DEEPLY RAMIFIED FIELDS, SEMITAME FIELDS, AND THE CLASSIFICATION OF DEFECT EXTENSIONS

FRANZ-VIKTOR KUHLMANN AND ANNA RZEPKA

Abstract. We study in detail the valuation theory of deeply ramified fields and introduce and investigate several other related classes of valued fields. Further, a classification of defect extensions of prime degree of valued fields that was earlier given only for the characteristic equal case is generalized to the case of mixed characteristic by a unified definition that works simultaneously for both cases. It is shown that deeply ramified fields and the other valued fields we introduce only admit one of the two types of defect extensions, namely the ones that appear to be more harmless in open problems such as local uniformization and the model theory of valued fields in positive characteristic. The classes of valued fields under consideration can be seen as generalizations of the class of tame valued fields. The present paper supports the hope that it will be possible to generalize to deeply ramified fields several important results that have been proven for tame fields and were at the core of partial solutions of the two open problems mentioned above.

1. Introduction

This paper owes its existence to the following well known deep open problems in positive characteristic:
1) resolution of singularities in arbitrary dimension,
2) decidability of the field $\mathbb{F}_q((t))$ of Laurent series over a finite field $\mathbb{F}_q$, and of its perfect hull.

Both problems are connected with the structure theory of valued function fields of positive characteristic $p$. The main obstruction here is the phenomenon of the defect, which we will define now.

By $(L|K,v)$ we denote a field extension $L|K$ where $v$ is a valuation on $L$ and $K$ is endowed with the restriction of $v$. The valuation ring of $v$ on $L$ will be denoted by $\mathcal{O}_L$, and that on $K$ by $\mathcal{O}_K$. Similarly, $\mathcal{M}_L$ and $\mathcal{M}_K$ denote the valuation ideals of $L$ and $K$. The value group of the valued field $(L,v)$ will be denoted by $vL$, and its residue field by $Lv$. The value of an element $a$ will be denoted by $va$, and its residue by $av$.

We will say that a valued field extension $(L|K,v)$ is unibranched if the extension of $v$ from $K$ to $L$ is unique. If $(L|K,v)$ is a finite unibranched extension, then

$$[L : K] = p^\nu \cdot (vL : vK)[Lv : Kv],$$

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where by the Lemma of Ostrowski $\nu$ is a nonnegative integer and $\hat{p}$ the characteristic exponent of $Kv$, that is, $\hat{p} = \text{char} \ Kv$ if it is positive and $\hat{p} = 1$ otherwise. The factor $d(L|K,v) = \hat{p}^\nu$ is called the defect of the extension $(L|K,v)$. If $\hat{p}^\nu > 1$, then $(L|K,v)$ is called a defect extension. If $d(L|K,v) = 1$, then we call $(L|K,v)$ a defectless extension. Nontrivial defect only appears when $\text{char} \ Kv = p > 0$, in which case $\hat{p} = p$.

Throughout this paper, when we talk of a defect extension $(L|K,v)$ of prime degree, we will always tacitly assume that it is a unibranched extension. Then it follows from (1) that $[L : K] = p = \text{char} \ Kv$ and that $(vL : vK) = 1 = [Lv : Kv]$; the latter means that $(L|K,v)$ is an immediate extension, i.e., the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto.

Via ramification theory, the study of defect extensions can be reduced to the study of purely inseparable extensions and of Galois extensions of degree $p = \text{char} \ Kv$. To this end, we fix an extension of $v$ from $K$ to its algebraic closure $\bar{K}$ of $K$. We denote the separable-algebraic closure of $K$ by $K^{\text{sep}}$. The absolute ramification field of $(K,v)$ (with respect to the chosen extension of $v$), denoted by $(K^r,v)$, is the ramification field of the normal extension $(K^{\text{sep}}|K,v)$. If $(K(a)|K,v)$ is a defect extension, then $(K^r(a)|K^r,v)$ is a defect extension with the same defect (see Proposition 2.13). On the other hand, $K^{\text{sep}}|K^r$ is a $p$-extension, so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree $p$.

Galois defect extensions of degree $p$ of valued fields of characteristic $p > 0$ (valued fields of equal characteristic) have been classified by the first author in [15]. There the extension is said to have dependent defect if it is related to a purely inseparable defect extension of degree $p$ in a way that we will explain in Section 3.3, and to have independent defect otherwise. Note that the condition for the defect to be dependent implies that the purely inseparable defect extension does not lie in the completion of $(K,v)$, hence if $(K,v)$ lies dense in its perfect hull, then it cannot have Galois defect extensions of prime degree with dependent defect.

The classification of defect extensions is important because work by M. Temkin (see e.g. [26]) and by the first author indicates that dependent defect appears to be more harmful to the above cited problems than independent defect. In the paper [5], S. D. Cutkosky and O. Piltant give an example of an extension of valued function fields consisting of a tower of two Galois defect extensions of prime degree where strong monomialization fails. As the valuation on these extensions is defined by use of generating sequences, it is hard to determine whether they have dependent or independent defect. However, work of Cutkosky, L. Ghezzi and S. ElHitti shows that both of them have dependent defect (see e.g. [6]); this again lends credibility to the hypothesis that dependent defect is the more harmful one.

An analogous classification of Galois defect extensions of degree $p$ of valued fields of characteristic 0 with residue fields of characteristic $p > 0$ (valued fields of mixed characteristic) has so far not been given. But such a classification is important for instance for the study of infinite algebraic extensions of the field $\mathbb{Q}_p$ of $p$-adic numbers, which in contrast to $\mathbb{Q}_p$ itself may well admit defect extensions. Indeed, $\mathbb{Q}_p^{ab}$, the maximal abelian extension of $\mathbb{Q}_p$, is such a field. Other examples will be given in Section 7. Moreover, we wish to study the valuation theory of deeply ramified fields (such as $\mathbb{Q}_p^{ab}$), which will be introduced below, in full generality without
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restriction to the equal characteristic case. For these fields in particular it is important to work out the similarities between the equal and the mixed characteristic cases.

The obvious problem for the definition of "dependent defect" in the mixed characteristic case is that a field of characteristic 0 has no nontrivial inseparable extensions. However, there is a characterization of independent defect equivalent to the one given in [15] that readily works also in the mixed characteristic case, and we use it to give a unified definition, as follows. Take a Galois defect extension $E = (L|K, v)$ of prime degree $p$. Then the set

$$\Sigma_\sigma := \left\{ v \left( \frac{\sigma f - f}{f} \right) \mid f \in L^\times \right\}$$

is a final segment of $vK$ and is independent of the choice of a generator $\sigma$ of $\text{Gal}(L|K)$ (see Theorem 3.5), and we will denote it by $\Sigma_E$. We will say that $E$ has independent defect if

$$\Sigma_E = \{ \alpha \in vK \mid \alpha > H_E \}$$

for some proper convex subgroup $H_E$ of $vK$; otherwise we will say that $E$ has dependent defect. If $(K, v)$ has rank 1, then condition (3) just means that $\Sigma_E$ consists of all positive elements in $vK$.

That our definition of "independent defect" in mixed characteristic is the right one is supported by the following observation. Take a valued field of positive characteristic. If it lies dense in its perfect hull, then by what we have said before, all Galois defect extensions of a valued field of equal characteristic must have independent defect. If the field is complete and of rank 1 (meaning that its value group can be seen as a subgroup of $\mathbb{R}$), then it is a perfectoid field. Such fields correspond via the so-called tilting construction to perfectoid field in mixed characteristic, and many important properties are preserved under the correspondence. So we expect that also perfectoid fields in mixed characteristic only admit independent defect extensions. This indeed holds with our definition.

For our purposes, the properties of completeness and rank 1 are irrelevant, and we prefer to work with a more flexible (and first order axiomatizable) notion. In fact, all perfectoid fields are deeply ramified, in the sense of [9]. Take a valued field $(K, v)$ with valuation ring $\mathcal{O}_K$. Choose any extension of $v$ to $K^{\text{sep}}$ and denote the valuation ring of $K^{\text{sep}}$ with respect to this extension by $\hat{\mathcal{O}}_K$. Then $(K, v)$ is a deeply ramified field if

$$\Omega_{\mathcal{O}_{K^{\text{sep}}}/\mathcal{O}_K} = 0,$$

where $\Omega_{B|A}$ denotes the module of relative differentials when $A$ is a ring and $B$ is an $A$-algebra. This definition does not depend on the chosen extension of the valuation from $K$ to $K^{\text{sep}}$.

According to [9, Theorem 6.6.12 (vi)], a nontrivially valued field $(K, v)$ is deeply ramified if and only if the following conditions hold:

(DRvg) whenever $\Gamma_1 \subsetneq \Gamma_2$ are convex subgroups of the value group $vK$, then $\Gamma_2/\Gamma_1$ is not isomorphic to $\mathbb{Z}$ (in other words, no archimedean component of $vK$ is discrete);

(DRvr) if char $Kv = p > 0$, then the homomorphism

$$\Omega_{\mathcal{O}_{K^{\text{sep}}}/\mathcal{O}_K} \ni x \mapsto x^p \in \mathcal{O}_{K^{\text{sep}}}$$

is surjective, where $\mathcal{O}_{K^{\text{sep}}}$ denotes the valuation ring of the completion of $(K, v)$. 


Axiom (DRvr) means that modulo $p\mathcal{O}_K$ every element in $\mathcal{O}_K$ is a $p$-th power.

By altering axiom (DRvg) we will now introduce new classes of valued fields, one of them containing the class of deeply ramified fields, and one contained in it in the case of positive residue characteristic. Note that axiom (DRvg) means that no archimedean component of $vK$ is isomorphic to $\mathbb{Z}$. We will call $(K,v)$ a 
\textbf{generalized deeply ramified field}, or in short a \textbf{gdr field}, if it satisfies axiom (DRvr) together with:

$(\text{DRvp})$ if $\text{char } Kv = p > 0$, then $vp$ is not the smallest positive element in the value group $vK$.

If $\text{char } Kv = p > 0$, then (DRvg) certainly holds whenever $vK$ is divisible by $p$.

We will call $(K,v)$ a \textbf{semitame field} if it satisfies axiom (DRvr) together with:

$(\text{DRst})$ if $\text{char } Kv = p > 0$, then the value group $vK$ is $p$-divisible.

We note:

\textbf{Proposition 1.1.} The properties (DRvg), (DRvp) and (DRst) are first order axiomatizable in the language of valued fields, and so are the classes of semitame, deeply ramified and gdr fields of fixed characteristic.

We will give the proof at the end of Section 6.

The notion of “semitame field” is reminiscent of that of “tame field”. Let us recall the definition of “tame”. For the purpose of this paper we will slightly generalize the notion of “tame extension” as defined in [18] (there, tame extensions were only defined over henselian fields). An algebraic unibranched extension $(L|K,v)$ will be called \textbf{tame} if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

$(\text{TE1})$ The ramification index $(vE : vK)$ is not divisible by $\text{char } Kv$.

$(\text{TE2})$ The residue field extension $Ev|Kv$ is separable.

$(\text{TE3})$ The extension $(E|K,v)$ is \textbf{defectless}, i.e., $[E : K] = (vE : vK)[Ev : Kv]$.

A henselian field $(K,v)$ is called a \textbf{tame field} if its algebraic closure with the unique extension of the valuation is a tame extension, and a \textbf{separably tame field} if its separable-algebraic closure is a tame extension. The absolute ramification field $(K^\flat,v)$ is the unique maximal tame extension of the henselian field $(K,v)$ by [7, Theorem (22.7)] (see also [23, Proposition 4.1]). Hence a henselian field is tame if and only if its absolute ramification field is already algebraically closed; in particular, every tame field is perfect.

In contrast to tame and separably tame fields, we do not require semitame fields to be henselian; in this way they become closer to deeply ramified fields. The other fundamental difference to tame fields is that semitame fields may admit defect extensions, but as we will see in Theorem 1.6 below, only those with independent defect. This justifies the hope that many of the results that have been proved for tame fields and applied to the problems we have cited in the beginning (see [18, 19]) can be generalized to the case of (henselian) semitame fields.

All valued fields of residue characteristic 0 are semitame and gdr fields, and they are deeply ramified fields if and only if (DRvg) holds. Likewise, all henselian valued fields of residue characteristic 0 are tame fields. In the present paper, we are not interested in the case of residue characteristic 0, so we will always assume that $\text{char } Kv = p > 0$. We will now summarize the basic facts about the connections between the properties we have introduced. The proofs will be provided in Section 6.
Theorem 1.2. 1) If \((K, v)\) is a nontrivially valued field with \(\text{char } Kv = p > 0\), then the following logical relations between its properties hold:

\[
\text{tame field} \Rightarrow \text{separably tame field} \Rightarrow \text{semitame field} \Rightarrow \text{deeply ramified field} \Rightarrow \text{gdr field}.
\]

2) For a valued field \((K, v)\) of rank 1 with \(\text{char } Kv = p > 0\), the three properties “semitame field”, “deeply ramified field” and “gdr field” are equivalent.

3) For a nontrivially valued field \((K, v)\) of characteristic \(p > 0\), the following properties are equivalent:
   a) \((K, v)\) is a semitame field,
   b) \((K, v)\) is a deeply ramified field,
   c) \((K, v)\) is a gdr field,
   d) \((K, v)\) satisfies (DR\(v\))
   e) the completion of \((K, v)\) is perfect,
   f) \((K, v)\) is dense in its perfect hull,
   g) \((K^p, v)\) is dense in \((K, v)\).

4) Every perfect valued field of positive characteristic is a semitame field.

In [21] the equivalence of assertions a) and f) of part 3) of this theorem is used to show that every valued field of positive characteristic that has only finitely many Artin-Schreier extensions is a semitame field. This proves that a nontrivially valued field of positive characteristic that is definable in an NTP\(_2\) theory is a semitame field, as it is shown in [4] that such a field has only finitely many Artin-Schreier extensions.

Take a valued field \((K, v)\) of characteristic 0 with residue characteristic \(p > 0\). Decompose \(v = v_0 \circ v_p \circ \overline{v}\), where \(v_0\) is the finest coarsening of \(v\) that has residue characteristic 0, \(v_p\) is a rank 1 valuation on \(Kv_0\), and \(\overline{v}\) is the valuation induced by \(v\) on the residue field of \(v_p\) (which is of characteristic \(p > 0\)). The valuations \(v_0\) and \(\overline{v}\) may be trivial. With this notation, we have:

Proposition 1.3. Under the above assumptions, the valued field \((K, v)\) is a gdr field if and only if \((Kv_0, v_p)\) is.

Note that by part 2) of Theorem 1.2, \((Kv_0, v_p)\) is already a semitame field once it is a gdr field.

From Theorem 1.2 and Proposition 1.3 it can be deduced that the three properties “semitame”, “deeply ramified” and “gdr” behave well for composite valuations.

Proposition 1.4. Take an arbitrary valued field \((K, v)\) and assume that \(v = w \circ \overline{w}\) with \(w\) and \(\overline{w}\) nontrivial. Then \((K, v)\) is a gdr field if and only if \((K, w)\) and \((Kw, \overline{w})\) are. If \(\text{char } Kw > 0\), then for \((K, v)\) to be a gdr field it suffices that \((K, w)\) is a gdr field. The same holds for “semitame” and “deeply ramified” in place of “gdr”.

If \(\text{char } Kw = 0\), then for \((K, v)\) to be a gdr field it suffices that \((Kw, \overline{w})\) is a gdr field.

The next theorem will show that we can reduce the study of several questions about semitame fields to considering their absolute ramification field.
Theorem 1.5. Take a valued field \((K, v)\), fix any extension of \(v\) to \(\bar{K}\), and let \((K', v)\) be the respective absolute ramification field of \((K, v)\). Then \((K', v)\) is a gdr field if and only if \((K, v)\) is, and \((K', v)\) is a semitame field if and only if \((K, v)\) is.

Note that even without the assumptions of the theorem, if \((K', v)\) is a gdr field, then it is already a deeply ramified field because \(vK'\) is divisible by every prime distinct from the residue characteristic. Hence if \((K, v)\) is a gdr field, then \((K', v)\) is a deeply ramified field. The converse is not true in general, since (DRvg) always holds in \((K', v)\) (as long as \(v\) is nontrivial), while it may not hold in \((K, v)\).

The next theorem addresses the connection of the properties we have defined with the classification of the defect. We denote by \((vK)_{vp}\) the smallest convex subgroup of \(vK\) that contains \(vp\) if \(\text{char } K = 0\), and set \((vK)_{vp} = vK\) otherwise. We call a valued field an \textbf{independent defect field} if all of its Galois defect extensions of prime degree have independent defect.

Theorem 1.6. 1) Take a valued field \((K, v)\) with \(\text{char } Kv = p > 0\). Then \((K, v)\) is a gdr field if and only if \((vK)_{vp}\) is \(p\)-divisible, \(Kv\) is perfect, and \((K, v)\) is an independent defect field.

2) A nontrivially valued field \((K, v)\) is semitame if and only if every unibranched Galois extension of prime degree is either tame or an extension with independent defect.

The classification of Galois defect extensions of prime degree in the equal characteristic case is also an important tool in the proof of Theorem 1.2 in [15], which we will state now. A valued field is called \textbf{algebraically maximal} (or \textbf{separable-algebraically maximal}) if it admits no nontrivial immediate algebraic (or separable-algebraic, respectively) extensions.

Theorem 1.7. A valued field of positive characteristic is a henselian and defectless field if and only if it is separable-algebraically maximal and each finite purely inseparable extension is defectless.

This theorem in turn is used in [14] for the construction of an example showing that a certain natural axiom system for the elementary theory of \(\mathbb{F}_p((t))\) (“henselian defectless valued field of characteristic \(p\) with residue field \(\mathbb{F}_p\) and value group a \(\mathbb{Z}\)-group”) is not complete.

A full analogue of Theorem 1.7 in mixed characteristic is not presently known. But we are able to show in Section 4 that the Galois extensions of prime degree with independent defect in mixed characteristic have the same properties as the ones in equal characteristic that have been used in [15] for the proof of Theorem 1.7. As a consequence, we are able to prove:

Theorem 1.8. Every algebraically maximal gdr field is a perfect, henselian and defectless field.

For the proof of this theorem we will need the following result which for the case of deeply ramified fields can be found in [9, Corollary 6.6.16 (i)]:

Theorem 1.9. Every algebraic extension of a deeply ramified field is again deeply ramified. The same holds for semitame fields and for gdr fields.
We will give the easy proof for the equal characteristic case in Section 6. For the mixed case we hope that eventually a direct valuation theoretical proof can be found. In view of Theorem 1.5 it suffices to prove that if \((K^r, v)\) is a gdr field and \((L|K^r, v)\) is a Galois extension of degree \(p\) with independent defect, then also \((L, v)\) is a gdr field. Understanding this implication without referring to the methods used in [9] would be important for the study of the more general class of independent defect fields. At this point, we are able to prove:

**Proposition 1.10.** 1) If \((K^r, v)\) is an independent defect field, then so is \((K, v)\).
2) A valued field \((K, v)\) of equal positive characteristic is an independent defect field if and only if every immediate purely inseparable extension of \((K, v)\) lies in its completion.

In contrast to the case of gdr fields, the property of being an independent defect field is not necessarily preserved under infinite algebraic extensions, as will be shown in Corollary 7.3 by the construction of a suitable algebraic extension of \(\mathbb{Q}_p\).

**Conjectures:** 1) If \((K, v)\) is an independent defect field, then also \((K^r, v)\) is an independent defect field.
2) A valued field \((K, v)\) of mixed characteristic with residue characteristic \(p\) is an independent defect field if and only if for every \(a \in \mathcal{O}_K\) for which the set \(\{v(a - c^p) \mid c \in K\}\) has no maximal element there is some \(c \in K\) such that \(v(a - c^p) \geq vp\).

Continuing the work presented in [5], the idea is presently investigated to employ higher ramification groups for the study of the ramification theory of 2-dimensional valued function fields. When working over valued fields with arbitrary value groups, the classical ramification numbers have to be replaced by **ramification jumps** which can be understood as cuts (or equivalently, final segments) in the value group (cf. Section 2.4).

While we are dealing with defect extensions \(\mathcal{E}\) of prime degree, in Theorem 3.5 we show that \(\Sigma_{\mathcal{E}}\) is a ramification jump. This allows us to characterize independent defect vis this ramification jump and its associated ramification ideal.

Moreover, for Galois defect extensions \((L|K, v)\) of prime degree we will compute in Section 5 the image of the valuation ideal \(\mathcal{M}_L\) under the trace of the extension. This allows us to characterize independent defect in yet another way, see Theorem 5.2. In summary, we obtain the following equivalent conditions for independent defect.

**Theorem 1.11.** Take a Galois defect extension \(\mathcal{E} = (K(a)|K, v)\) of prime degree with Galois group \(G\). Then the following assertions are equivalent:

a) \(\Sigma_{\mathcal{E}} = \{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}\) for some proper proper convex subgroup \(H_{\mathcal{E}}\) of \(vK\), i.e., \(\mathcal{E}\) has independent defect,

b) the ramification jump \(\Sigma_{-}(G)\) is equal to \(\{\alpha \in vK \mid \alpha > H_{\mathcal{E}}\}\) for some proper proper convex subgroup \(H_{\mathcal{E}}\) of \(vK\),

c) the ramification ideal \(I_{-}(G)\) is a valuation ideal \(\mathcal{M}_H\) of \(L\) that contains the valuation ideal \(\mathcal{M}_L\) of \((L, v)\),

d) the distance of \(\mathcal{E}\) is idempotent,

e) the trace \(Tr_{L|K}(\mathcal{M}_L)\) is a valuation ideal \(\mathcal{M}_H\) of \(K\) that is contained in \(\mathcal{M}_K\).
If the rank of \((K, v)\) is 1, then \(H_E\) can only be equal to \(\{0\}\) and \(M_H\) can only be equal to \(M_L\).

In a subsequent paper we will prove in addition that \(E\) has independent defect if and only if \(\Omega_{\Omega_L|\Omega_K} = 0\).

To shorten expressions, we will often write “independent defect extension” in place of “extension with independent defect”.

2. Preliminaries

2.1. Cuts, distances and defect. We recall basic notions and facts connected with cuts in ordered abelian groups and distances of elements of valued field extensions. For the details and proofs see Section 2.3 of [15] and Section 3 of [24].

Take a totally ordered set \((T, <)\). For a nonempty subset \(S\) of \(T\) and an element \(t \in T\) we will write \(S < t\) if \(s < t\) for every \(s \in S\). A set \(S \subseteq T\) is called an initial segment of \(T\) if for each \(s \in S\) every \(t < s\) also lies in \(S\). Similarly, \(S \subseteq T\) is called a final segment of \(T\) if for each \(s \in S\) every \(t > s\) also lies in \(S\). A pair \((\Lambda^L, \Lambda^R)\) of subsets of \(T\) is called a cut in \(T\) if \(\Lambda^L\) is an initial segment of \(T\) and \(\Lambda^R = T \setminus \Lambda^L\); it then follows that \(\Lambda^R\) is a final segment of \(T\). To compare cuts in \((T, <)\) we will use the lower cut sets comparison. That is, for two cuts \(\Lambda_1 = (\Lambda_1^L, \Lambda_1^R), \Lambda_2 = (\Lambda_2^L, \Lambda_2^R)\) in \(T\) we will write \(\Lambda_1 < \Lambda_2\) if \(\Lambda_1^L \subseteq \Lambda_2^L\), and \(\Lambda_1 \leq \Lambda_2\) if \(\Lambda_1^L \subseteq \Lambda_2^L\).

For any \(t \in T\) define the following principal cuts:

\[
\begin{align*}
  s^- & := \{t \in T | t < s\}, \{t \in T | t \geq s\}, \\
  s^+ & := \{t \in T | t \leq s\}, \{t \in T | t > s\}.
\end{align*}
\]

We identify the element \(s\) with \(s^+\). Therefore, for a cut \(\Lambda = (\Lambda^L, \Lambda^R)\) in \(T\) and an element \(s \in T\) the inequality \(\Lambda < s\) means that for every element \(t \in \Lambda^L\) we have \(t < s\). Similarly, for any subset \(M\) of \(T\) we define \(M^+\) to be a cut \((\Lambda^L, \Lambda^R)\) in \(T\) such that \(\Lambda^L\) is the smallest initial segment containing \(M\), that is,

\[
M^+ = \{\{t \in T | \exists m \in M t \leq m\}, \{t \in T | t > M\}\}.
\]

Likewise, we denote by \(M^-\) the cut \((\Lambda^L, \Lambda^R)\) in \(T\) such that \(\Lambda^L\) is the largest initial segment disjoint from \(M\), i.e.,

\[
M^- = \{\{t \in T | t < M\}, \{t \in T | \exists m \in M t \geq m\}\}.
\]

For every extension \((L|K, v)\) of valued fields and \(z \in L\) define

\[
v(z - K) := \{v(z - c) | c \in K\}.
\]

The set \(v(z - K) \cap vK\) is an initial segment of \(vK\) and thus the lower cut set of a cut in \(vK\). However, it is more convenient to work with the cut

\[
\text{dist} (z, K) := (v(z - K) \cap vK)^+ \quad \text{in the divisible hull } v\overline{K} \text{ of } vK.
\]

We call this cut the distance of \(z\) from \(K\). The lower cut set of dist \((z, K)\) is the smallest initial segment of \(v\overline{K}\) containing \(v(z - K) \cap vK\). If \((F|K, v)\) is an algebraic subextension of \((L|K, v)\) then \(v\overline{F} = v\overline{K}\). Thus dist \((z, K)\) and dist \((z, F)\) are cuts in the same group and we can compare these cuts by set inclusion of the lower cut sets. Since \(v(z - K) \subseteq v(z - F)\) we deduce that

\[
\text{dist} (z, K) \leq \text{dist} (z, F).
\]
If \( \text{char} \, K = p > 0 \) and \( z \in K \), then \( K^p \) is a subfield of \( K \), and the expressions
\[
v(z - K^p) \quad \text{and} \quad \text{dist} \,(z, K^p)
\]
are covered by our above definitions. We generalize this to the case where \( \text{char} \, K = 0 \) with the same definitions but note that \( v(z - K^p) \cap vK \) is not necessarily an initial segment of \( vK \).

If \( S \) is any subset of an abelian group \( T \), then for every \( t \in T \) and \( n \in \mathbb{Z} \) we set
\[
t + nS := \{ t + ns \mid s \in S \};
\]
in particular, \( -S = \{ -s \mid s \in S \} \). If \( \Lambda = (\Lambda^L, \Lambda^R) \) is a cut in a divisible ordered abelian group \( \Gamma \) and \( n > 0 \), then \( n\Lambda^L \) is again an initial segment of \( \Gamma \); we denote by \( n\Lambda \) the cut in \( \Gamma \) with the lower cut set \( n\Lambda^L \). Further, we define \( -\Lambda \) to be the cut \((-\Lambda^R, -\Lambda^L)\).

We say that the distance \( \text{dist} \,(z, K) \) is idempotent if
\[
n \cdot \text{dist} \,(z, K) = \text{dist} \,(z, K)
\]
for some natural number \( n \geq 2 \) (and hence for all \( n \in \mathbb{N} \)). The following characterization of cuts distances is a consequence of \([15, \text{Lemma 2.14}]\):

**Lemma 2.1.** A cut in \( \widetilde{K} \) is idempotent if and only if it is equal to \( H^- \) or \( H^+ \) for some convex subgroup \( H \) of \( \widetilde{K} \).

If \( y \) is another element of \( L \) then we define:
\[
z \sim_K y \iff v(z - y) > \text{dist} \,(z, K).
\]
The next lemma shows, among other things, that the relation \( \sim_K \) is symmetrical.

**Lemma 2.2.** Take a valued field extension \((L|K,v)\) and elements \( z, y \in L \).
1) If \( z \sim_K y \), then \( v(z - c) = v(y - c) \) for all \( c \in K \) such that \( v(z - c) \in vK \), and \( \text{dist} \,(z, K) = \text{dist} \,(y, K) \) and \( y \sim_K z \).
2) If \( (K(z)|K,v) \) is immediate, then \( v(z - K) \) has no largest element and is a subset of \( vK \).
3) If \( v(z - K) \cap vK \) has no maximal element, then \( z \sim_K y \) holds if and only if \( v(z - c) = v(y - c) \) for every \( c \in K \) such that \( v(z - c) \in K \).

**Proof.** 1): This is part (1) of Lemma 2.17 in \([15]\).
2): The first assertion follows from \([10, \text{Theorem 1}]\). To prove the second assertion, take \( c \in K \); we wish to show that \( v(z - c) \in vK \). By assumption there is \( d \in K \) such that \( v(z - d) > v(z - c) \). Hence \( v(z - c) = \min \{ v(z - c), v(z - d) \} = v(c - d) \in vK \).
3): This follows from part (2) of Lemma 2.17 in \([15]\). \(\square\)

For any \( \alpha \in vK \) and each cut \( \Lambda \) in \( \widetilde{K} \) we set \( \alpha + \Lambda := (\alpha + \Lambda^L, \alpha + \Lambda^R) \). An immediate consequence of the above definitions is the following lemma:

**Lemma 2.3.** Take an extension \((L|K,v)\) of valued fields. Then for every element \( c \in K \) and \( y, z \in L \),
\[
a) \quad \text{dist} \,(z + c, K) = \text{dist} \,(z, K),
b) \quad \text{dist} \,(cz, K) = vc + \text{dist} \,(z, K),
c) \quad \text{if} \, z \sim_K y \, \text{then} \, z + c \sim_K y + c,
d) \quad \text{if} \, c \neq 0 \, \text{and} \, z \sim_K y \, \text{then} \, cz \sim_K cy.
\]
The next two facts are important properties of distances of elements in valued field extensions. For the proof of the next lemma see [2, Lemma 7] and [15, Lemma 2.5].

**Lemma 2.4.** Take an arbitrary immediate extension \((F|K,v)\) and a finite defectless unibranched extension \((L|K,v)\). Then the extension of \(v\) from \(F\) to \(F.L\) is unique, \((F.L,F,v)\) is defectless, \((F.L|L,v)\) is immediate, and for every \(a \in F \setminus K\) we have that
\[
\text{dist}(a,K) = \text{dist}(a,L).
\]
Moreover,
\[
[F.L:F] = [L:K],
\]
i.e., \(F|K\) and \(L|K\) are linearly disjoint.

For the proof of the following results see [2, Lemmas 5 and 9].

**Lemma 2.5.** Take a unibranched extension \((F|K,v)\) and an extension of \(v\) to the algebraic closure of \(F\). Take \(K_h\) to be the henselization of \(K\) with respect to this fixed extension of \(v\). Then for every \(a \in F\) we have that
\[
[K(a):K] = [K_h(a):K_h],
\]
that \(d(K(a)|K,v) = d(K_h(a)|K_h,v)\) as well as
\[
dist(a,K) = dist(a,K_h).
\]

A valued field \((K,v)\) is said to be **separably defectless** if every finite separable extension of \((K,v)\) is defectless, and **inseparably defectless** if every finite purely inseparable extension of \((K,v)\) is defectless. The following is Lemma 4.15 of [15].

**Lemma 2.6.** Every finite extension of an inseparably defectless field is again an inseparably defectless field.

For the proof of the next proposition, see [15], Proposition 2.8.

**Proposition 2.7.** Take a henselian field \((K,v)\) and a tame extension \(N\) of \(K\). Then for any finite extension \(L|K\),
\[
d(L|K,v) = d(L.N|N,v).
\]
In particular, \((K,v)\) is defectless (separably defectless, inseparably defectless) if and only if \((K^r,v)\) is defectless (separably defectless, inseparably defectless).

For the following theorem, see [10, Theorem 1] and [15, Theorem 2.19].

**Theorem 2.8.** If \((L|K,v)\) is an immediate extension of valued fields, then for every element \(a \in L \setminus K\) we have that \(v(a - K)\) is an initial segment of \(vK\) and that \(v(a - K)\) has no maximal element. In particular, \(va < \text{dist}(a,K)\).

The following partial converse of this theorem also holds (cf. e.g. [24, Lemma 2.1]):

**Lemma 2.9.** Assume that \((K(a)|K,v)\) is a unibranched extension of prime degree such that \(v(a - K)\) has no maximal element. Then the extension \((K(a)|K,v)\) is immediate and hence a defect extension.

The property that the set \(v(a - K)\) has no maximal element does not in general imply that \((K(a)|K,v)\) is immediate. However, the next lemma (cf. e.g. [24, Lemma 2.1]) shows that if in addition \((K,v)\) is henselian and \(a\) is algebraic over \(K\), then \((K(a)|K,v)\) is a defect extension.
Lemma 2.10. If $(L|K,v)$ is a defectless unibranch extension, then for every element $a \in K$ the set $v(a - K)$ admits a maximal element.

The next lemma follows from [10, Lemma 8] and [24, Lemma 5.2]. We use the Taylor expansion

$$(6) \quad f(X) = \sum_{i=0}^{n} \partial_if(c)(X - c)^i$$

where $\partial_if$ denotes the $i$-th Hasse-Schmidt derivative (also called formal derivative) of $f$.

Lemma 2.11. Take a nontrivial extension $(K(a)|K,v)$ of degree $p^k$. Assume that $v(a - K)$ has no maximal element and in addition, for every polynomial $g \in K[X]$ of degree $< [K(a) : K]$ there is $\alpha \in v(a - K)$ such that for all $c \in K$ with $v(a - c) \geq \alpha$, the value $v_g(c)$ is fixed. Then for every nonconstant polynomial $f \in K[X]$ of degree $< p^k$ there are $\gamma \in v(a - K)$ and $h = p^\ell$ with $0 \leq \ell < k$ such that for all $c \in K$ with $v(a - c) \geq \gamma$ and all $i$ with $1 \leq i \leq \deg f$, we have:

the value $v_{\partial_i f(c)}$ is fixed, equal to $\partial_i f(a)$,

$$(7) \quad v_{\partial_h f(c)} + h \cdot v(x - c) < v_{\partial_i f(c)} + i \cdot v(x - c)$$

whenever $i \neq h$,

$$(8) \quad v(f(a) - f(c)) = v_{\partial_h f(c)} + h \cdot v(a - c),$$

and

$$(9) \quad \text{dist}(f(a), K) = v_{\partial_h f(c)} + h \cdot \text{dist}(a, K).$$

The following is Lemma 2.4 of [15].

Lemma 2.12. Take a valued field $(K,v)$, a finite extension $(L|K,v)$ and a coarsening $w$ of $v$ on $L$. If $(K,v)$ is henselian, then so is $(K,w)$. If $(L|K,v)$ is defectless, then so is $(L|K,w)$.

2.2. The absolute ramification field.

Proposition 2.13. Take an immediate unibranch extension $(K(a)|K,v)$. Extend $v$ to the algebraic closure of $K$ and let $(K^h,v)$ be the henselization and $(K^r,v)$ the absolute ramification field of $(K,v)$ with respect to this extension. Then $(K^r(a)|K^r,v)$ is an immediate extension with

$$(10) \quad [K^r(a) : K^r] = [K^h(a) : K^h] = [K(a) : K],$$

$$(11) \quad d(K^r(a)|K^r,v) = d(K^h(a)|K^h,v) = d(K(a)|K,v),$$

$$(12) \quad \text{dist}(a, K^r) = \text{dist}(a, K^h) = \text{dist}(a, K).$$

Proof. Since $(K(a)|K,v)$ is a unibranch extension, we know from Lemma 2.5 that $[K^h(a) : K^h] = [K(a) : K]$ as well as $d(K^h(a)|K^h,v) = d(K(a)|K,v)$ and $\text{dist}(a, K^h) = \text{dist}(a, K)$. Since $(K(a)|K,v)$ is an immediate unibranch extension by assumption,

$$(13) \quad [K^h(a) : K^h] = [K(a) : K] = d(K(a)|K,v) = d(K^h(a)|K^h,v),$$

showing that also $(K^h(a)|K^h,v)$ is immediate.

Further, $(K^r|K^h,v)$ is a tame and hence defectless extension. Thus by Proposition 2.4, $(K^r(a)|K^r,v)$ is immediate with $[K^r(a) : K^r] = [K^h(a) : K^h]$ and
\[ \text{dist}(a, K^r) = \text{dist}(a, K^h). \] 

By Proposition 2.7, \( d(K^r(a)|K^r, v) = d(K^h(a)|K^h, v). \)

For the proof of the following results, see Lemma 2.9 of [15].

**Lemma 2.14.** Take any valued field \((K, v)\) and let \(K^h\) and \(K^r\) be its henselization and its absolute ramification field with respect to any extension of \(v\) to the algebraic closure of \(K\). If \(\text{char } K^v = 0\), then \(K^r\) is algebraically closed. If \(\text{char } K^v = p > 0\), then every finite extension of \(K^r\) is a tower of normal extensions of degree \(p\). Further, if \(L|K\) is a finite extension, then there is already a finite tame extension \(N\) of \(K^h\) such that \(L.N|N\) is such a tower.

The proof of this lemma uses the fact that if \(\text{char } K^v = p > 0\), then \(K^\text{sep}|K^r\) is a \(p\)-extension. From this we can also conclude that \(K^r\) contains all \(p\)-th roots of unity. The following is Lemma 14 of [17].

**Lemma 2.15.** A henselian field of characteristic 0 and residue characteristic \(p > 0\) contains an element \(C\) such that \(C^p - 1 = -p\) if and only if it contains a primitive \(p\)-th root \(\zeta_p\) of unity.

We therefore know that in the case of mixed characteristic, the henselian field \(K^r\) contains such an element \(C\). It satifies:

\[ C^p = -pC \quad \text{and} \quad vC = \frac{vp}{p-1}. \]

Further, it is well known that

\[ v(\zeta_p - 1) = \frac{vp}{p-1}. \]

(see e.g. the proof of Lemma 14 of [17]).

### 2.3. 1-units and \(p\)-th roots in valued fields of mixed characteristic.

Throughout this section, \((K, v)\) will be a valued field of characteristic zero and residue characteristic \(p > 0\), with valuation ring \(\mathcal{O}\) and valuation ideal \(\mathcal{M}\). We assume that \(v\) is extended to the algebraic closure \(\tilde{K}\) of \(K\).

A **1-unit** in \((K, v)\) is an element of the form \(u = 1 + b\) with \(b \in \mathcal{M}\); in other words, \(u\) is a unit in \(\mathcal{O}\) with residue 1. We will call the value \(v(u - 1)\) the **level** of the 1-unit \(u\).

**Lemma 2.16.** 1) If \(b_1, \ldots, b_n \in \mathcal{O}\), then

\[ (b_1 + \ldots + b_n)^p \equiv b_1^p + \ldots + b_n^p \mod p\mathcal{O}. \]

2) Take elements \(b_1, \ldots, b_n \in K\) of values \(\geq -\frac{vp}{p}\). Then

\[ (b_1 + \ldots + b_n)^p \equiv b_1^p + \ldots + b_n^p \mod \mathcal{O}_K. \]

3) Take \(\eta \in \tilde{K}\) such that \(\eta^p = a \in \mathcal{O}_K\). Then for every \(c \in K\) such that \(v(\eta - c) \geq \frac{vp}{p}\) we have that \(a \equiv c^p \mod p\mathcal{O}\).

**Proof.** 1): We have:

\[ (b_1 + b_2)^p = b_1^p + \sum_{i=1}^{p-1} \binom{p}{i} b_1^{p-i}b_2^i + b_2^p. \]
Since the binomial coefficients under the sum are all divisible by $p$ and since $b_1, b_2 \in \mathcal{O}$, all summands on the right hand side for $1 \leq i \leq p - 1$ lie in $p\mathcal{O}$, which proves our assertion in the case of $n = 2$. The general case follows by induction on $n$.

2): If $vb_1 \geq -\frac{vp}{p}$ and $vb_2 \geq -\frac{vp}{p}$, then $vb_1b_{i-1}^i - vb_i \geq -vp$ for $1 \leq i \leq p - 1$, so all summands in the sum on the right hand side of (15) have non-negative value. As for part 1), the assertion now follows by induction on $n$.

3): For $c \in K$ with $v(\eta - c) > 0$ we have that $vc \geq 0$ and, by part 1):

\[(16) \quad (\eta - c)^p \equiv \eta^p - c^p \equiv a - c^p \mod p\mathcal{O}_{K(\eta)} .
\]

If $v(\eta - c) \geq \frac{vp}{p}$, then $v(\eta - c)^p \geq vp$, i.e., $a - c^p \equiv (\eta - c)^p \equiv 0 \mod p\mathcal{O}_{K(\eta)}$. \hfill \Box

**Lemma 2.17.** Take $\eta \in \bar{K}$ such that $\eta^p \in K$ and $v\eta = 0$. Then for $c \in K$ such that $v(\eta - c) > 0$, $v(\eta - c) < \frac{1}{p - 1} vp$ holds if and only if $v(\eta^p - c^p) < \frac{p}{p - 1} vp$, and if this is the case, then $v(\eta^p - c^p) = pv(\eta - c)$. If $v(\eta - c) > \frac{1}{p - 1} vp$, then $v(\eta^p - c^p) = vp + v(\eta - c)$.

**Proof.** Take any $c \in K$ such that $0 < v(\eta - c)$. Then $vc = v\eta = \frac{vp}{p} = 0$. We have that

\[\eta^p = (\eta - c + c)^p = (\eta - c)^p + \sum_{i=1}^{p-1} \binom{p}{i} (\eta - c)^i c^{p-i} + c^p .\]

Since $vc = 0$ and the binomial coefficients under the sum all have value $vp$, the unique summand with the smallest value is $p(\eta - c)c^{p-1}$. Therefore,

\[(17) \quad v(\eta^p - c^p) \geq \min\{v(\eta - c)^p, vp(\eta - c)\} = \min\{pv(\eta - c), vp + v(\eta - c)\} ,
\]

with equality holding if $pv(\eta - c) \neq vp + v(\eta - c)$. We observe that

\[(18) \quad v(\eta - c) < \frac{vp}{p - 1} \iff pv(\eta - c) < vp + v(\eta - c) ,
\]

and the same holds for “$>$” in place of “$<$”. Assume that $v(\eta - c) < \frac{vp}{p - 1}$. Then by (18) and (17),

\[v(\eta^p - c^p) = pv(\eta - c) < \frac{p}{p - 1} vp .
\]

Now assume that $v(\eta - c) \geq \frac{1}{p - 1} vp$. Then by (18), $pv(\eta - c) \geq vp + v(\eta - c)$, and (17) yields that

\[v(\eta^p - c^p) \geq vp + v(\eta - c) \geq vp + \frac{1}{p - 1} vp = \frac{p}{p - 1} vp .
\]

Finally, if $v(\eta - c) > \frac{1}{p - 1} vp$, then from (17) and the subsequent remark we conclude that

\[v(\eta^p - c^p) = vp + v(\eta - c) .
\]

Taking $\eta$ to be a 1-unit $u$, we obtain:

**Corollary 2.18.** Assume that $u$ is a 1-unit. Then the level of $u$ is smaller than $\frac{1}{p - 1} vp$ if and only if the level of $u^p$ is smaller than $\frac{p}{p - 1} vp$, and if this is the case, then $v(u^p - 1) = pv(u - 1)$.\hfill \Box
Lemma 2.19. Take \( \eta \in \bar{K} \) such that \( \eta^p \in K \). If there is some \( c \in K \) such that
\[
v(\eta - c) > v\eta + \frac{vp}{p-1},
\]
then \( \eta \) lies in the henselization of \((K, v)\) within \((\bar{K}, v)\).

Proof. If \( \eta \in K \), then there is nothing to show, so let us assume that \( \eta \notin K \). Every root of \( X^p - \eta^p \) is of the form \( \eta \zeta_i^p \) with \( 0 \leq i \leq p-1 \). For \( 0 \leq i \neq j \leq p-1 \) we have that
\[
v(\eta \zeta_i^p - \eta \zeta_j^p) = v\eta + jv\zeta_p + v(\zeta_i - \zeta_j) = v\eta + \frac{vp}{p-1},
\]
where the last equality holds since \( v\zeta_p = 0 \) and \( v(\zeta - 1) = \frac{vp}{p-1} \) for every primitive \( p \)-th root of unity \( \zeta \). Hence if (19) holds, then it follows from Krasner’s Lemma that \( \eta \in K(c)_{h} = K^h \), where \( K^h \) denotes the henselization of \((K, v)\) within \((\bar{K}, v)\). \( \square \)

The following construction will play an important role in Section 3.4. Take a 1-unit \( \eta \in \bar{K} \) such that \( \eta^p \in K \). Then also \( \eta^p \) is a 1-unit. Assume that \( K \) contains an element \( C \) as in Lemma 2.15. Consider the substitution \( X = CY + 1 \) for the polynomial \( X^p - \eta^p \). We then obtain the polynomial \((CY + 1)^p - \eta^p \). Dividing this polynomial by \( C^p \) and using the fact that \( C^p \neq -pC \), we obtain the polynomial
\[
f_\eta(Y) = Y^p + g(Y) - Y - \frac{\eta^p - 1}{C^p},
\]
where
\[
g(Y) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p} Y^i.
\]

Note that \( g(Y) \in \mathcal{M}_K[Y] \) since \( C \in K \) and \( vC = \frac{vp}{p-1} \). We see that an element \( \tilde{\eta} \) is a root of \( X^p - \eta^p \) if and only if the element \( \frac{\tilde{\eta} - 1}{C} \) is a root of \( f_\eta \). Thus the roots of \( f_\eta \) are of the form \( \zeta_i \frac{\eta - 1}{C} \) with \( 0 \leq i \leq p-1 \).

Set
\[
\vartheta_\eta := \frac{\eta - 1}{C}.
\]

Then \( K(\eta) = K(\vartheta_\eta) \), with \( f_\eta \) the minimal polynomial of \( \vartheta_\eta \) over \( K \).

Lemma 2.20. In a henselian field \((K, v)\) of mixed characteristic with residue characteristic \( p \) which contains a primitive \( p \)-th root of unity, every 1-unit of level greater than \( \frac{p}{p-1}vp \) is a \( p \)-th power.

Proof. By Lemma 2.15, \( K \) contains an element \( C \) as in that lemma. Take a 1-unit \( u \in K \) of level greater than \( \frac{p}{p-1}vp \). Apply the above transformation to the polynomial \( X^p - u \) with \( \eta^p = u \). By our assumption on \( u \) we have that \( \frac{\eta^p - 1}{C^p} \in \mathcal{M}_K \). Hence \( f_\eta(Y) \) is equivalent modulo \( \mathcal{M}_K[Y] \) to \( Y^p - Y \), which splits in the henselian field \( K \). Therefore, \( \eta \in K \). \( \square \)
2.4. Higher ramification groups. Take a henselian field \((K,v)\). Assume that \(L/K\) is a Galois extension, and let \(G = \text{Gal}(L/K)\) denote its Galois group. For ideals \(I\) of \(\mathcal{O}_L\), we consider the \((\text{upper series of})\) higher ramification groups

\[
G_I := \left\{ \sigma \in G \mid \frac{\sigma f - f}{f} \in I \text{ for all } f \in L^\times \right\}
\]

(see [28], §12). For every ideal \(I\) of \(\mathcal{O}_L\), \(G_I\) is a normal subgroup of \(G\) ([28] (d) on p.79). The function

\[
\varphi : I \mapsto G_I
\]

preserves \(\subseteq\), that is, if \(I \subseteq J\), then \(G_I \subseteq G_J\). As \(\mathcal{O}_L\) is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the higher ramification groups are linearly ordered by inclusion. Note that in general, \(\varphi\) will neither be injective, nor surjective.

We define functions from the set of all subgroups of \(G\) to the set of all ideals of \(\mathcal{O}_L\) in the following way. Given a subgroup \(H \subseteq G\), we define

\[
I_-(H) := \bigcap_{H \subseteq G_I} I_I \quad \text{and} \quad I_+(H) := \bigcup_{G_I \subseteq H} I_I.
\]

Note that \(H' \subseteq H''\) implies that \(I_-(H') \subseteq I_-(H'')\) and \(I_+(H') \subseteq I_+(H'')\).

For every ideal \(J \subseteq \mathcal{O}_L\), \(I_-(G_J)\) is the smallest ideal \(I \subseteq J\) such that \(G_J = G_I\). Any ideal of the form \(I_-(G_J)\) will be called a ramification ideal. But note that in general, \(I_+(G_J)\) may not be the largest ideal \(I \supseteq J\) such that \(G_J = G_I\).

The function

\[
v : I \mapsto \Sigma_I := \{vf \mid 0 \neq f \in I\}
\]

is an order preserving bijection from the set of all ideals of \(\mathcal{O}_L\) onto the set of all final segments of the value group \(vL\) (contained in its nonnegative part \((vL)^{\geq 0}\)).

The set of final segments of \(vL\) is again linearly ordered by inclusion. The inverse of the above function is the order preserving function

\[
\Sigma \mapsto I_\Sigma := \{a \in L \mid va \in \Sigma\} \cup \{0\}.
\]

If \(I\) is a ramification ideal, then we call \(\Sigma_I\) a ramification jump.

We will write

\[
G_\Sigma := G_{I_\Sigma} = \left\{ \sigma \in G \mid v\frac{\sigma f - f}{f} \in \Sigma \cup \{\infty\} \text{ for all } f \in L^\times \right\}
\]

Given any subgroup \(H\) of \(G\), we define

\[
\Sigma_-(H) := \bigcap_{H \subseteq G_\Sigma} \Sigma \quad \text{and} \quad \Sigma_+(H) := \bigcup_{G_\Sigma \subseteq H} \Sigma.
\]

As intersections and unions of final segments, \(\Sigma_-(H)\) and \(\Sigma_+(H)\) are themselves final segments.

Note that \(H' \subseteq H''\) implies that \(\Sigma_-(H') \subseteq \Sigma_-(H'')\) and \(\Sigma_+(H') \subseteq \Sigma_+(H'')\). Further, we observe that for every ramification group \(G'\),

\[
\Sigma_-(G') \subseteq \Sigma_+(G'),
\]

since there is some \(\Sigma\) such that \(G' = G_\Sigma\). If \(G' \subsetneq G''\) are two distinct ramification groups, then

\[
\emptyset = \Sigma_-(1) \subseteq \Sigma_-(G') \subseteq \Sigma_+(G') \subseteq \Sigma_-(G'') \subseteq \Sigma_+(G'') \subseteq \Sigma_+(G) = (vL)^{\geq 0}.
\]
We have that
\[ I_-(H) = I_{\Sigma_-(H)}, \quad I_+(H) = I_{\Sigma_+(H)}. \]

The collection of these ideals and final segments reveals information on the valuation theoretical structure of the extension \((L|K, v)\).

3. Defect extensions of prime degree

We will investigate defect extensions \((L|K, v)\) of prime degree \(p\). By what we have already stated in the Introduction, such extensions are immediate unibranched extensions; moreover, \(p = \text{char } Kv > 0\). By Theorem 2.8, for every \(a \in L \setminus K\) the set \(v(a - K)\) is an initial segment of \(vK\) without maximal element, and \(\text{dist}(a, K) > va\).

In the following, we distinguish two cases:
• the equal characteristic case where \(\text{char } K = p\),
• the mixed characteristic case where \(\text{char } K = 0\) and \(\text{char } Kv = p\).

We fix an extension of \(v\) from \(L\) to the algebraic closure \(\tilde{K}\) of \(K\).

In a first section, we investigate the set \(\Sigma_{\sigma}\) defined in (2) for \(\sigma \in \text{Gal}(K)\).

3.1. The set \(\Sigma_{\sigma}\). We start with the following two easy but helpful observations.

**Lemma 3.1.** Let \((K(a)|K, v)\) be any algebraic extension of valued fields. If \(\sigma \in \text{Gal}(K)\) is such that \(\sigma a \neq a\), then
\[
\left\{ v \frac{\sigma(a - c) - (a - c)}{a - c} \mid c \in K \right\} = \left\{ v \frac{\sigma a - a}{a - c} \mid c \in K \right\} = -v(a - K) + v(\sigma a - a) .
\]

*Proof.* The first equality holds since \(\sigma c = c\), and the second holds since
\[
v \frac{\sigma a - a}{a - c} = -v(a - c) + v(\sigma a - a) .
\]

**Lemma 3.2.** Take a nontrivial immediate unibranched extension \((K(a)|K, v)\). Then the following assertions hold.
1) For each \(\sigma \in \text{Gal}(K)\) and \(c \in K\),
\[ v(a - c) < v(\sigma a - a) . \]
2) For each \(\sigma \in \text{Gal}(K)\) and \(c \in K\) such that \(\sigma a \neq a\),
\[ \text{dist}(a, K) \leq v(\sigma a - a) . \]

*Proof.* 1): Since the extension is immediate and \(a \notin K\), the set \(v(a - K)\) has no maximal element. Thus it suffices to prove that \(v(a - c) \leq v(\sigma a - a)\). If this were not true, then for some \(\sigma \in \text{Gal}(K)\) and \(c \in K\), \(v(a - c) > v(\sigma a - a)\). But this implies that
\[ v\sigma(a - c) = v(\sigma a - c) = \min\{v(\sigma a - a), v(a - c)\} = v(\sigma a - a) , \]
which contradicts our assumption that \(K(a)|K\) is a unibranched extension, as \(v\sigma\) is also an extension of \(v\) from \(K\) to \(K(a)\).

2): This is an immediate consequence of part 1).
Lemma 3.3. Take a defect extension \((K(a)|K,v)\) of prime degree and any \(f \in K(a)^\times\). Then for all \(\sigma \in \text{Gal}(K)\) such that \(\sigma a \neq a\) there is some \(c \in K\) such that

\[
(30) \quad v(\sigma f - f) > -v(a - c) + v(\sigma a - a) .
\]

Proof. As stated in the Introduction, \((K(a)|K,v)\) is immediate with \([K(a):K] = p = \text{char} \, Kv\). The element \(f \in K(a)^\times\) can be written as \(f(a)\) for \(f(X) \in K[X]\) of degree smaller than \(p\). By Theorem 2.8, \(v(a - K)\) has no maximal element. Hence by \([2, \text{Lemma 11}]\), we can choose \(\gamma \in v(a - K)\) so large that for all \(c \in K\) with \(v(a - c) \geq \gamma\), all values \(v\partial_i f(c)\) are fixed and equal to \(v\partial_i f(a)\) whenever \(0 \leq i < p\), and that (7) and (8) hold by Lemma 2.11. Since \(\text{deg} \, f < p = [L:K]\), we have that \(h = 1\). It suffices to restrict our attention to those \(c \in K\) for which \(v(a - c) \geq \gamma\). Then we have that

\[
(31) \quad v\partial_i f(a)(a - c) = v\partial_i f(c)(a - c) < v\partial_i f(c)(a - c)^i = v\partial_i f(a)(a - c)^i
\]

for all \(i > 1\). From part 1) of Lemma 3.2 we infer that

\[
0 < v\left(\frac{\sigma a - a}{a - c}\right) < v\left(\frac{\sigma a - a}{a - c}\right)^i
\]

for all \(i > 1\). Using this together with (31), we obtain:

\[
v\partial_i f(a)(\sigma a - a) = v\partial_i f(a)(a - c)\left(\frac{\sigma a - a}{a - c}\right) < v\partial_i f(a)(a - c)^i\left(\frac{\sigma a - a}{a - c}\right)^i = v\partial_i f(a)(\sigma a - a)^i .
\]

It follows that

\[
v(\sigma f(a) - f(a)) = v(f(\sigma a) - f(a)) = v\left(\sum_{i=1}^{\text{deg} \, f} \partial_i f(a)(\sigma a - a)^i\right)
\]

\[
= v\partial_i f(a)(\sigma a - a) = vf = \partial_i f(c) + v(\sigma a - a) .
\]

Since \(h = 1\), (8) shows that

\[
v\partial_i f(c) + v(a - c) = v(f(a) - f(c)) \geq \min\{vf(a),vf(c)\} .
\]

The value on the right hand side is fixed, but the value of the left hand side increases with \(v(a - c)\). Since \(v(a - K)\) has no maximal element, we can choose \(\gamma\) so large that the value on the left hand side is larger than the one on the right hand side, which can only be the case if \(vf(a) = vf(c)\), whence \(vf(a) < v\partial_i f(c) + v(a - c)\). Consequently,

\[
v\frac{\sigma f(a) - f(a)}{f(a)} = vf = \partial_i f(c) + v(\sigma a - a) - vf(a) > -v(a - c) + v(\sigma a - a) .
\]

\(\square\)

Theorem 3.4. Take a defect extension \(\mathcal{E} = (L|K,v)\) of prime degree. Then the following assertions hold.

1) For every generator \(a \in L\) of the extension and every \(\sigma \in \text{Gal}(K)\) such that \(\sigma a \neq a\) we have:

\[
(32) \quad \Sigma_\sigma = -v(a - K) + v(\sigma a - a) .
\]

2) The set \(\Sigma_\sigma\) is a final segment of \(vK^{>0} = \{\alpha \in vK \mid \alpha > 0\}\).
Proof. 1): The inclusion “⊇” in (32) follows from Lemma 3.1. To show the reverse inclusion, we use Lemma 3.3. The element \( f \in K(a)^{\times} \) can be written as \( f(a) \) for \( f(X) \in K[X] \) of degree smaller than \( p \). Since \( v(a-K) \) is an initial segment of \( vK \), \(-v(a-K)\) is a final segment of \( vK \). Thus we can infer from (30) that

\[
v \frac{\sigma f - f}{f} \in -v(a-K) + v(\sigma a - a).
\]

This proves the inclusion “⊆”.

2): Since \( E \) is a unibranched extension, we have that \( vf = v f \) and hence \( v(\sigma f - f) \geq vf \) for all \( f \in L^{\times} \), showing that \( v \frac{\sigma f - f}{f} \in vL^{>0} = vK^{>0} \). Since \(-v(a-K)\) is a final segment of \( vK \), the same holds for \( \Sigma_{\sigma} = -v(a-K) + v(\sigma a - a) \). \( \square \)

3.2. Galois defect extensions of prime degree. A Galois extension of degree \( p \) of a field \( K \) of characteristic \( p > 0 \) is an Artin-Schreier extension, that is, generated by an Artin-Schreier generator \( \vartheta \) which is the root of an Artin-Schreier polynomial \( X^p - X - c \) with \( c \in K \). A Galois extension of degree \( p \) of a field \( K \) of characteristic 0 is a Kummer extension, that is, generated by a Kummer generator \( \eta \) which satisfies \( \eta^p \in K \). If \( (L|K,v) \) is a Galois defect extension of degree \( p \) of fields of characteristic 0, then a Kummer generator of \( L|K \) can be chosen to be a 1-unit. Indeed, choose any Kummer generator \( \eta \), since \((L|K,v)\) is immediate, we have that \( v \eta \in v(K(\eta)) = vK \), so there is \( c \in K \) such that \( vc = -v\eta \). Then \( vpc = 0 \), and since \( vce \in K(\eta)v = Kv \), there is \( d \in K \) such that \( dv = (vcd)^{-1} \). Then \( v(\eta v c) = 0 \) and \((vcd) v = 1 \). Hence \( vcd \) is a 1-unit. Furthermore, \( K(\eta v d) = K(\eta) \) and \( (\eta v d)^p = \eta^p v^p d^p \in K \). Thus we can replace \( \eta \) by \( \eta v d \) and assume from the start that \( \eta \) is a 1-unit. It follows that also \( \eta^p \in K \) is a 1-unit.

Throughout this article, whenever we speak of “Artin-Schreier extension” we refer to fields of positive characteristic, and with “Kummer extension” we refer to fields of characteristic 0.

**Theorem 3.5.** Take a Galois defect extension \( E = (L|K,v) \) of prime degree with Galois group \( G \). The set \( \Sigma_{\sigma} \) does not depend on the choice of the generator \( \sigma \) of \( G \). Writing \( \Sigma_{\sigma} \) for \( \Sigma_{\sigma} \), we have that \( \Sigma_{E} \) is a final segment of \( vK^{>0} \) and satisfies

\[
\Sigma_{E} = \Sigma_{+}(1) = \Sigma_{-}(G).
\]

Hence the ramification ideal \( I_{-}(G) = I_{+}(1) \) is equal to the ideal of \( O_{L} \) generated by the set

\[
\left\{ \frac{\sigma f - f}{f} \mid f \in L^{\times} \right\},
\]

for any generator \( \sigma \) of \( G \).

If \((L|K,v)\) an Artin-Schreier defect extension with any Artin-Schreier generator \( a \), then

\[
(33) \quad \Sigma_{E} = -v(a-K).
\]

If \( K \) contains a primitive root of unity and \((L|K,v)\) is a Kummer extension with Kummer generator \( a \) which is a 1-unit, then

\[
(34) \quad \Sigma_{E} = \frac{vp}{p-1} - v(a-K).
\]
Proof. Assume first that \((L|K, v)\) an Artin-Schreier defect extension with Artin-Schreier generator \(a\). Then for every generator \(\sigma\) of \(G\), we have that \(\sigma(\vartheta) = \vartheta + i\) for some \(i \in \mathbb{F}_p\) and thus, \(v(\sigma \vartheta - \vartheta) = vi = 0\). Hence equation (32) shows that \(\Sigma_\sigma\) does not depend on the choice of \(\sigma\) and that (33) holds.

Now assume that \(K\) contains a primitive root of unity and \((L|K, v)\) is a Kummer extension with Kummer generator \(a\) which is a 1-unit. Then \(\sigma a - a = \zeta - 1\) for some primitive root of unity \(\zeta\), and by equation (32),

\[
v(\sigma a - a) = va + v(\zeta - 1) = \frac{vp}{p - 1}.
\]

This shows that also in this case, \(\Sigma_\sigma\) does not depend on the choice of \(\sigma\), and that (34) holds.

If \(\Sigma \subsetneq \Sigma_\sigma\), then \(\sigma \notin G_\Sigma\) and hence \(G_\Sigma = 1\). If \(\Sigma_\sigma \subsetneq \Sigma\), then \(\sigma \notin G_\Sigma\) and hence \(G_\Sigma = G\). Since \(-v(a - K)\) has no smallest element, equations (33) and (34) show that the same is true for \(\Sigma_\sigma\). Therefore, \(\Sigma_\sigma\) is the union of all final segments properly contained in it, whence

\[
\Sigma_\sigma = \bigsqcup_{G_\Sigma = 1} \Sigma = \bigsqcup_{G_\Sigma \subsetneq 1} \Sigma = \Sigma_+(1).
\]

Trivially, \(\Sigma_{\sigma}\) is the intersection of all final segments that contain it, so

\[
\Sigma_{\sigma} = \bigcap_{G = G_{\Sigma}} \Sigma = \bigcap_{G_{\Sigma} \subseteq G} \Sigma = \Sigma_{-}(G).
\]

We define the distance of \(\sigma\) to be the cut

\[
\text{dist } \sigma := (\Sigma_{\sigma})^+.
\]

in \(\tilde{vK}\). By applying the distance operator to the right hand sides of equations (33) and (34), we obtain:

**Corollary 3.6.** If \(\mathcal{E}\) is an Artin-Schreier defect extension, then

\[
\text{dist } \mathcal{E} = \text{dist } (a, K)
\]

for every Artin-Schreier generator of \(\mathcal{E}\). Consequently, all Artin-Schreier generators of \(\mathcal{E}\) have the same distance.

If \(\mathcal{E}\) is a Kummer extension, then

\[
\text{dist } \mathcal{E} = -\frac{vp}{p - 1} + \text{dist } (a, K)
\]

for every Kummer generator \(a\) which is a 1-unit. Consequently, all Kummer generators of \(\mathcal{E}\) that are 1-units have the same distance.

Under a certain additional assumption, also Kummer defect extensions have generators whose distance is equal to \(\text{dist } \mathcal{E}\) and whose minimal polynomials resemble Artin-Schreier polynomials. Details will be worked out in Section 3.4.

**Proposition 3.7.** Take a Galois defect extension \(\mathcal{E} = (L|K, v)\) of prime degree.

1) We have that

\[
\text{dist } \mathcal{E} \leq 0^-.
\]

If \(\mathcal{E}\) is an Artin-Schreier defect extension, then

\[
\text{dist } (a, K) \leq 0^-
\]
for every Artin-Schreier generator a. If $E$ is an Kummer defect extension, then
\[ 0 < \text{dist}(a, K) \leq \left( \frac{vp}{p-1} \right)^{-} \]
for every Kummer generator a which is a 1-unit.

2) The extension $E$ has independent defect if and only if
\[ \text{dist} E = H^{-} \]
for some proper convex subgroup $H$ of $\tilde{v}K$. In particular, if the value group of $(K, v)$ is archimedean, then $E$ has independent defect if and only if $\text{dist} E = 0^{-}$.

3) An Artin-Schreier defect extension with Artin-Schreier generator $a$ has independent defect if and only if
\[ \text{dist} (a, K) = H^{-} \]
and a Kummer defect extension of prime degree with Kummer generator $a$ which is a 1-unit has independent defect if and only if
\[ \text{dist} (a, K) = \frac{vp}{p-1} + H^{-}, \]
for some proper convex subgroup $H$ of $\tilde{v}K$.

Proof. 1): Inequality (35) follows from part 2) of Theorem 3.4 together with the definition of $\Sigma_{E}$ in Theorem 3.5. From inequality (35) we obtain inequality (36) and the second inequality in (37) by an application of Corollary 3.6. The first inequality in (37) follows from Theorem 2.8 since $va = 0$.

2): By definition, the lower cut set of $\text{dist} E$ is the smallest initial segment of $\tilde{v}K$ containing $-\Sigma_{E} \cap vK = -\Sigma_{E}$. Since $-\Sigma_{E}$ is an initial segment of $vK$, $\text{dist} E = H^{-}$ implies that $-\Sigma_{E} = \{ \alpha \in vK \mid \alpha < H \} = \{ \alpha \in vK \mid \alpha < H_{E} \}$ where $H_{E} := H \cap vK$ is a convex subgroup of $vK$. For the converse, assume that $\Sigma_{E} = \{ \alpha \in vK \mid \alpha > H_{E} \}$ for a convex subgroup $H_{E}$ of $vK$. Then $-\Sigma_{E} = \{ \alpha \in vK \mid \alpha < H_{E} \}$ and by definition, the left cut set of $(-\Sigma_{E})^{+}$ is the smallest initial segment of $vK$ containing $\{ \alpha \in vK \mid \alpha < H_{E} \}$. This is the same as the largest initial segment of $vK$ disjoint from $H_{E}$. As it is convex, it is also the largest initial segment of $vK$ disjoint from the convex hull $H$ of $H_{E}$ in $\tilde{v}K$, which by definition shows that $\text{dist} E = H^{-}$. Further, being the convex hull of the subgroup $H_{E}$ of $vK$, $H$ is a convex subgroup of $vK$.

The final assertion of part 2) follows from the fact that the only proper convex subgroup in an archimedean ordered abelian group is $\{0\}$.

3): This follows from part 2) together with Corollary 3.6. $\square$

We choose any extension of $v$ from $K(a)$ to $\tilde{K}$ and take $(K^{r}, v)$ to be the absolute ramification field of $(K, v)$. By equation (11) of Proposition 2.13, $(K^{r}(a)|K^{r}, v)$ is again a Galois defect extension of degree $p$.

**Proposition 3.8.** Take a Galois defect extension $E = (L|K, v)$ of prime degree, an absolute ramification field as above, and an intermediate field $L$ of $K^{r}|K$. Then also $E_{L} = (L(a)|L, v)$ is a Galois defect extension of degree $p$, and
\[ \text{dist} E_{L} = \text{dist} E, \]
and $E'$ has independent defect if and only $E$ has. Further, if $(L,v)$ is an independent defect field, then so is $(K,v)$.

Proof. As $L$ is an intermediate field of $K'|K$, we have that $K = L'$ by general ramification theory, and equation (11) shows that also $E_L$ is a Galois defect extension of prime degree. Assume that $a$ is a generator of $E$ as in Theorem 3.5. By equation (12), $\text{dist}(a,L) = \text{dist}(a,K)$. In view of Corollary 3.6, we obtain that $\text{dist} E_L = \text{dist} E$. From this, the second assertion follows by part 2) of Proposition 3.7. The final assertion is an immediate consequence of what has already been proven. □

We will now prove the equivalence of assertions a), b), c) and d) in Theorem 1.11. The equivalence of assertions a) and b) follows from the fact that $\Sigma E = \Sigma - (G)$. Further, the equivalence of assertions b) and c) in Theorem 1.11 because equation (3) holds if and only if the ideal $I(G) = I_{H} E$ is the valuation ring of the coarsening of $v$ on $L$ with respect to the convex subgroup $H_E$. For the proof of the equivalence of assertions a) and d), we observe that by Lemma 2.1, idempotence of $\text{dist} E$ is equivalent to it being equal to $H^-$ or $H^+$ for some convex subgroup $H$ of $vK$. By Proposition 3.7, $\text{dist} E = H^+$ is not possible. Now the equivalence follows from the above corollary.

For Artin-Schreier defect extensions, a different definition was given for dependent and independent defect in [15]. We will show in the next section that our new definition is consistent with the previous one.

3.3. Artin-Schreier defect extensions. In this section, we consider the case of a valued field $(K,v)$ of positive characteristic $p$ and an Artin-Schreier defect extension $(L|K,v)$ with Artin-Schreier generator $\vartheta$, that is, $\vartheta^p - \vartheta \in K$. The following definition was introduced in [15]: if there is an immediate purely inseparable extension $(K(\eta)|K,v)$ of degree $p$ such that

\begin{equation}
\vartheta \sim_K \eta,
\end{equation}

then we say that the Artin-Schreier defect extension has dependent defect; otherwise it has independent defect. Note that (39) implies that $\text{dist}(\eta,K) < \infty$, that is, $\eta$ does not lie in the completion of $(K,v)$, since otherwise it would follow that $\vartheta = \eta$.

The above definition does not depend on the Artin-Schreier generator of the extension $L|K$. Indeed, by [15, Lemma 2.26], $\vartheta' \in L$ is another Artin-Schreier generator of $L|K$ if and only if $\vartheta' = i\vartheta + c$ for some $i \in \mathbb{F}_p$ and $c \in K$. If we set $\eta' = i\eta + c$, then $K(\eta) = K(\eta')$ and $v(\vartheta' - \eta') = v(i(\vartheta - \eta)) = v(\vartheta - \eta) > \text{dist} (\vartheta,K)$, that is, $\vartheta' \sim_K \eta'$.

Our above definition is consistent with the new one given in the previous section, as follows from the equivalence of assertions a) and d) of Theorem 1.11 together with the following fact, which is [15, Proposition 4.2]).

Proposition 3.9. An Artin-Schreier defect extension is independent (in the sense as defined in [15]) if and only if its distance (defined as the distance of any Artin-Schreier generator) is idempotent.
The name “dependent defect” was chosen because the existence of Artin-Schreier defect extensions with dependent defect depends on the existence of purely inseparable defect extensions of degree $p$. The following proposition makes this dependence more precise:

**Proposition 3.10.** Take a valued field $(K, v)$ of positive characteristic $p$.

1) If $(K(\eta)|K, v)$ is a purely inseparable defect extension of degree $p$ not contained in the completion of $(K, v)$, then for every $b \in K$ of high enough value and every root of the polynomial $Y^p - Y - \left(\frac{\eta}{b}\right)^p$, the extension $(K(\vartheta)|K, v)$ is an Artin-Schreier extension with dependent defect and Artin-Schreier generator $\vartheta$ such that $\vartheta \sim_K \eta/b$.

2) Take an Artin-Schreier extension $(L|K, v)$ with dependent defect. Then there exists a purely inseparable defect extension $(K(\eta)|K, v)$ of degree $p$ not contained in the completion of $(K, v)$ and an Artin-Schreier generator $\vartheta$ of $L|K$ such that $\eta^p = \vartheta^p - \vartheta$ and $\vartheta \sim_K \eta$.

3) $(K, v)$ is an independent defect field if and only if every immediate purely inseparable extension of $(K, v)$ lies in its completion.

**Proof.** Assertion 1) follows from [15, Proposition 4.3], and assertion 3) follows from assertions 1) and 2).

In order to prove assertion 2), take an Artin-Schreier extension $(L|K, v)$ with dependent defect and an arbitrary Artin-Schreier generator $\vartheta_0$. Then by Proposition 3.9, $\text{dist}(\vartheta_0, K)$ is not idempotent, i.e., $p \text{dist}(\vartheta_0, K) < \text{dist}(\vartheta_0, K)$ in view of part 1) of Proposition 3.7. This means that there is some $c \in K$ such that $v(\vartheta_0 - c) > p \text{dist}(\vartheta_0, K)$. Set $a := (\vartheta_0 - c)^p - (\vartheta_0 - c) \in K$ so that $\vartheta := \vartheta_0 - c$ becomes a root of the Artin-Schreier polynomial $X^p - X - a$. Then by [15, Theorem 4.5 (c)]), the root $\eta$ of the polynomial $X^p - a$ generates an immediate extension which does not lie in the completion of $(K, v)$ and $\vartheta \sim_K \eta$ holds. \qed

### 3.4. Kummer defect extensions of prime degree $p$

In this section we will study the case of a valued field $(K, v)$ of characteristic 0 with $\text{char} K v = p > 0$ and $K$ is henselian.

Our goal in this section is to show that also in the mixed characteristic case, under a certain additional assumption, there are generators $z$ of the defect extension $E$ such that $\Sigma_E = -v(z - K)$ and consequently, $\text{dist} E = \text{dist}(z, K)$. We will use the construction from Section 2.3 that associates to $\eta$ an element $\vartheta_\eta$ whose minimal polynomial $f_\eta$ given in (20) bears some resemblance with an Artin-Schreier polynomial. To this end, we assume that $K$ contains an element $C$ as in (13). For the construction we do not need that the extension $E$ is Galois, but if $(K, v)$ is henselian then by Lemma 2.15 it contains a primitive $p$-th root of unity as it contains $C$, which then yields that the extension is indeed Galois.

**Theorem 3.11.** Take a valued field $(K, v)$ of mixed characteristic containing an element $C$ as in (13), and a Kummer defect extension $E = (K(\eta)|K, v)$ of prime degree $p$ with $\eta$ a 1-unit such that $\eta^p \in K$. Define $\vartheta_\eta$ by (22). Then

$$\Sigma_E = -v(\vartheta_\eta - K)$$
The following assertions are equivalent:
a) $E$ has independent defect,
b) $\text{dist}(\vartheta_\eta, K) = H^-$ for some proper convex subgroup $H$ of $\widehat{vK}$, 
c) $\text{dist}(\eta, K) = \frac{vp}{p-1} + H^-$ for some proper convex subgroup $H$ of $\widehat{vK}$.

Proof. Take $\sigma \in \text{Gal}(K)$ such that $\sigma(\eta) = \zeta_p \eta$. Then
\begin{equation}
(42) \quad v(\sigma(\eta) - \eta) = v\eta + v(\zeta_p - 1) = \frac{vp}{p-1},
\end{equation}
where the last equality follows from (14) together with the fact that $\eta$ is 1-unit. Using equation (13), we deduce:
\begin{equation}
(43) \quad \text{dist}(\vartheta_\eta, K) = -\frac{vp}{p-1} + \text{dist}(\eta, K).
\end{equation}
The leftmost and the rightmost inequalities of (41) follow from equation (37). This completes the proof of (41).

Finally, we prove the equivalences. From part 2) of Proposition 3.7 we know that $E$ has independent defect if and only if $\text{dist} E = H^-$ for some proper convex subgroup $H$ of $\widehat{vK}$. Since $\text{dist} E = \text{dist}(\vartheta_\eta, K)$ as we have already proved, the latter is just assertion b). Equation (43) shows that assertion b) is equivalent to assertion c).

Note that if assertion b) of the theorem holds, then
\begin{equation}
(44) \quad vp / \notin H,
\end{equation}
as follows from (41).

4. Properties of Galois extensions of prime degree with independent defect

Throughout this section we will assume that $(K,v)$ is a valued field of residue characteristic $p > 0$. Except in Proposition 4.7, we also assume that $K$ contains a primitive $p$-th root of unity if $\text{char} K = 0$.

The following is Lemma 4.9 of [15]:
Proposition 4.1. Assume that char $K = p$ and $(K(\vartheta)|K,v)$ is an independent Artin-Schreier defect extension with Artin-Schreier generator $\vartheta$ of distance $0^\sim$. Then every algebraically maximal immediate extension (and in particular, every maximal immediate extension) of $(K,v)$ contains an independent Artin-Schreier defect extension $(K(\vartheta')|K,v)$ of distance $0^\sim$ and such that $\vartheta \sim_K \vartheta'$.

Here is the analogue of this result in the case of mixed characteristic:

Proposition 4.2. Assume that char $K = 0$ and that $(K(\eta)|K,v)$ is an independent defect extension of distance $0^\sim$, generated by a 1-unit $\eta$ with $\eta^p \in K$. Then every algebraically maximal immediate extension of $(K,v)$ contains an independent defect extension $(K(\eta')|K,v)$ of prime degree and distance $0^\sim$, where $\eta'$ is also a $p$-th root of a 1-unit in $K$ and $\eta \sim_K \eta'$.

Proof. Take an algebraically maximal immediate extension $(M,v)$ of $(K,v)$. We note that $(M,v)$ is henselian. If $\eta \in M$, then the assertion is trivial.

Assume that $\eta \notin M$. Then $(M(\eta)|M,v)$ is an extension of degree $p$ with $\eta^p \in M$. Since $M$ is algebraically maximal, $(M(\eta)|M,v)$ is defectless. Indeed, otherwise $(M(\eta)|M,v)$ would be a defect extension of degree $p$, hence a nontrivial immediate extension, a contradiction to the maximality of $(M,v)$. Therefore by Lemma 2.10, the set $v(\eta - M)$ admits a maximal element. Since by Theorem 2.8 the set $v(\eta - K)$ has no maximal element, we have that $v(\eta - K) \subsetneq v(\eta - M)$. Hence there is an element $b \in M$ such that $v(\eta - b) > \text{dist} (\eta,K)$. Since equation (43) yields that $\text{dist} (\eta,K) = \left(\frac{vp}{p-1}\right)^-$ we may deduce that

$$v(a - b) \geq \frac{vp}{p-1}. \tag{45}$$

If $b^p \in K$, we set $\eta' = b$.

Now assume that $b^p \notin K$. Since $\eta$ is a 1-unit, so is $b$ and thus,

$$v\left(\frac{\eta}{b} - 1\right) = v(\eta - b) \geq \frac{vp}{p-1}.$$  

The element $\frac{\eta}{b}$ is 1-unit of level $\geq \frac{1}{p-1} vp$, hence by Corollary 2.18, $\eta^p$ is 1-unit of level $\geq \frac{p}{p-1} vp$. As $(M|K,v)$ is immediate, there is some $c \in K$ such that

$$v\left(\frac{\eta^p}{b^p} - c\right) > v\left(\frac{\eta^p}{b^p} - 1\right) \geq \frac{p}{p-1} vp. \tag{46}$$

Then $c$ is also a 1-unit, and we have that

$$v\left(\frac{\eta^p}{b^pc} - 1\right) = v\left(\frac{\eta^p}{b^p} - c\right) \geq \frac{p}{p-1} vp.$$

Therefore, by Lemma 2.20 the 1-unit $\frac{\eta^p}{b^pc}$ admits a $p$-th root $u$ in the henselian field $M$. Then $bu \in M$ with

$$(bu)^p = b^p \frac{\eta^p}{b^pc} = \frac{\eta^p}{c} \in K.$$

Since $\frac{\eta^p}{b^pc}$ is 1-unit of level $\geq \frac{p}{p-1} vp$, (46) yields that the same holds for $c$. Since

$$c = \frac{\eta^p}{(bu)^p},$$
Corollary 2.18 shows that the level of the 1-unit $\frac{p}{bu}$ is $\geq \frac{1}{p-1}vp$. We obtain that
\[ v(\eta - bu) = v\left(\frac{\eta}{bu} - 1\right) \geq \frac{vp}{p-1}, \]
and we set $\eta' = bu$.

In both cases we have now achieved that $\eta'$ is a 1-unit which is a $p$-th root of an element in $K$ such that $v(\eta - \eta') \geq \frac{vp}{p-1}$, which by inequality (37) of Proposition 3.7 yields that $v(\eta - \eta') > \text{dist} (\eta, K)$. From part 1) of Lemma 2.2 we obtain that
\[ \text{dist} (\eta', K) = \text{dist} (\eta, K), \]
which by part 3) of Proposition 3.7 shows that like $(K(\eta)|K, v)$, also $(K(\eta')|K, v)$ is an independent defect extension of distance $0^-$.

From Propositions 4.1 and 4.2 we obtain the following result.

**Corollary 4.3.** Assume that there is a maximal immediate extension of $(K, v)$ in which $K$ is relatively algebraically closed. Then $(K, v)$ admits no independent Galois defect extension of prime degree and distance $0^-$. 

We wish to generalize the previous result to the case of independent defect extensions with arbitrary distance.

**Lemma 4.4.** Assume that for every coarsening $w$ of $v$ (including the valuation $v$ itself) such that $Kw$ is of positive characteristic there is a maximal immediate extension $(M_w, w)$ of $(K, w)$ in which $K$ is relatively algebraically closed. Then $(K, v)$ admits no independent Galois defect extension of prime degree.

**Proof.** The case of equal positive characteristic has been settled in Lemma 4.11 of [15]. Hence we assume now that $(K, v)$ is of characteristic $0$ with residue characteristic $p > 0$ and containing a primitive $p$-th root of unity.

Suppose that $(L|K, v)$ is an independent Galois defect extension of prime degree. By Corollary 4.3, its distance cannot be $0^-$. Hence it is equal to $H^-$ for some nontrivial proper convex subgroup $H$ of $vK$. Denote by $v_H$ the coarsening of $v$ with respect to $H$, and by $M_{v_H}$ its valuation ideal. From (44) we know that $vp \notin H$, so we have that $p \in M_{v_H}$ and therefore, $\text{char } K v_H = p$. By Lemma 2.12, a coarsening of a henselian valuation is again henselian, so $(K, v_H)$ is henselian.

By our assumption, we can write $L = K(\vartheta_\eta)$ with $\vartheta_\eta$ as in Section 3.4. Then $\text{dist} (\vartheta_\eta, K) = H^-$, which means that $v(\vartheta_\eta - K)$ is cofinal in $(v\overline{K})^{<0} \setminus H$. It follows that $v_H(\vartheta_\eta - K)$ is cofinal in $v\overline{K}^{<0}/H = (v_H\overline{K})^{<0}$. Since $v_H\overline{K}$ is divisible, $(v_H\overline{K})^{<0}$ has no largest element. Thus in particular, $v_H(\vartheta_\eta - K)$ has no maximal element. Together with Lemma 2.9, this shows that $(L|K, v_H)$ is an immediate extension of henselian fields. Hence, $(L|K, v_H)$ is a Galois defect extension of prime degree and distance $0^-$. On the other hand, by assumption $(K, v_H)$ admits a maximal immediate extension in which $K$ is relatively algebraically closed. Therefore, Corollary 4.3 shows that $(K, v_H)$ admits no Galois defect extension of prime degree and distance $0^-$, a contradiction. 

**Lemma 4.5.** Take a coarsening $w$ of $v$ (possibly the valuation $v$ itself) such that $(K, w)$ admits a maximal immediate extension $(M_w, w)$ in which $K$ is relatively algebraically closed. If $(L|K, v)$ is a finite separable and defectless extension, then
(M_w,L,w) is a maximal immediate extension of (L,w) such that L is relatively algebraically closed in M_w,L.

Proof. Since (L|K,v) is defectless by assumption, the same is true for the extension (L|K,w) by Lemma 2.12. We note that (K,w) is henselian since it is assumed to be relatively algebraically closed in the henselian field (M_w,w). Hence we may apply Lemma 2.4: since (M_w|K,w) is immediate and (L|K,w) is defectless, (M_w,L|L,w) is immediate and M_w|K and L|K are linearly disjoint. The latter implies that L is relatively algebraically closed in M_w,L (for the proof of this fact, see [20] or [22, Chapter 24]). On the other hand, [27, Theorem 31.22] shows that (M_w,L,w) is a maximal field, being a finite extension of a maximal field, and it is therefore a maximal immediate extension of (L,w).

Proposition 4.6. If (K,v) is algebraically maximal and (L|K,v) is a finite separable and defectless extension, then (L,v) admits no independent Galois defect extension of prime degree.

Proof. Take a coarsening w of v such that Kw is of positive characteristic. Note that every immediate extension of (K,w) is also immediate under the finer valuation v. Since (K,v) is algebraically maximal, this yields that also (K,w) is algebraically maximal.

Take (M_w,w) to be a maximal immediate extension of (K,w). Then K is relatively algebraically closed in M_w. Lemma 4.5 yields that (M_w,L,w) is a maximal immediate extension of (L,w) such that L is relatively algebraically closed in M_w,L.

This shows that for every coarsening w of v such that Lw is of positive characteristic there is a maximal immediate extension of (L,w) in which L is relatively algebraically closed. By Lemma 4.4 this proves that (L,v) admits no independent Galois defect extension of prime degree.

Proposition 4.7. Assume that (K,v) is a valued field of positive residue characteristic p. Then the following are equivalent
a) (K,v) is henselian and defectless,
b) (K,v) is algebraically maximal and in every finite tower of extensions of degree p over K^r every defect extension of degree p is separable and independent.

Proof. Assume first that a) holds. Since K is henselian and defectless, it admits in particular no immediate algebraic extension, that is, (K,v) is algebraically maximal.

Take now a finite tower L of extensions of degree p over K^r. Choose generators a_1, ..., a_s of the extension L|K^r and set K' = K(a_1, ..., a_s). Then (K'|K,v) is finite, hence by assumption a defectless extension. Since the extension (K'|K,v) is tame, Proposition 2.7 yields that

1 = d(K'|K,v) = d(K'.K^r|K^r,v) = d(L|K^r,v).

Hence L|K^r is a defectless extension, and so is every extension of degree p in the tower L|K^r. This shows that condition b) holds.

Suppose now that (K,v) satisfies condition b). Since (K,v) is algebraically maximal, it is henselian. Take a finite extension (L|K,v). We wish to show that the extension is defectless. Take K' to be the relative separable-algebraic closure of K in L. By Lemma 2.14, there is a finite tame extension N of K such that K'.N|N is a tower N = N_0 \subset N_1 \subset \ldots \subset N_m = K'.N of Galois extensions N_i=N_{i-1} of degree p. If char K = 0, we can assume in addition that N contains a
primitive $p$-th root of unity, replacing $N$ by $N(ζ_p)$ if necessary; since $p$ does not divide $[N(ζ_p) : N]$, this is also a tame extension of $(K, v)$.

We first show that the extension $(K' | K, v)$ is defectless. Proposition 2.7 shows that $d(K'.N | N, v) = d(K' | K, v)$, so it suffices to show that $(K'.N | N, v)$ is defectless. We observe that also $K^r = N_0.K^r ⊆ N_1.K^r ⊆ ... N_m.K^r = K'.K^r$ is a tower of Galois extensions $N_i.K^r | N_i−1.K'$ of degree $p$. Assume that $(N_i−1 | N, v)$ is a defectless extension for some $i \leq m$ and consider the extension $(N_i | N_{i−1}, v)$. Condition b) implies that the extension $(N_i, K^r | N_i−1, K', v)$ is either defectless or an independent Galois defect extension. Since $(K, v)$ is algebraically maximal and $(N_i−1 | K, v)$ is a finite separable defectless extension, Proposition 4.6 shows that $(N_i | N_{i−1}, v)$ cannot be an independent defect extension. Therefore, also $(N_i, K^r | N_i−1, K', v)$ cannot be an independent defect extension. Hence by assumption, this extension is defectless. From Proposition 2.7 it thus follows that $(N_i | N_{i−1}, v)$ is defectless. This shows that also $(N_i | N, v)$ is a defectless extension. By induction on $i$ we obtain that $(K'.N | N, v)$ is a defectless extension.

The above conclusion together with Proposition 2.7 yields that

\[
d(L/K, v) = d(L.K^r | K^r, v) = d(L.K^r | K'.K^r, v).
\]

Since $L/K'$ is purely inseparable, $L.K^r | K'.K^r$ is a tower of purely inseparable extensions of degree $p$. On the other hand, assumption b) implies that every defect extension of degree $p$ in the tower $L.K^r | K^r$ is separable. Thus every extension in the tower $L.K^r | K'.K^r$ is defectless. This shows that $d(L.K^r | K'.K^r, v) = 1$ and together with equation (47) yields that $(L/K, v)$ is a defectless extension. \square

Note that if $\text{char} K = p > 0$, then condition b) holds if and only if $(K, v)$ is separable-algebraically maximal and inseparably defectless. Indeed, assume that $(K, v)$ satisfies b). Then it is separable-algebraically maximal. If $(K, v)$ would admit a purely inseparable defect extension $(L, v)$, then Proposition 2.7 would yield that $(L.K^r | K', K^r, v)$ were also a purely inseparable defect extension, which contradicts our assumption that every defect extension of degree $p$ in the tower $L.K^r | K'$ is separable.

Suppose now that $(K, v)$ is separable-algebraically maximal and inseparably defectless. Then $(K, v)$ is algebraically maximal, and by Proposition 2.7, $(K^r, v)$ is inseparably defectless. Take a finite extension $(L | K^r, v)$. By Lemma 2.14, $L | K^r$ is a finite tower of normal extensions of degree $p$. As $(K^r, v)$ is inseparably defectless, Lemma 2.6 yields that every purely inseparable extension of degree $p$ in this tower is defectless. Moreover, since every finite extension of $K^r$ does not admit purely inseparable defect extensions, it also admits no dependent Artin-Schreier defect extensions. This yields that every defect extension of degree $p$ in the tower $L | K^r$ is independent.

We have now shown that in the case of valued fields of positive characteristic, our above characterization of henselian defectless fields is equivalent to Theorem 1.2 of [15].

5. The trace of defect extensions of prime degree

In this section we will consider the trace on a Galois defect extension $E = (L | K, v)$ of prime degree. If $L/K$ is an Artin-Schreier extension, then we write $L = K(θ)$ where $θ$ is an Artin-Schreier generator. If $L/K$ is a Kummer extension, then we
write \( L = K(\eta) \) where \( \eta \) is a Kummer generator, that is, \( \eta^p \in K \); as explained at the beginning of Section 3.2, we can assume that \( \eta \) is a 1-unit.

The proof of the following fact can be found in [11, Section 6.3].

**Lemma 5.1.** Take a separable field extension \( K(a)|K \) of degree \( n \) and let \( f(X) \in K[X] \) be the minimal polynomial of \( a \) over \( K \). Then

\[
\text{Tr}_{K(a)|K} \left( \frac{a^m}{f'(a)} \right) = \begin{cases} 
0 & \text{if } 1 \leq m \leq n-2 \\
1 & \text{if } m = n-1.
\end{cases}
\]

\( \Box \)

For arbitrary \( d \in K \), we note:

\[
d(a-c)^{p-1} \in M_{K(a)} \iff vd > -(p-1)v(a-c).
\]

Take \( \Lambda \) to be the smallest final segment of \( \tilde{v}K \) containing \(- (p-1)v(a-K)\). Then the above equation yields that

\[
vd \in \Lambda \iff \exists c \in K : d(a-c)^{p-1} \in M_{K(a)}.
\]

First we consider the equal characteristic case. By Lemma 5.1,

\[
\text{Tr}_{K(\vartheta)|K} (\vartheta^i) = \begin{cases} 
0 & \text{if } 1 \leq i \leq p-2 \\
-1 & \text{if } i = p-1.
\end{cases}
\]

This also holds for \( \vartheta - c \) for arbitrary \( c \in K \) in place of \( \vartheta \) since it is also an Artin-Schreier generator. In particular,

\[
\text{Tr}_{K(\vartheta)|K} (d(\vartheta-c)^{p-1}) = -d
\]

By (50) and the first equation of Corollary 3.6 it follows that

\[
\text{Tr}_{K(\vartheta)|K} (M_{K(\vartheta)}) \supseteq \{ d \in K | vd > -(p-1) \text{dist} (\vartheta, K) \}
= \{ d \in K | vd > -(p-1) \text{dist} E \}.
\]

Now we consider the mixed characteristic case. Since \( \eta^p \in K \), we have that

\[
\text{Tr}_{K(\eta)|K} (\eta^i) = 0
\]

for \( 1 \leq i \leq p-1 \). For \( c \in K \) and \( 0 \leq j \leq p-1 \), we compute:

\[
(\eta - c)^j = \sum_{i=1}^{j} \binom{j}{i} \eta^{i}(-c)^{j-i} + (-c)^j.
\]

Thus for every \( d \in K \),

\[
\text{Tr}_{K(\eta)|K} (d(\eta-c)^j) = pd(-c)^j.
\]

If \( vd > -(p-1) \text{dist} (\eta, K) \), then we may choose \( c \in K \) with \( vd > -(p-1)v(\eta-c) \); this remains true if we make \( v(\eta-c) \) even larger. Since \( \eta \) is a 1-unit, there is \( c \in K \) such that \( v(\eta-c) > 0 \), which implies that \( vc = 0 \). Hence we may choose \( c \in K \) with \( vd > -(p-1)v(\eta-c) \) and \( vc = 0 \). Applying (53) with \( j = p-1 \), we find that \( \text{Tr}_{K(\eta)|K} (d(-c)^{-(p-1)}(\eta-c)^{p-1}) = pd \). We obtain, using the second equation of Corollary 3.6:

\[
\text{Tr}_{K(\eta)|K} (M_{K(\eta)}) \supseteq \{ pd | d \in K \text{ and } vd > -(p-1) \text{dist} (\eta, K) \}
= \{ d \in K | vd > -(p-1) \text{dist} (\eta, K) + vp \}
= \{ d \in K | vd > -(p-1) \text{dist} E \}.
\]
In order to prove the opposite inclusions in (52) and (54), we have to find out enough information about the elements \( g(a) \in K(a) \) that lie in \( M_{K(a)} \). Using the Taylor expansion, we write

\[
g(a) = \sum_{i=0}^{p-1} \partial_i g(c)(a-c)^i.
\]

By Lemma 2.11 there is \( c \in K \) such that among the values \( v\partial_i g(c)(a-c)^i \), \( 0 \leq i \leq p-1 \), there is precisely one of minimal value, and the same holds for all \( c' \in K \) with \( v(a-c') \geq v(a-c) \). In particular, we may assume that \( v(a-c) > va \). For all such \( c \), we have:

\[
v g(a) = \min_{0 \leq i \leq p-1} v\partial_i g(c)(a-c)^i.
\]

Hence for \( g(a) \) to lie in \( M_{K(a)} \) it is necessary that \( v\partial_i g(c)(a-c)^i > 0 \), or equivalently,

\[
(55) \quad v\partial_i g(c) > -iv(a-c)
\]

for \( 0 \leq i \leq p-1 \) and \( c \in K \) as above.

In the equal characteristic case, for \( g(\vartheta) \in M_{K(\vartheta)} \) and \( c \in K \) as above, we find:

\[
\text{Tr}_{K(\vartheta)|K}(g(\vartheta)) = \sum_{i=0}^{p-1} \text{Tr}_{K(\vartheta)|K}(\partial_i g(c)(\vartheta-c)^i) = -\partial_{p-1} g(c).
\]

Since \( \partial_{p-1} g(c) > -(p-1)v(\vartheta-c) \) by (55), this proves the desired equality in (52).

In the mixed characteristic case, for \( g(\eta) \in M_{K(\eta)} \) and \( c \in K \) as above, we find:

\[
\text{Tr}_{K(\eta)|K}(g(\eta)) = \sum_{j=0}^{p-1} \text{Tr}_{K(\eta)|K}(g_j(c)(\eta-c)^j) = p \sum_{j=0}^{p-1} \partial_j g(c)(-c)^j.
\]

As we assume that \( v(\eta-c) > 0 \), we have that \( vc = 0 \) and \( -iv(\eta-c) \geq -(p-1)v(\eta-c) \) for \( 0 \leq i \leq p-1 \). Hence by (55), \( v \sum_{j=0}^{p-1} \partial_j g(c)(-c)^j \geq -(p-1)v(\eta-c) \). This proves the desired equality in (54). We have now proved the first assertion of the following theorem:

**Theorem 5.2.** Take a Galois defect extension \( \mathcal{E} = (L|K,v) \) of prime degree \( p \). Then

\[
\text{Tr}_{L|K}(M_L) = \{ d \in K \mid vd > -(p-1) \text{ dist } \mathcal{E} \}.
\]

The extension \( \mathcal{E} \) has independent defect if and only if for some proper convex subgroup \( H \) of \( vK \),

\[
\text{Tr}_{L|K}(M_L) = \{ d \in K \mid vd > \alpha \text{ for all } \alpha \in H \} = \mathcal{M}_{v_H},
\]

where \( \mathcal{M}_{v_H} \) is the valuation ideal of the coarsening \( v_H \) of \( v \) on \( K \) whose value group is \( vK/(H \cap vK) \). In particular, if \( H = \{0\} \) (which is always the case if the rank of \( (K,v) \) is 1), then this means that

\[
\text{Tr}_{L|K}(M_L) = M_K.
\]

In the mixed characteristic case, \( \mathcal{M}_{v_H} \) will always contain \( p \), so that \( \text{char } K v_H = p \).
Proof. Equation (56) is already proven. By the equivalence of assertions a) and d) in Theorem 1.11, which we have already proved, the extension $E$ has independent defect if and only if $\text{dist} E$ is idempotent. This in turn is equivalent to $(p - 1) \text{dist} E = \text{dist} E$ and $\text{dist} E = H^-$ for some proper convex subgroup $H$ of $\bar{v}K$, or in other words,

$$-(p - 1) \text{dist} E = H^+, \tag*{56}$$

which turns (56) into the first equation of (57). This equation means that $d$ is an element of the valuation ideal $M_{vH}$ of the coarsening $vH$ of $v$ whose value group has divisible hull $v\bar{K}/H$. Hence the second equation in (57) holds.

The last statement of the theorem follows from assertion (44). □

From this theorem we obtain the equivalence of assertions a) and e) in Theorem 1.11.

6. Semitame, deeply ramified and gdr fields

Throughout this section, we will consider a valued field $(K, v)$ of residue characteristic $p > 0$, if not stated otherwise.

To start with, we state a few simple observations.

**Lemma 6.1.** 1) If $\text{char} K = p > 0$, then

$$O_K/pO_K \ni x \mapsto x^p \in O_K/pO_K \tag*{58}$$

is surjective if and only if $K$ is perfect; in particular, (DR

vr) holds if and only if $\hat{K}$ is perfect.

2) If (58) is surjective, then (DR

vr) holds.

3) Assume that $\text{char} K = 0$. Then the following assertions are equivalent:

a) (58) is surjective,

b) for every $\hat{a} \in O_{\hat{K}}$ there is $c \in O_K$ such that $\hat{a} \equiv c^p \mod pO_K(\hat{a})$,

c) (DR

vr) holds.

4) If $(K, v)$ satisfies (DR

vr), then so does every extension of $(K, v)$ within its completion.

**Proof.** 1): From $\text{char} K = p > 0$ it follows that $pO_K = \{0\}$, hence the surjectivity of the homomorphism in (5) means that every element in $O_K$ is a $p$-th power. Hence the same is true for every element in $K$, i.e., $K$ is perfect. Replacing $K$ by $\hat{K}$ in (58), we thus obtain that $\hat{K}$ is perfect.

2): Assume first that $\text{char} K = p > 0$. Then by part 1) the surjectivity of (58) implies that $K$ is perfect. Since the completion of a perfect field is again perfect, it follows that $\hat{K}$ is perfect. Hence again by part 1), (DR

vr) holds.

Now assume that $\text{char} K = 0$. Take $\hat{a} \in O_{\hat{K}}$. Then there exists $a \in K$ such that $\hat{a} \equiv a \mod pO_{\hat{K}}$. By assumption, there is some $c \in O_K$ such that $\hat{a} \equiv c^p \mod pO_K$. It follows that $\hat{a} \equiv a \equiv c^p \mod pO_K$, showing that (DR

vr) also holds in this case.

3): Assume that $\text{char} K = 0$. Trivially, b) implies a), and part 2) of our lemma shows that a) implies c). To show that c) implies b), take $\hat{a} \in O_{\hat{K}}$. Then by (DR

vr) there is $\hat{c} \in O_{\hat{K}}$ such that $\hat{a} \equiv \hat{c}^p \mod pO_{\hat{K}}$. We take $c \in O_K$ such that $\hat{c} \equiv c \mod pO_K$. Then $\hat{a} \equiv \hat{c}^p \equiv c^p \mod pO_K$, whence $\hat{a} \equiv c^p \mod pO_{K(\hat{a})}$.
4): Take \((L|K,v)\) to be a subextension of \((\hat{K}|K,v)\). Then \(\hat{L} = \hat{K}\), and in the case of \(\text{char } K = p > 0\) our assertion follows from part 1).

Now assume that \((K,v)\) is of mixed characteristic and satisfies (DRvr). Then by the implication \(c \Rightarrow b\) of part 3), for every \(\hat{a} \in \mathcal{O}_K = \mathcal{O}_L\) there is \(c \in \mathcal{O}_\mathcal{K} \subseteq \mathcal{O}_L\) such that \(\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}\). Hence (58) is surjective in \((L,v)\), and the implication \(a \Rightarrow c\) of part 3) shows that \((L,v)\) satisfies (DRvr).

\[\square\]

**Lemma 6.2.** 1) If \((K,v)\) satisfies (DRvr), then the following assertions hold:

a) The residue field \(Kv\) is perfect.

b) If char \(K = p > 0\), then \(vK\) is \(p\)-divisible and \((K,v)\) is a semitame field.

2) Assume that \((K,v)\) is a gdr field of mixed characteristic. Then the convex hull \((vK)_{vp}\) of \(Z_{vp}\) in \(vK\) is \(p\)-divisible. If in addition \((vK)_{vp} = vK\), then \((K,v)\) is a semitame field.

3) Assume that \((K,v)\) is a gdr field of mixed characteristic and that \(a \in K\) with \(va \in (vK)_{vp}\). Then there is \(c \in K\) such that

\[v(a - c^p) \geq va + vp.\]

**Proof.** 1): To prove part a), take any \(a \in \mathcal{O}\). By assumption, there is \(\hat{c} \in \mathcal{O}_K\) such that \(a \equiv \hat{c}^p \mod p\mathcal{O}_K\). From this we obtain that \(av = \hat{c}^pv = (\hat{c}v)^p \in Kv = KwK\). Hence \(Kv\) is perfect.

To prove part b), assume that char \(K = p > 0\). Then by part 1) of Lemma 6.1, (DRvr) implies that \(K\) is perfect, so \(vK = v\hat{K}\) is \(p\)-divisible and (DRst) holds, showing that \((K,v)\) is a semitame field.

2): First, let us show that every \(a \in vK\) with \(0 \leq \alpha < vp\) is divisible by \(p\). Take \(a \in \mathcal{O}\) such that \(va = \alpha\). From (DRvr) we obtain that there is \(\hat{c} \in \mathcal{O}_K\) such that \(a \equiv \hat{c}^p \mod p\mathcal{O}_K\). Since \(va < vp\), this yields that \(va = v\hat{c}^p = pv\hat{c}\), showing that \(\alpha = va\) is divisible by \(p\) in \(v\hat{K} = vK\).

By assumption, \(vp\) is not the smallest positive element in \(vK\), hence there is \(\alpha \in vK\) such that \(0 < \alpha < vp\), and we know that \(\alpha\) is divisible by \(p\). We may assume that \(2\alpha \geq vp\) since otherwise we replace \(\alpha\) by \(vp - \alpha\). In this way we make sure that \((vK)_{vp}\) is equal to the smallest convex subgroup containing \(\alpha\). This implies that for every \(\beta \in (vK)_{vp}\) there is some \(n \in \mathbb{Z}\) such that \(0 \leq \beta - n\alpha < vp\). Then by what we have already shown, \(\beta - n\alpha\) is divisible by \(p\). Since also \(\alpha\) is divisible by \(p\), the same is consequently true for \(\beta\).

If in addition \((vK)_{vp} = vK\), then \(vK\) is \(p\)-divisible, and since (DRvr) holds by assumption, \((K,v)\) is a semitame field.

3): Since \(va \in (vK)_{vp}\), part 2) shows that there is \(b \in K\) such that \(pbv = va\). Hence \(vb^{-p}a = 0\) and since \((K,v)\) is a gdr field, there is \(d \in K\) such that \(v(b^{-p}a - dp) \geq vp\), whence

\[v(a - (bd)^p) = pvb + v(b^{-p}a - dp) \geq va + vp.\]

With \(c := bd\), this yields (59).

\[\square\]

**Lemma 6.3.** If \((vK)_{vp}\) is \(p\)-divisible and \(Kv\) is perfect, then \(v(\eta - K)\) does not admit a maximal element smaller than \(\frac{vp}{p}\).

**Proof.** Take \(c \in K\) such that \(0 < v(\eta - c) < \frac{vp}{p}\). Then \(v(\eta - c)^p < vp\) and from (16) it follows that \(v(\eta^p - c^p) = v(\eta - c)^p < vp\). Since \((vK)_{vp}\) is \(p\)-divisible, there is some \(d_1 \in K\) such that \(vd_1^p = -v(\eta^p - c^p)\). Then \(vd_1^p(\eta^p - c^p) = 0\), and since
$Kv$ is perfect, there is some $d_2 \in K$ such that $d_2^n v = (d_2^n (\eta^p - c^p))^{-1}$. Then $v(d_2^n d_2^n (\eta^p - c^p)) = 0$ and $(d_2^n d_2^n (\eta^p - c^p)) v = 1$. With $d = (d_1 d_2)^{-1}$ it follows that $v(d^{-p}(\eta^p - c^p) - 1) > 0$, whence $v(\eta^p - c^p - d^p) > v(\eta^p - c^p)$. Again by (16), we obtain that $(\eta - c - d)^p \equiv \eta^p - c^p - d^p \mod pO$, and it follows that $v(\eta - c - d) > v(\eta - c)$. \hfill $\square$

**Proposition 6.4.** 1) Assume that $(vK)_v$ is $p$-divisible, $Kv$ is perfect, and $(K, v)$ is an independent defect field. Then $(K, v)$ is a gdr field.

2) If every separable unibranched extension of degree $p$ of $(K, v)$ is either tame or an independent defect extension, then $(K, v)$ is a semitame field.

**Proof.** 1): From our assumption that $(vK)_v$ is $p$-divisible it follows that (DRvp) holds. It remains to show that $(K, v)$ satisfies (DRrv).

Assume first that char $K > 0$. Then by assumption, $vK$ is $p$-divisible and $Kv$ is perfect, hence the perfect hull of $K$ is an immediate extension of $(K, v)$. Our assumption that $(K, v)$ is an independent defect field implies that $(K, v)$ has no Artin-Schreier extension with dependent defect. This yields that the perfect hull of $K$ lies in its completion (see Corollary 4.6 of [15]). It follows that the completion is perfect and hence $(K, v)$ satisfies (DRrv) by part 1) of Lemma 6.1.

Now assume that char $K = 0$. Assume further that $b \in K$ is not a $p$-th power, and take $\eta \in \bar{K}$ with $\eta^p = b$. Then by Lemma 6.3, $v(\eta - K)$ has a maximal element $\geq \frac{v}{p}$, or it has no maximal element at all. In the first case, part 3) of Lemma 2.16 shows the existence of $c \in K$ such that $b \equiv c^p \mod pO_K$. In the second case, we know from Lemma 2.10 that $(K(\eta)|K, v)$ is a defect extension. By assumption, it is independent, so $\text{dist}(\eta, K) = \frac{vp}{p-1} + H^-$ for some proper convex subgroup $H$ of $\bar{K}$ with $vp \notin H$. Hence again there is some $c \in K$ such that $v(\eta - c) \geq \frac{vp}{p}$, which by part 3) of Lemma 2.16 gives us $b \equiv c^p \mod pO_K$. This shows that (58) is surjective. Hence by part 2) of Lemma 6.1, (DRrv) holds.

2): Our assumptions yield that $vK$ is $p$-divisible (so (DRst) holds), and $Kv$ is perfect. Indeed, if $a \in vK$ is not divisible by $p$ and $a \in K$ with $va = a$, then taking a $p$-th root of $a$ induces an extension that is neither tame nor immediate. The same holds if $a \in K$ is such that $av$ does not have a $p$-th root in $Kv$. Since defect extensions of degree $p$ are not tame, our assumption yields that every separable defect extension of degree $p$ is independent. Hence we obtain from part 1) that (DRvr) holds. \hfill $\square$

**Proof of Theorem 1.2.** 1): Assume that $(K, v)$ is nontrivially valued. The implication tame field $\Rightarrow$ separably tame field is obvious, and so is the implication semitame field $\Rightarrow$ deeply ramified field. To prove the implication deeply ramified field $\Rightarrow$ gdr field, we first observe that if char $K = p > 0$, then $vp = \infty$ which is not the smallest positive element of $vK$. If char $K = 0$, then we take $\Gamma_1$ to be the largest convex subgroup of $vK$ not containing $vp$, and $\Gamma_2$ to be the smallest convex subgroup of $vK$ containing $vp$. If $vp$ were the smallest positive element of $vK$, then we would have that $\Gamma_1 = \{0\}$ and $\Gamma_2 = Zvvp$, whence $\Gamma_2/\Gamma_1 \simeq Z$ in contradiction to (DRvg).

Now assume that $(K, v)$ is a separably tame field. If char $K > 0$, then by [18, Corollary 3.12], $(K, v)$ is dense in its perfect hull. Then the completion of the perfect hull is also the completion of $(K, v)$. Since the completion of a perfect valued field
is again perfect, we obtain that the completion of \((K, v)\) is perfect. Now part 1) of Lemma 6.1 shows that \((K, v)\) is a semitame field. If \(\text{char } K = 0\), then the separably tame field \((K, v)\) is a tame field. Hence every finite extension of \((K, v)\) is a tame extension. Thus by part 2) of Proposition 6.4, \((K, v)\) is a semitame field.

2): Assume that \((K, v)\) is a gdr field of rank 1 and mixed characteristic. Since the rank is 1, we have that \((vK)_{v^p} = vK\). Hence by part 2) of Lemma 6.2, \((K, v)\) is a semitame field. This together with part 1) of our theorem shows the required equivalence in the case of mixed characteristic. For the case of equal characteristic, it will be shown in the proof of part 3).

3): Assume that \((K, v)\) is a nontrivially valued field of characteristic \(p > 0\).

The implications a)\(\Rightarrow\)b)\(\Rightarrow\)c) have already been shown in part 1).

c)\(\Rightarrow\)d): This holds by definition.

d)\(\Rightarrow\)e): This holds by part 1) of Lemma 6.1.

e)\(\Rightarrow\)f): If the completion of \((K, v)\) is perfect, then it contains the perfect hull of \(K\); since \((K, v)\) is dense in its completion, it is then also dense in its perfect hull.

f)\(\Rightarrow\)g): If \((K, v)\) is dense in its perfect hull, then in particular it is dense in \(K^{1/p} = \{a^{1/p} \mid a \in K\}\). Since \(x \mapsto x^p\) is an isomorphism which preserves valuation divisibility, the latter holds if and only if \((K^p, v)\) is dense in \((K, v)\).

g)\(\Rightarrow\)f): Assume that \((K^p, v)\) is dense in \((K, v)\). Since for each \(i \in \mathbb{N}\), \(x \mapsto x^{p^i}\) is an isomorphism which preserves valuation divisibility, it follows that \((K^{1/p^{i-1}}, v)\) is dense in \((K^{1/p^i}, v)\). By transitivity of density we obtain that \((K, v)\) is dense in \((K^{1/p^i}, v)\) for each \(i \in \mathbb{N}\), and hence also in its perfect hull.

f)\(\Rightarrow\)e): This implication was already shown in the proof of part 1) of our theorem.

e)\(\Rightarrow\)a): Assume that \(\hat{K}\) is perfect. The extension \((\hat{K}|K, v)\) is immediate, so \(vK = v\hat{K}\), which is \(p\)-divisible. Hence (DRst) holds. By part 1) of Lemma 6.1, also (DRvr) holds.

4): The assertion follows from the implication f)\(\Rightarrow\)a) of part 3) as a perfect field is equal to its perfect hull.

Our next goal is the proof of Propositions 1.3 and 1.4, for which we need some preparation.

**Lemma 6.5.** Assume that \((K, v)\) is of mixed characteristic, and let \(v_0\) be the coarsening of \(v\) with respect to \((vK)_{v^p}\), that is, the finest coarsening that has a residue field of characteristic 0. Further, denote by \(w\) the valuation induced by \(v\) on \(Kv_0\).

Then \((K, v)\) is a gdr field if and only if \((Kv_0, w)\) is a gdr field.

**Proof.** First assume that \((K, v)\) is a gdr field. Then \(vp\) is not the smallest positive element in \(vK\), which implies that \(wp\) is not the smallest element in \(w(Kv_0)\). Take any \(b \in \mathcal{O}_{Kv_0}\). Then choose \(a \in \mathcal{O}_K\) such that \(av_0 = b\). Since \((K, v)\) is a gdr field, there is some \(c \in \mathcal{O}_K\) such that \(a - c^p \in p\mathcal{O}_K\). It follows that \(cv_0 \in \mathcal{O}_{Kv_0}\) with \(b = (cv_0)^p = (a - c^p)v_0 \in p\mathcal{O}_{Kv_0}\), showing that \((Kv_0, w)\) satisfies (DRvr) by part 3) of Lemma 6.1. Hence \((Kv_0, w)\) is a gdr field.

Now assume that \((Kv_0, w)\) is a gdr field. Then \(wp\) is not the smallest element in \(w(Kv_0)\), which implies that \(vp\) is not the smallest positive element in \(vK\). Take any \(a \in \mathcal{O}_K\). Then \(av_0 \in \mathcal{O}_{Kv_0}\), and there is some \(d \in \mathcal{O}_{Kv_0}\) such that \(av_0 - d^p \in p\mathcal{O}_{Kv_0}\). Choose \(c \in \mathcal{O}_K\) such that \(cv_0 = d\). It follows that \(a - c^p \in p\mathcal{O}_K\). We have now shown that \((K, v)\) is a gdr field.
**Proposition 6.6.** Assume that \((K,v)\) is a gdr field of mixed characteristic, and take \(a \in \mathcal{O}_K\) such that \(va \in (vK)_p\).

1) Assume that \(va = 0\). Then for every \(c \in \mathcal{O}_K\) with \(0 < v(a - c^p) \in (vK)_p\) there is \(d \in \mathcal{O}_K\) such that

\[
v(a - d^p) = vp + \frac{1}{p} v(a - c^p).
\]

2) Assume that \(va \in (vK)_p\) and that \(\text{dist} (a,K^p) < va + \frac{p}{p-1} vp\). Then

\[
va + vp < \text{dist} (a,K^p) = va + \frac{p}{p-1} vp + H^-
\]

where \(H\) is a convex subgroup of \(K\) not containing \(vp\).

**Proof.** 1) Set \(\alpha := v(a - c^p) > 0\). Since \((K,v)\) is a gdr field, part 3) of Lemma 6.2 shows that there is \(\tilde{d} \in K\) such that:

\[
(60) \quad v(a - c^p - \tilde{d}^p) \geq vp + \alpha.
\]

It follows that \(v\tilde{d}^p = \alpha\). Since \(vc = va = 0\),

\[
(61) \quad v((c + \tilde{d})^p - c^p - \tilde{d}^p) = v \sum_{i=1}^{p-1} p^{p-i} \tilde{d} = vp + v\tilde{d} = vp + \frac{\alpha}{p}.
\]

From (60) and (61), we obtain for \(d := c + \tilde{d}\):

\[
v(a - d^p) = \min \left\{ vp + \alpha, vp + \frac{\alpha}{p} \right\} = vp + \frac{\alpha}{p}.
\]

2) First we prove the assertion in the case of \(va = 0\). Since \((K,v)\) is a gdr field, there is some \(c \in K\) such that \(v(a - c^p) \geq vp\), so \(\text{dist} (a,K^p) \geq vp\).

We will use the following observation. If \((vK)_p \ni v(a - c^p) \geq \frac{p}{p-1} vp - \varepsilon > 0\) for some \(c \in K\) and positive \(\varepsilon \in vK\), then by part 1) there is \(d \in \mathcal{O}_K\) such that

\[
v(a - d^p) = vp + \frac{v(a - c^p)}{p} \geq vp + \frac{vp}{p-1} - \frac{1}{p} \varepsilon = \frac{p}{p-1} vp - \frac{1}{p} \varepsilon.
\]

By assumption, \(\text{dist} (a,K^p) < \frac{p}{p-1} vp\). Hence the set of all convex subgroups \(H'\) of \(K\) such that \(v(a - K^p) \cap \left(\frac{p}{p-1} vp + H'\right) = 0\) is nonempty as it contains \(\{0\}\). The set is closed under arbitrary unions, so it contains a maximal subgroup \(H\), the union of all subgroups in the set. Since \(vp \in v(a - K^p)\), we see that \(H\) cannot contain \(vp\).

Take any positive \(\delta \notin H\). Then by the definition of \(H\), there is some \(n \in \mathbb{N}\) such that \(v(a - K^p)\) contains a value \(\geq \frac{p}{p-1} vp - n\delta\). We set \(\varepsilon := \min \left\{ \frac{p}{p-1} vp - vp, n\delta \right\}\) and observe that there is \(c \in K\) such that

\[
v(a - c^p) \geq \frac{p}{p-1} vp - \varepsilon \geq vp > 0.
\]

Note that \(v(a - c^p) \in (vK)_p\) since \(\text{dist} (a,K^p) < \frac{p}{p-1} vp\). Using our above observation, by induction starting from \(c_0 = c\) we find \(c_i \in K\) such that

\[
v(a - c_i^p) \geq \frac{p}{p-1} vp - \frac{1}{p^i} \varepsilon.
\]
We choose some \( j \in \mathbb{N} \) such that \( \frac{a}{p^j} < 1 \). Then
\[
\frac{1}{p^j} \varepsilon \leq \frac{n}{p^j} \delta < \delta
\]
and consequently,
\[
v(a - e_j^p) > \frac{p}{p-1} vp - \delta.
\]
This together with the definition of \( H \) shows that
\[
(62) \quad vp < \text{dist} (a, K^p) = \frac{p}{p-1} vp + H^-.
\]
If \( 0 \neq va \in (vK)_p \), then since \((K, v)\) is a gdr field, part 2) of Lemma 6.2 shows that there is \( b \in K \) such that \( v b^p = va \). By what we have already shown, (62) holds for \( b^{-p}a \) in place of \( a \). We have that
\[
v(a - (bc)^p) = vb^p + v(b^{-p}a - c^p) = va + v(b^{-p}a - c^p),
\]
whence
\[
\text{dist} (a, K^p) = va + \text{dist} (b^{-p}a, K^p),
\]
which together with (62) for \( b^{-p}a \) in place of \( a \) proves assertion 2) of our lemma. \( \square \)

**Proof of Proposition 1.3.**

In view of Lemma 6.5, where we take \( w = v_p \circ \varphi \), it suffices to prove the proposition under the additional assumption that \( v_0 \) is trivial, that is, \( vK = (vK)_p \). Then the assertion is trivial if \( \varphi \) is trivial, so we assume that it is not. This implies that \( vp \) is not the smallest positive element in \( vK \).

Let us first assume that \((K, v)\) is a gdr field. Then \( \frac{a}{p} \in vK \) by part 2) of Lemma 6.2, so \( \frac{vK}{p} \in v_p K \), showing that \( v_p p \) is not the smallest positive element in \( v_p K \). It remains to show that \((K, v_p)\) satisfies (DRvr); by part 3) of Lemma 6.1 it suffices to prove that (58) is surjective in \((K, v_p)\). Take any \( a \in \mathcal{O}_{v_p} \). Since \((K, v)\) is a gdr field, by part 3) of Lemma 6.2 there is \( c \in K \) such that \( v(a - c^p) \geq va + vp \), whence \( v_p(a - c^p) \geq vp a + v_p p \geq v_p p \).

Now assume that \((K, v_p)\) is a gdr field. We know already that (DRvp) holds in \((K, v)\), so it remains to show that (58) holds. Take \( a \in \mathcal{O}_v \subseteq \mathcal{O}_{v_p} \). Since \((K, v_p)\) is a gdr field, part 2) of Proposition 6.6 implies that there is some \( c \in K \) such that \( v_p(a - c^p) > v_p p \), whence \( v(a - c^p) > vp \). \( \square \)

**Proof of Proposition 1.4.**

Take an arbitrary valued field \((K, v)\) and assume that \( v = w \circ \varphi \) with \( w \) and \( \varphi \) nontrivial. Assume first that \( \text{char} K > 0 \). Then by part 3) of Theorem 1.2, the properties “semitame”, “deeply ramified” and “gdr” are equivalent, so we have to prove the assertions of the lemma only for “gdr”.

As \( w \) is nontrivial and a coarsening of \( v \), the topologies generated by \( v \) and \( w \) are equal, and it follows that \((K, v)\) is dense in its perfect hull if and only if the same holds for \((K, w)\). By the equivalence of assertions c) and f) in part 3) of Theorem 1.2, it follows that \((K, v)\) is a gdr field if and only if \((K, w)\) is a gdr field. If the latter is the case, then from part 1) of Lemma 6.2 we see that \( Kw \) is perfect, and as it is of positive characteristic like \( K \), we obtain from part 3) of Theorem 1.2 that \((Kw, \varpi)\) is also a gdr field.

Now we assume that \( \text{char} K = 0 \) and prove the assertions for the property “gdr”. If \( \text{char} Kv = 0 \), then \((K, v), (K, w)\) and \((Kw, \varpi)\) all are gdr fields. So we
may assume that \( \text{char} \ K v > 0 \). We decompose \( v \) and \( w \) as in the paragraph before Proposition 1.3: \( v = v_0 \circ v_p \circ \overline{v} \) and \( w = w_0 \circ w_p \circ \overline{w} \). First we discuss the case where \( \text{char} \ K w > 0 \). Then \( v_0 = w_0 \), \( v_p = w_p \), and \( \overline{w} \) is a (possibly trivial) coarsening of \( \overline{v} \).

Hence it follows from Proposition 1.3 that \((K, v)\) is a gdr field if and only if \((K, w)\) is a gdr field. If the latter is the case, then because of \( \text{char} \ K w > 0 \) it follows as before that \((K w, \overline{w})\) is also a gdr field.

Now we discuss the case where \( \text{char} \ K w = 0 \). Then \((K, w)\) is trivially a gdr field, and \( w_0 \) is a coarsening of \( v_0 \). Consequently, for the decomposition \( \overline{w} = \overline{w}_0 \circ \overline{w}_p \circ \overline{w} \) we obtain that \( \overline{w}_p = v_p \), \( \overline{v} = \overline{v}_p \), and \( \overline{w}_0 \) is possibly trivial, with \( w \circ \overline{w}_0 = v_0 \). It follows that \((K v_0, v_p) = ((K w)\overline{w}_0, \overline{w}_p)\). Using Proposition 1.3, we conclude that \((K, v)\) is a gdr field if and only if \((K w, \overline{w})\) is a gdr field.

It remains to consider the properties “semitame” and “deeply ramified”. We observe that if \( \text{char} \ K v = p > 0 \), then \( v K \) is \( p \)-divisible if and only if the same is true for \( w K \) and \( \overline{w}(K w) \). Likewise, all archimedean components of \( v K \) are densely ordered if and only if the same is true for \( w K \) and \( \overline{w}(K w) \). From what we have proved before, it thus follows that \((K, v)\) is a semitame (or deeply ramified) field if and only if both \((K, v)\) and \((K w, \overline{w})\) are semitame (or deeply ramified, respectively) fields.

Further, we recall that in the case of \( \text{char} \ K w > 0 \), \((K, w)\) being a gdr field implies that \( K w \) is perfect, and so \( \overline{w}(K w) \) is \( p \)-divisible and thus all of its archimedean components are densely ordered. This proves that \((K, v)\) is a semitame (or deeply ramified) field already if \((K, w)\) is.

We will now prepare the proof of Theorems 1.5 and 1.6.

**Lemma 6.7.** Every algebraic extension of a deeply ramified field of positive characteristic is again a deeply ramified field.

**Proof.** By part 3) of Theorem 1.2, a valued field \((K, v)\) of positive characteristic is a deeply ramified field if and only if its completion \((\hat{K}, v)\) is perfect. Take any algebraic extension \((L|K, v)\). Then the completion \((\hat{L}, v)\) of \((L, v)\) contains \((\hat{K}, v)\). Since \( \hat{K} \) is perfect, so is \( L \cdot \hat{K} \). Since \((\hat{L}, v)\) is also the completion of \((L \cdot \hat{K}, v)\), it is perfect too.

**Lemma 6.8.** Assume that \((K, v)\) is a gdr field of mixed characteristic. Further, take a defect extension \((K(\eta)|K, v)\) with \( \eta p \in K \) such that \( v \eta = 0 \). Then

\[
\text{dist}(\eta, K) = \frac{\eta p}{p - 1} + H^{-},
\]

where \( H \) is a convex subgroup of \( \hat{K} \) not containing \( v p \).

**Proof.** Suppose that there is some \( c \in K \) such that \( v(\eta - c) \geq \frac{\eta p}{p - 1} \). Since the defect extension \((K(\eta)|K, v)\) is immediate, \( v(\eta - c) \) has no maximal element, and so there will also be some element \( c \in K \) such that \( v(\eta - c) > \frac{\eta p}{p - 1} \). Then by Lemma 2.19, \( \eta \) lies in some henselization \( K^h \). But this is impossible since by Lemma 2.5, the unibranch extension \((K(\eta)|K, v)\) is linearly disjoint from \( K^h|K \). We conclude that \( \text{dist}(\eta, K) < \frac{\eta p}{p - 1} \). By Lemma 2.17, this is equivalent to \( \text{dist}(\eta p, K^p) < \frac{p}{p - 1} \eta p \). Therefore, we can apply Proposition 6.6 to \( a = \eta p \). We find that

\[
\text{dist}(\eta p, K^p) = \frac{p}{p - 1} \eta p + H^{-}
\]
where $H$ is a convex subgroup of $\vec{v}K$ not containing $vp$. As $\vec{v}K$ is $p$-divisible, we can again apply Lemma 2.17 to obtain that (64) is equivalent to (63).

\[\square\]

**Proposition 6.9.** Every gdr field is an independent defect field.

**Proof.** In the case of residue characteristic 0, the assertion is trivial. So we assume that $(K,v)$ is a gdr field of positive residue characteristic.

Assume first that char $K > 0$. Then by part 3) of Theorem 1.2, the perfect hull of $(K,v)$ lies in its completion. Now part 3) of Proposition 3.10 shows that $(K,v)$ is an independent defect field.

Finally, assume that char $K = 0$, and take a Galois defect extension $(L/K,v)$ of prime degree. As shown in the beginning of Section 3.2, we can assume that $L = K(\eta)$ with a Kummer generator $\eta$ which is a 1-unit. So we can apply Lemma 6.8 to find that (63) holds for some convex subgroup $H$ of $\vec{v}K$. By part 3) of Proposition 3.7 it follows that $(K(\eta)|K,v)$ has independent defect. This proves that $(K,v)$ is an independent defect field.

\[\square\]

**Lemma 6.10.** Fix any extension of $v$ from $K$ to $\tilde{K}$, and let $(K^r,v)$ be the respective absolute ramification field of $(K,v)$. If $(K^r,v)$ is a gdr field, then so is $(K,v)$, and if $(K^r,v)$ is a semitame field, then so is $(K,v)$.

**Proof.** Assume that $(K^r,v)$ is a gdr field and hence an independent defect field by Proposition 6.9. By parts 1) and 2) of Lemma 6.2, $(vK^r)_vp$ is $p$-divisible and $K^r v$ is perfect. Since $vK^r/vK$ has no $p$-torsion and $K^r v|Kv$ is separable, it follows that $(vK)_vp$ is $p$-divisible and $Kv$ is perfect. From Proposition 3.8 we know that $(K,v)$ is an independent defect field. Part 1) of Proposition 6.4 now shows that $(K,v)$ is a gdr field.

Now assume that $(K^r,v)$ is a semitame field. Then by part 1) of Theorem 1.2, $(K^r,v)$ is a gdr field, hence so is $(K,v)$. Since $vK^r$ is $p$-divisible and the order of every element in $vK^r/vK$ is coprime to $p$, also $vK$ is $p$-divisible. Hence by definition, $(K^r,v)$ is a semitame field.

\[\square\]

**Lemma 6.11.** Assume that $(K,v)$ is a henselian gdr field of mixed characteristic and $(L/K,v)$ is a finite extension. Then the following assertions hold.

1) If $[L : K] = [Lv : Kv]$, then also $(L,v)$ is a gdr field.

2) Take a prime $q$ different from $p$. Assume that $L = K(a)$ with $a^q \in K$, $va \notin Kv$ and $q = (vL : vK)$. Then also $(L,v)$ is a gdr field.

**Proof.** We assume that $(K,v)$ is a gdr field of mixed characteristic with residue characteristic $p > 0$. If $(L/K,v)$ is an algebraic extension, then it is also of mixed characteristic with residue characteristic $p$, and like $(K,v)$, it satisfies (DRvp).

Hence by part 3) of Lemma 6.1, $(L,v)$ will be a gdr field once (58) is surjective.

In order to prove part 1), we take a finite extension $(L/K,v)$ such that $[L : K] = [Lv : Kv]$. Since $Kv$ is perfect by part 1) of Lemma 6.2, $Lv|Kv$ is separable and we write $Lv = Kv(\xi)$ with $\xi \in Lv$. Since also $Lv$ is perfect, there are $\xi_0, \ldots, \xi_n \in Kv$ with $n = [Lv : Kv] - 1$ such that $\xi = (\xi_n\xi^n + \ldots + \xi_1 \xi + \xi_0)^p$. Let $F$ be the extension of $\mathbb{F}_p$ generated by the coefficients of the minimal polynomial of $\xi$ over $Kv$ and the elements $\xi_0, \ldots, \xi_n$. As a finitely generated extension of the perfect field $\mathbb{F}_p$, $F$ is separably generated, that is, it admits a transcendence basis $t_1, \ldots, t_k$ such that $F|\mathbb{F}_p(t_1, \ldots, t_k)$ is separable-algebraic. We have that $F \subseteq Kv$, so we may choose $x_1, \ldots, x_k \in K$ such that $x_i v = t_i$. Then $v\mathbb{Q}(x_1, \ldots, x_k) = v\mathbb{Q} = \mathbb{Z}vp$. 


and \( \mathbb{Q}(x_1, \ldots, x_k) = F_p(t_1, \ldots, t_k) \) (cf. [3, chapter VI, §10.3, Theorem 1]). Using Hensel’s Lemma, we find an extension \( K_0 \) of \( \mathbb{Q}(x_1, \ldots, x_k) \) within the henselian field \( K \) such that \( K_0v = F \) and \( vK_0 = vQ(x_1, \ldots, x_k) = \mathbb{Z}_v p \).

Using Hensel’s Lemma again, we find \( a \in L \) such that \( av = \xi, [K_0(a) : K_0] = [F(\xi) : F] \) and \( vK_0(a) = vK_0 = \mathbb{Z}_v p \). By construction, \( \xi^{1/p} \in F(\xi) \), so we can choose \( b \in K_0(a) \) such that \( bv = \xi^{1/p} \). Then \( av = (bv)^p = b^pv \), so \( v(a - b^p) > 0 \) and thus \( v(a - b^p) \geq vp \).

We observe that since \( F \) contains all coefficients of the minimal polynomial of \( \xi \) over \( Kv \),

\[
[Kv(\xi) : Kv] = [F(\xi) : F] = [K_0(a) : K_0] \geq [K(a) : K] \geq [Kv(\xi) : Kv].
\]

Hence equality holds everywhere; in particular, \( K(a) = L \). Also, we obtain that \( 1, a, \ldots, a^n \) is a basis of \( K(a)/K \) with the residues \( 1, av, \ldots, a^n v \) linearly independent over \( Kv \). Hence if we write an arbitrary element of \( K(a) \) as \( \sum_{i=0}^n c_i a^i \) with \( c_i \in K \), then

\[
v \sum_{i=0}^n c_i a^i = \min_{0 \leq i \leq n} vc_i.
\]

Thus, for the sum to have non-negative value, all \( c_i \) must have non-negative value. Since \( (K, v) \) is a gdr field, we then have \( d_i \in K \) such that \( c_i \equiv d_i^p \mod pO_K \). So we obtain from Lemma 2.16 that

\[
\sum_{i=0}^n c_i a^i \equiv \sum_{i=0}^n d_i^p (b_p)^i \equiv \left( \sum_{i=0}^n d_i b^i \right)^p \mod pO_L,
\]

where the last equivalence holds by part 1) of Lemma 2.16. This shows that (58) is surjective, which proves that \( (L, v) \) is a gdr field.

In order to prove part 2), we take a prime \( q \) different from \( p \) and a finite extension \( (L/K, v) \) such that \( L = K(a) \) with \( a^q \in K \), \( \alpha := va \notin vK \) and \( q = (vL : vK) \). We obtain that \( [K(a) : K] = q = (vK(a) : vK) \). As \( p \) and \( q \) are coprime, also \( pva = v(a^q) \) generates \( v(a^q) \) over \( vK \), and \( K(a) = K(a^q) \). So \( 1, a^q, \ldots, a^{q(p-1)} \) is a basis of \( K(a)/K \) with the values \( v1, va^q, \ldots, va^{q(p-1)} \) belonging to distinct cosets of \( vK \). Hence if we write an arbitrary element \( b \) of \( K(a) \) as \( b = \sum_{i=0}^{q-1} c_i a^{pi} \) with \( c_i \in K \), then

\[
v \sum_{i=0}^{q-1} c_i a^{pi} = \min_{0 \leq i < q} vc_i + iva^q.
\]

Assume that the sum has non-negative value. Then all \( c_i a^{pi} \) must have non-negative value. But for \( i > 0 \), this does not imply that \( vc_i \geq 0 \); we only know that \( vc_i a^{pi} > 0 \) since \( iva^q \notin vK \), whence \( va^q > -vc_i \).

Suppose that \( va \) is not equivalent to an element in \( vK \) modulo \( (vL)_{vp} \). Then the same holds for \( vc_i + piva \) in place of \( va \), for \( 1 \leq i < q \), so that \( vc_i a^{pi} \notin (vL)_{vp} \). In this case, \( b \) is equivalent to \( c_0 \) modulo \( pO_L \). Since \( (K, v) \) is a gdr field, there is \( d_0 \in K \) such that \( b \equiv c_0 \equiv d_0^p \mod pO_K \). Hence we may now assume that \( va \) is equivalent to an element \( \delta \in vK \) modulo \( (vL)_{vp} \). We choose \( d \in K \) with \( vd = \delta \) and replace \( a \) by \( a/d \), so from now on we can assume that \( va \in (vL)_{vp} \).

As \( (K, v) \) is a gdr field, \( (vK)_{vp} \) is \( p \)-divisible by part 2) of Lemma 6.2. It follows that \( p(vK)_{vp} \) lies dense in \( (vL)_{vp} \) and thus there is \( b_i \in K \) such that \( -vc_i \leq pvb_i \leq
va^{p_i}$, whence $va_i b_i^{p_i} \geq 0$ and $vb_i^{-1} a^{p_i} \geq 0$. Again since $(K, v)$ is a gdr field, there are $d_i \in K$ such that $c_i b_i^{p_i} \equiv d_i^p \mod p\mathcal{O}_K$. So we obtain that
\[
\sum_{i=0}^{q-1} c_i a^{p_i} = \sum_{i=0}^{q-1} (c_i b_i^{p_i})(b_i^{-1} a^{p_i}) \equiv \sum_{i=0}^{q-1} d_i^p b_i^{-1} a^{p_i} \equiv \left( \sum_{i=0}^{q-1} d_i b_i a^i \right)^p \mod p\mathcal{O}_L,
\]
where the last equivalence holds by part 1) of Lemma 2.16. Again, this shows that (58) is surjective, which proves that $(L, v)$ is a gdr field. \hfill \Box

\textbf{Proof of Theorem 1.5:}

The case of residue characteristic 0 is trivial, so we assume that char $K^v = p > 0$. It has been proven already in Lemma 6.10 that if $(K^v, v)$ is a gdr field, then so is $(K, v)$, and if $(K^v, v)$ is a semitame field, then so is $(K, v)$. Let us now assume that $(K, v)$ is a gdr field; we aim to show that so is $(K^v, v)$.

First we consider the case of equal characteristic $p > 0$. Then by part 3) of Theorem 1.2, $(K, v)$ is a deeply ramified field. Hence by Lemma 6.7 also $(K^v, v)$ is a deeply ramified field and thus a gdr field.

Now we consider the case of a gdr field $(K, v)$ of mixed characteristic with char $K^v = p > 0$. In this part of the proof we will freely make use of facts from ramification theory; for details, see [7, 8, 18].

We let $L$ be a maximal extension of $K$ inside of $K^v$ that is again a gdr field; since the union over an ascending chain of gdr fields is again a gdr field, $L$ exists by Zorn’s Lemma.

First we will show that $(L, v)$ is henselian. As in Proposition 1.3, we decompose $v = v_0 \circ v_p \circ \pi$, where $v_0$ is the finest coarsening of $v$ that has residue characteristic 0, $v_p$ is a rank 1 valuation on $Kv_0$, and $\pi$ is the valuation induced by $v$ on the residue field of $v_p$ (which is of characteristic $p$). The valuations $v_0$ and $\pi$ may be trivial. As $K^v|K$ is algebraic, the restrictions of the respective valuations to any intermediate field of $K^v|K$ and the respective residue fields have the same properties. On any of these intermediate fields, $v$ is henselian if and only if $v_0$, $v_p$, and $\pi$ are.

Suppose that $v_0$ is not henselian on $L$. As $(K^v, v)$ is henselian, so is $(K^v, v_0)$ which therefore contains a henselization $L^{h(v_0)}$ of $L$ with respect to $v_0$. As henselizations are immediate extensions, we know that $L^{h(v_0)}v_0 = Lv_0$; by Proposition 1.3, $(Lv_0, v_p)$ is a gdr field. Using the same proposition again, we find that also $(L^{h(v_0)}, v_0)$ is a gdr field. By the maximality of $L$ we conclude that $L^{h(v_0)} = L$, so $v_0$ is henselian on $L$.

Next, suppose that $v_p$ is not henselian on $Lv_0$. As $(K^v, v)$ is henselian, so is $(K^v v_0, v_p)$ which therefore contains a henselization $Lv_0^{h(v_p)}$ of $Lv_0$ with respect to $v_p$. From Proposition 1.3 we infer that $(Lv_0, v_p)$ is a gdr field. As its rank is 1, its henselization lies in its completion. Hence by part 4) of Lemma 6.1, $(Lv_0^{h(v_p)}, v_p)$ satisfies (DRvr). Since (DRvp) holds in $(Lv_0, v_p)$, it also holds in $(Lv_0^{h(v_p)}, v_p)$, so the latter is a gdr field. The extension $Lv_0^{h(v_p)}|Lv_0$ is separable-algebraic, so we can use Hensel’s Lemma to find an extension $L'$ of $L$ within $K^v$ such that $L'v_0 = Lv_0^{h(v_p)}$. Using Proposition 1.3 again, we find that $(L', v)$ is a gdr field. Hence $L' = L$ by the maximality of $L$, that is, $Lv_0 = Lv_0^{h(v_p)}$ showing that $(Lv_0, v_p)$ is henselian.

Finally, suppose that $\pi$ is not henselian on $Lv_0v_p$. As $(K^v, v)$ is henselian, so is $(K^v v_0 v_p, \pi)$ which therefore contains a henselization $Lv_0^{h(\pi)}$ of $Lv_0v_p$ with respect
to \( \overline{\tau} \). Suppose that \( \text{L}v_0v_p^{h(\tau)}|\text{L}v_0v_p \) is nontrivial, so it contains a finite separable subextension. Using Hensel’s Lemma, we lift it to a subextension \( F[L \mid K^r]L \) such that \( [F : L] = [Fv_0v_p : Lv_0v_p] \). By what we have shown already, \( (L, v_0v_p) \) is henselian, and by definition it is of mixed characteristic. Therefore, we can employ part 1) of Lemma 6.11 to deduce that \( (F, v_0v_p) \) is a gdr field. By Proposition 1.3, also \( (F, v) \) is a gdr field. This contradiction to the maximality of \( L \) shows that \( \text{L}v_0v_p^{h(\tau)} = \text{L}v_0v_p \), that is, \( (\text{L}v_0v_p, \overline{\tau}) \) is henselian. Altogether, we have now shown that \( (L, \overline{\tau}) \) is henselian.

The residue field of \( K^r \) is the separable-algebraic closure of \( K \). Suppose that \( L \) is not separable-algebraically closed, so it admits a finite separable-algebraic extension. Using Hensel’s Lemma, we lift it to a subextension \( F[L \mid K^r]L \) such that \( [F : L] = [Fv : Lv] \). Again by part 1) of Lemma 6.11, \( (F, v) \) is a gdr field. This contradiction to the maximality of \( L \) shows that \( L \) is separable-algebraically closed.

The value group of \( K^r \) is the closure of \( vK \) under division by all primes other than \( p \). Suppose that \( vL \neq vK^r \). Then there is some prime \( q \neq p \) and \( a \in K^r \) with \( va \notin vK \) and \( q = \langle vL : vK \rangle \). By part 1) of Lemma 6.11, also \( (L(a), v) \) is a gdr field, which again contradicts the maximality of \( (L, v) \). We conclude that \( vL = vK^r \).

By what we have shown, \( Lv = K^r v \) and \( vL = vK^r \). Since \( K \subseteq L \subseteq K^r \), we know that \( K^r = L^r \), so the fact that \( (L^r \mid L, v) \) is a tame extension together with the equality of the value groups and residue fields implies that \( L = L^r = K^r \). We have proved that \( (K^r, v) \) is a gdr field.

Assume now that \( (K, v) \) is a semitame field. Then by part 1) of Theorem 1.2, \( (K, v) \) is a gdr field. As we have shown above, it follows that the same is true for \( (K^r, v) \). Since \( vK \) is \( p \)-divisible, \( vK^r \) is \( p \)-divisible too. Hence by definition, \( (K^r, v) \) is a semitame field.

\textit{Proof of Theorem 1.6:} 1): Assume that \( (K, v) \) is a gdr field. The assertions on \( vK \) and \( Kv \) have been proven in Lemma 6.2. Further, by Proposition 6.9, \( (K, v) \) is an independent defect field.

For the converse, we may assume that \( \text{char}Kv > 0 \) since every valued field with residue characteristic 0 is a semitame field. Now our assertion is the content of part 1) of Proposition 6.4.

2): The assertion is trivial if \( \text{char}Kv = 0 \), so we may assume that \( \text{char}Kv > 0 \).

First, we assume that \( (K, v) \) is a semitame field. Then by part 1) of Lemma 6.2, \( Kv \) is perfect. Since also \( vK \) is \( p \)-divisible by assumption, equation (1) shows that every unibranched extension \( (L \mid K, v) \) of degree \( p \) of \( (K, v) \) satisfies \( vL : vK = 1 \), so it either has defect \( p \), or \( [Lv : Kv] = p \) with \( Lv[Kv] \) a separable extension. In the latter case, the extension has no defect and is tame. Otherwise, it is a defect extension of degree \( p \). Then, as \( (K, v) \) is a gdr field by Theorem 1.2, part 1) of our theorem shows that it must be an independent defect extension.

The converse is the content of part 2) of Proposition 6.4.

\textit{Proof of Theorem 1.9.} By [9, Corollary 6.6.16 (i)], every algebraic extension of a deeply ramified field is again a deeply ramified field. For the convenience of the reader, we gave the easy proof for the case of deeply ramified fields of positive characteristic in Lemma 6.7, and for extensions within the absolute ramification
field, it can be deduced from Theorem 1.5 as follows. If \((L|K, v)\) is an extension within \(K^r\), then \(L^r = K^r\); if \((K, v)\) is a deeply ramified field, then it is a gdr field and Theorem 1.5 shows that also \((L, v)\) is a gdr field. On the other hand, condition (DRvg) is preserved under algebraic extensions, so \((L, v)\) is a deeply ramified field.

It remains to deal with semitame fields and with gdr fields. For semitame fields the proof is immediate as they are just the deeply ramified fields with \(p\)-divisible value groups. Both properties are preserved under algebraic extensions.

Now take a gdr field \((K, v)\). Every valued field of residue characteristic 0 is a gdr field, so we may assume that \(\text{char } Kv = p > 0\). If \((K, v)\) is of equal positive characteristic, then it is a deeply ramified field by part 3) of Theorem 1.2 and has already been dealt with above. Thus we assume that \((K, v)\) is of mixed characteristic. With \(v_0\) and \(w\) as in Lemma 6.5 we know from that lemma that \((Kv_0, w)\) is a gdr field. Hence by part 2) of Theorem 1.2, it is a semitame field. Now take any algebraic extension \((L|K, v)\). Then also \((L_{v_0}|Kv_0, w)\) is an algebraic extension, and by what we have shown already, \((L_{v_0}, w)\) is again a semitame field, and thus again by part 2) of Theorem 1.2 a gdr field. Hence by Lemma 6.5, \((L, v)\) is a gdr field. \(\square\)

**Proof of Theorem 1.8.** Every algebraically maximal field with residue characteristic 0 is henselian and defectless. Therefore, we may assume that \((K, v)\) is an algebraically maximal gdr field of positive residue characteristic \(p\). If \(\text{char } K = p\), then by part 3) of Theorem 1.2, \((K, v)\) is dense in its perfect hull. But as it is algebraically maximal, this extension must be trivial, i.e., \(K\) is perfect.

Take an absolute ramification field \((K^r, v)\) of \((K, v)\) and a finite tower \(K^r = L_0 \subset L_1 \subset \ldots \subset L_n\) of extensions of degree \(p\) over \(K^r\). By Theorem 1.9, every \((L_i, v)\) is a gdr field. Hence Theorem 1.6 yields that among the extensions \((L_i|L_{i-1}, v)\), \(1 \leq i \leq n\), every separable defect extension is independent. Now Proposition 4.7 shows that \((K, v)\) is henselian and defectless. \(\square\)

**Proof of Proposition 1.10.** Part 1) follows from Proposition 3.8. Part 2) is the content of part 3) of Theorem 3.10.

**Proof of Proposition 1.1.** It is well known that first order properties of the value group \(vK\) of a valued field \((K, v)\) can be encoded in \((K, v)\) in the language of valued fields. The axiomatization for (DRvp) and (DRst) is straightforward. Further, (DRvg) holds in an ordered abelian group \((G, <)\) if and only if for each positive \(\alpha \in G\) there is \(\beta \in G\) such that \(2\beta \leq \alpha \leq 3\beta\).

If \((K, v)\) is of mixed characteristic, then (DRvr) is equivalent to the surjectivity of (58), and this in turn holds if and only if for each \(a \in K\) with \(va \geq 0\) there is \(b \in K\) such that \(v(a - b^p) \geq vp\). Hence the classes of semitame, deeply ramified and gdr fields of mixed characteristic are first order axiomatizable.

If \((K, v)\) is of equal positive characteristic, then part 3) of Theorem 1.2 shows that semitame, deeply ramified and gdr fields form the same class, which can be axiomatized by saying that \((K^p, v)\) is dense in \((K, v)\), or in other words, for every \(\alpha \in vK\) and every \(a \in K\) there is \(b \in K\) such that \(v(a - b^p) \geq \alpha\).

In the case of equal characteristic 0, (DRvp), (DRvr) and (DRst) are trivial and all valued fields are semitame and gdr fields, while the class of deeply ramified fields consists of those which satisfy (DRvg). \(\square\)
7. Two constructions

In this section we give constructions for independent and dependent defect extensions in mixed characteristic. First, we show how to construct a semitame field with an independent defect extension of degree $p$.

**Theorem 7.1.** Consider the field $\mathbb{Q}_p$ of $p$-adic numbers together with the $p$-adic valuation $v_p$. Set $a_0 := p$ and by induction, choose $a_i \in \widehat{\mathbb{Q}_p}$ such that $a_i^p = a_{i-1}$ for $i \in \mathbb{N}$. Then $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$ together with the unique extension of $v$ is a semitame field and hence a deeply ramified field.

Further, take $\vartheta \in \widehat{\mathbb{Q}_p}$ such that $\vartheta^p - \vartheta = \frac{1}{p}$. Then $(K(\vartheta)|K,v)$ is an independent defect extension of degree $p$.

**Proof.** By choice of the $a_i$, $\frac{v_p}{p} = va_i \in v\mathbb{Q}_p(a_i)$. Therefore, $p^i \leq (v\mathbb{Q}_p(a_i) : v\mathbb{Q}_p) \leq (v\mathbb{Q}_p(a_i) : \mathbb{Q}_p) \leq p^i$.

Hence equality holds everywhere, and $[\mathbb{Q}_p(a_i)v : \mathbb{Q}_p] = 1$. We thus obtain that $v\mathbb{Q}_p(a_i) = \frac{1}{p}v\mathbb{Q}_p$ and $\mathbb{Q}_p(a_i)v = \mathbb{Q}_p v$. Consequently, $vK = \bigcup_{i \in \mathbb{N}} v\mathbb{Q}_p(a_i) = \frac{1}{p^\infty}\mathbb{Z}$ and $Kv = \mathbb{Q}_p v$.

This shows that $vK$ is $p$-divisible and that its only proper convex subgroup is $H = \{0\}$. In order to show that $(K,v)$ is a semitame field it remains to show that it satisfies (DRvr).

Take $b \in \mathcal{O}_K$. Then $b \in \mathbb{Q}_p(a_i)$ for some $i \in \mathbb{N}$ and we can write:

$$b \equiv \sum_{j=0}^n c_j a_i^j \mod p\mathcal{O}_{\mathbb{Q}_p(a_i)}$$

with $n < [\mathbb{Q}_p(a_i) : \mathbb{Q}_p] = p^i$ and $c_j \in \{0, \ldots, p-1\}$. Since $c_j^p \equiv c_j \mod p\mathcal{O}_{\mathbb{Q}_p}$ and $a_i^{p^p} = a_i$, we can compute:

$$\left(\sum_{j=0}^n c_j a_i^j\right)^p \equiv \sum_{j=0}^n c_j^p (a_i^p)^j \equiv \sum_{j=0}^n c_j a_i^j \equiv b \mod p\mathcal{O}_{\mathbb{Q}_p(a_i)}.$$

In view of part 2) of Lemma 6.1, this proves that $(K,v)$ satisfies (DRvr) and is therefore a semitame field.

Now we take $\vartheta$ as in the assertion of our theorem. Our first aim is to show that the extension $(K(\vartheta)|K,v)$ is nontrivial and immediate. For each $i \in \mathbb{N}$, we set

$$b_i = \sum_{j=1}^i \frac{1}{a_j} \in K(a_i)$$
and compute, using part 2) of Lemma 2.16:

\[(\vartheta - b_i)^p - (\vartheta - b_i) = \vartheta^p - \sum_{j=1}^{i} \frac{1}{a_j^p} - \vartheta + \sum_{j=1}^{i} \frac{1}{a_j} \]

\[= \frac{1}{p} - \sum_{j=0}^{i-1} \frac{1}{a_j} + \sum_{j=1}^{i} \frac{1}{a_j} = \frac{1}{a_i} \mod \mathcal{O}_{\mathbb{Q}_p(a_i)}.

It follows that \(v(\vartheta - b_i) < 0\) and

\[-\frac{vp}{p^i} = \frac{1}{a_i} = \min\{pv(\vartheta - b_i), v(\vartheta - b_i)\} = pv(\vartheta - b_i),\]

whence

\[(65) \quad v(\vartheta - b_i) = -\frac{vp}{p^{i+1}}.

We have that

\[p \leq (v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i)) \leq (v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i))[\mathbb{Q}_p(a_i, \vartheta)v : \mathbb{Q}_p(a_i)v] \leq [K(a_i, \vartheta) : K] \leq p.\]

Thus equality holds everywhere and we have that \((v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i)) = p\) as well as \(\mathbb{Q}_p(a_i)v = \mathbb{Q}_p[a_i]v = \mathbb{Q}_p^v\). The former shows that \(v\mathbb{Q}_p(a_i, \vartheta) = \frac{1}{p\mathbb{Q}_p}\mathbb{Q}_p\), which implies that for all \(i \in \mathbb{N}, \vartheta \not\in \mathbb{Q}_p(a_i)\). Hence \(\vartheta \not\in K\), and we have:

\[vK(\vartheta) = \bigcup_{i \in \mathbb{N}} v\mathbb{Q}_p(a_i, \vartheta) = \frac{1}{p^\infty} \mathbb{Z} = Kv \quad \text{and} \quad K(\vartheta)v = \mathbb{Q}_p^v = vK.\]

This shows that \((K(\vartheta)|K, v)\) is nontrivial and immediate, as asserted. As we have already proven that \((K, v)\) is a semitame field, it follows from Theorem 1.6 that the extension has independent defect.

What we have just presented is the mixed characteristic analogue of the following example given in, e.g., [13, Example 12]. Take \(K\) to be the perfect hull of \(\mathbb{F}_p((t))\), that is, \(K = \mathbb{F}_p((t))(t^{1/p^i} | i \in \mathbb{N})\). Take \(\mathbb{V}\) to be the \(t\)-adic valuation on \(\mathbb{F}_p((t))\); since it is henselian, there is a unique extension to \(K\) and \((K, v)\) is again henselian. The Artin-Schreier extension \((K(\vartheta)|K, v)\) generated by a root \(\vartheta\) of the polynomial \(X^p - X - \frac{i}{p}\) is nontrivial and immediate. As \(K\) is perfect, it does not admit any dependent Artin-Schreier defect extension, so the extension \((K(\vartheta)|K, v)\) has independent defect. In fact, \((K, v)\) is a semitame field.

We turn to the construction of a dependent defect extension of degree \(p\). The following is an analogue of Example 3.22 of [16].

**Theorem 7.2.** Set \(a_0 := -\frac{1}{p} \in \mathbb{Q}_p\) and by induction, choose \(a_i \in \mathbb{Q}_p\) such that \(a_i^p - a_i = -a_{i-1}\) for \(i \in \mathbb{N}\). Consider \(K := \mathbb{Q}_p(a_i | i \in \mathbb{N})\) together with the unique extension of \(v\). Then \(vK\) is \(p\)-divisible. Further, take \(\eta \in \mathbb{Q}_p\) such that

\[\eta^p = \frac{1}{p}.

Then \((K(\eta)|K, v)\) is a dependent defect extension of degree \(p\). Consequently, \((K, v)\) does not satisfy \((DRv)\).
Proof. By induction on $i$, we again obtain that $v a_i = \frac{1}{p} v p$. As in Theorem 7.1 we deduce that $v \mathbb{Q}_p(a_i) = \frac{1}{p^2} v \mathbb{Q}_p$ and $\mathbb{Q}_p(a_i)v = \mathbb{Q}_p v$, and for $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$ we obtain that $vK = \frac{1}{p^2} v \mathbb{Q}_p$ and $Kv = \mathbb{Q}_p v$. In particular, the only proper convex subgroup of $vK$ is $H = \{0\}$.

We set
$$b_i = \sum_{j=1}^{i} a_j \in K(a_i)$$
and compute, using part 2) of Lemma 2.16:
$$\begin{align*}
(\eta - b_i)^p &= \eta^p - \sum_{j=1}^{i} a_j^p = \frac{1}{p} + \sum_{j=1}^{i} (a_{j-1} - a_j) \\
&= \frac{1}{p} + \sum_{j=0}^{i-1} a_j - \sum_{j=1}^{i} a_j = -a_i \mod \mathcal{O}_{\mathbb{Q}_p(a_i)}.
\end{align*}$$
It follows that
$$v_{\mathbb{Q}_p} = va_i = pv(\eta - b_i),$$
whence
$$v(\eta - b_i) = \frac{vp}{p^t + 1}.$$  \tag{66}$$
As in the proof of Theorem 7.1 we deduce that $(K(\eta)|K,v)$ is nontrivial and immediate. It remains to show that its defect is dependent.

From (66) we see that $\text{dist}(\eta, K) \geq 0^-$. Suppose that $\text{dist}(\eta, K) > 0^-$. Then there is an element $c \in K$ such that $v(\eta - c) > v(\eta - b_i)$ for every $i \in \mathbb{N}$. Hence,
$$v(c - b_i) = \min\{v(\eta - c), v(\eta - b_i)\} = v(\eta - b_i) = \frac{vp}{p^t + 1}. \tag{67}$$
Since $c \in K$, we have that $c \in \mathbb{Q}_p(a_i)$ for some $i \in \mathbb{N}$. Then we obtain that $c - b_i \in \mathbb{Q}_p(a_i)$, but equation (67) shows that $v(c - b_i) = \frac{vp}{p^t + 1} \notin v\mathbb{Q}_p(a_i)$, a contradiction. Therefore, $\text{dist}(\eta, K) = 0^-.$

Since $(K(\eta)|K,v)$ is immediate, there is $d \in K$ such that $d\eta$ is a 1-unit. We have that $vd = -v\eta = \frac{vp}{p}$ and
$$\text{dist}(d\eta, K) = vd + 0^- = \frac{vp}{p} + 0^- < \frac{vp}{p-1} + 0^-.$$ As $K(\eta) = K(d\eta)$, this shows that $(K(\eta)|K,v)$ is a dependent defect extension of degree $p$. \qed

This second example shows that in order to obtain a semitame field it is not sufficient to just make the value group $p$-divisible and the residue field perfect, not even if one starts from a discretely valued field.

Since $\mathbb{Q}_p$ is a defectless field and a fortiori an independent defect field, but $(K,v)$ admits a Kummer extension with dependent defect, this example also shows:

**Corollary 7.3.** The property of being an independent defect field is not necessarily preserved under infinite algebraic extensions.
References

Institute of Mathematics, University of Silesia in Katowice, Bankowa 14, 40-007 Katowice, Poland

Email address: anna.rzepka@us.edu.pl