinguished pair. Let $f(x)$ be the minimal polynomial of θ_1 over K of degree n_1. Then by virtue of Lemma 3.4, for any β belonging to $\widetilde{K}$ with $deg\ \beta \leqslant n_1 - 1$, we have

$$v(g(\beta)) = \frac{n}{n_1} v(f(\beta)). \tag{17}$$

By hypothesis, the sets $N_j(g)$ have a maximum element for $1 \leqslant j \leqslant n-1$. It now follows from (17) that $N_j(f) = \{v(f(\beta)) \mid \beta \in \widetilde{K},\ deg\ \beta \leqslant j\}$ has a maximum element for $1 \leqslant j \leqslant n_1 - 1$. Therefore by induction hypothesis, $M_j(\theta_1)$ and hence $M_j(\theta)$ will have a maximum element for $1 \leqslant j \leqslant n_1 - 1$. As $v(\theta - \theta_1)$ is the maximum element of $M_j(\theta)$ for $n_1 \leqslant j \leqslant n-1$, we see that (iii)$\Rightarrow$(ii).

Remark 3.6. Suppose that $K(\theta)/K$ is a defectless extension. In view of Lemma 3.3, there exists θ_1 such that (θ, θ_1) is a distinguished pair. By successive applications of Lemma 3.2, it follows that there exist distinguished pairs (θ, θ_1), $(\theta_1, \theta_2), \ldots, (\theta_{r-1}, \theta_r)$ with θ_r in K and $deg\ \theta_i = n_i$ (say). Using induction on $n_0 = deg\ \theta$ and applying Lemmas 3.5, 3.4, it can be quickly shown (as in the proof of (ii)$\Rightarrow$(iii) above) that

$$\max N_j(g) = v(g(\theta_i)) \quad \text{when} \quad n_i \leqslant j \leqslant n_{i-1} - 1,\ 1 \leqslant i \leqslant r.$$

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