ON FINITE INERTIAL EXTENSIONS OF A VALUED FIELD

L. van den Dries (January 2003)

Let \((K, v)\) be a valued field, and \((L, v)\) a finite extension of \((K, v)\) within the absolute inertia field of \((K, v)\). Let \(A\) denote the valuation ring of \((K, v)\) and \(B\) the valuation ring of \((L, v)\).

**Proposition.** There is \(\eta \in B\) such that

1. \(L = K(\eta)\), the (monic) minimum polynomial \(h(X)\) of \(\eta\) over \(K\) lies in \(A[X]\), and \(\eta\) and \(h'(\eta)\) are units of \(B\);
2. for each finite \(S \subseteq B\) there is \(u \in A[\eta]\) such that \(u\) is a unit of \(B\) and \(S \subseteq A[\eta, 1/u]\);
3. if \((K, v)\) has finite rank, then there is \(u \in A[\eta]\) such that \(u\) is a unit of \(B\) and \(B = A[\eta, 1/u]\).

In particular, \(\eta\) is integral over \(A\) and is a henselian element over \((K, v)\). The proposition implies (a strong form) of Kuhlmann’s Conjecture 1 in [K]. This result seems to be slightly different from other (strong) solutions given in notes by Roquette (May 2000), and Knaf & Kuhlmann (Oct. 2002), so is perhaps worth recording. Roquette shows that each element of \(B\) lies in a subring of \(B\) of the form \(A[\eta, r, s]\) with \(r, s\) henselian over \((K, v)\) (and \(\eta\) as in the proposition). Applied to the element \(1/u\) in (3), this yields:

**Corollary.** If \((K, v)\) has finite rank, then \(B = A[\eta, r, s]\) with \(\eta\) the (henselian) element of the proposition, and \(r, s\) henselian over \((K, v)\).

**Proof of the Proposition.** Let \(\eta\) be the element constructed in the proof of lemma 4 of [K], whose notations we use in what follows. We also use some lemmas from [L]. It is shown in [K] that (1) holds. Let \(m\) be the maximal ideal of \(A\), let \(A^*\) be the integral closure of \(A\) in \(L\), and let \(m^*\) be the maximal ideal of \(A^*\) such that \(B = A_{m^*}\). Let \(G(X) \in A[X]\) be a monic polynomial with image \(g(X)\) in \(K_v[X]\) where \(K_v = A/m\). By lemma 4 in [K] and lemma 12.5.7 in [L] the ring \(A[\eta] \subseteq A^*\) has exactly two maximal ideals, namely \((m, \eta)A[\eta]\) and \(n := (m, G(\eta))A[\eta]\). Note that \((m, \eta)A[\eta]\) is not contained in \(m^*\) since \(\eta\) is a unit in \(B\), and that \(n\) is contained in \(m^*\), so \(m^* \cap A[\eta] = n\). Thus, using Lemma 12.5.17 in [L] for the first inclusion, we have

\[
A^* \subseteq \frac{1}{h'(\eta)} A[\eta] \subseteq A[\eta]_n \subseteq A_{m^*} = B.
\]

Hence by Lemma 12.5.16 in [L] we have

\[
B = A[\eta]_n.
\]

This yields (2) by using common denominators.
Assume now that \((K, v)\) has finite rank. Then \(A\) has only finitely many prime ideals, so \(A^*\) has only finitely many prime ideals, and hence \(A[\eta]\) has only finitely many prime ideals. So we can take \(u_1 \in A[\eta]\setminus \mathfrak{n}\) such that \(u_1\) lies in each prime ideal of \(A[\eta]\) not contained in \(\mathfrak{n}\). Put \(u := u_1 h'(\eta)\). Then \(A[\eta]_u\) is a local ring, because the prime ideals of \(A[\eta]_u\) are exactly the \(pA[\eta]_u\) with \(p\) a prime ideal of \(A[\eta]\) that does not contain \(u\), and each such \(p\) is contained in \(\mathfrak{n}\). From

\[
A^* \subseteq \frac{1}{h'(\eta)} A[\eta] \subseteq A[\eta]_u \subseteq A^*_{m^*} = B
\]

and Lemma 12.5.16 in [L] we obtain \(B = A[\eta]_u = A[\eta, 1/u]\) as requested in (3). This finishes the proof.

Remarks

If \((K, v)\) has rank 1, then \((\mathfrak{m}, \eta)A[\eta]\) is the only prime ideal of \(A[\eta]\) not contained in \(\mathfrak{n}\), so we can take \(u_1 := \eta\) in that case. Thus \(B = A[\eta, 1/(\eta h'(\eta))]\) in the rank 1 case.

Parts (1) and (2) of the proposition go through in a more general setting: Let \(A\) be a local domain integrally closed in its field of fractions \(K\), let \(I(A)\) be the integral closure of \(A\) in \(K^{sep}\), and choose a maximal ideal \(p\) of \(I(A)\). (In the case that \(A\) is the valuation ring of a valuation \(v\) on \(K\), this is equivalent to choosing an extension of \(v\) to a valuation on \(K^{sep}\).) Let

\[
G^i := \{\sigma \in Gal(K^{sep}|K) : x - \sigma x \in p \text{ for all } x \in I(A)\}
\]

be the inertia group of \(p\), let \(L|K\) be a finite extension contained in the fixed field of \(G^i\), let \(A^* := I(A) \cap L\) be the integral closure of \(A\) in \(L\), put \(m^* := p \cap A^*\) and \(B := A^*_{m^*}\), so \(B\) is a local domain dominating \(A\) and integrally closed in its fraction field \(L\). Then lemma 4 in [K] and its proof go through (modulo obvious changes), and yield an element \(\eta \in A^*\) with property (1) in the proposition. The proof of the proposition shows that (2) holds as well.

[K] F.-V. Kuhlmann, A conjecture about finite extensions within the absolute inertia field of a valued field (March 2000)