Abstract. We study the valuation theory of deeply ramified fields and introduce and investigate several other related classes of valued fields. Further, a classification of defect extensions of prime degree of valued fields that was earlier given only for the characteristic equal case is generalized to the case of mixed characteristic. It is shown that deeply ramified fields and the other valued fields we introduce only admit one of the two types of defect extensions, namely the ones that appear to be more harmless in open problems such as local uniformization and the model theory of valued fields in positive characteristic. The classes of valued fields under consideration can be seen as generalizations of the class of tame valued fields. The present paper supports the hope that it will be possible to generalize to deeply ramified fields several important results that have been proven for tame fields and were at the core of partial solutions of the two mentioned problems.

1. Introduction

This paper owes its existence to the following well known deep open problems in positive characteristic:

1) resolution of singularities in arbitrary dimension,

2) decidability of the field \( \mathbb{F}_q((t)) \) of Laurent series over a finite field \( \mathbb{F}_q \), and of its perfect hull.

Both problems are connected with the structure theory of valued function fields of positive characteristic \( p \). The main obstruction here is the phenomenon of the defect, which we will define now.

By \((L|K,v)\) we denote a field extension \( L|K \) where \( v \) is a valuation on \( L \) and \( K \) is endowed with the restriction of \( v \). The valuation ring of \( v \) on \( L \) will be denoted by \( \mathcal{O}_L \), and that on \( K \) by \( \mathcal{O}_K \). Similarly, \( \mathcal{M}_L \) and \( \mathcal{M}_K \) denote the valuation ideals of \( L \) and \( K \). The value group of the valued field \((L,v)\) will be denoted by \( vL \), and its residue field by \( Lv \). The value of an element \( a \) will be denoted by \( va \), and its residue by \( av \).

We will say that a valued field extension \((L|K,v)\) is a \textbf{uv-extension} if the extension of \( v \) from \( K \) to \( L \) is unique. If \((L|K,v)\) is a finite uv-extension, then

\[
[L : K] = \tilde{p}^\nu \cdot (vL : vK)[Lv : Kv],
\]

Date: 25. 11. 2018.

2010 Mathematics Subject Classification. 12J10, 12J25.

Key words and phrases. deeply ramified fields, semitame fields, tame fields, defect, higher ramification groups.

The second author was partially supported by a Polish Opus grant 2017/25/B/ST1/01815.
where by the Lemma of Ostrowski $\nu$ is a nonnegative integer and $\tilde{p}$ the characteristic exponent of $Kv$, that is, $\tilde{p} = \text{char } Kv$ if it is positive and $\tilde{p} = 1$ otherwise. The factor $d(L|K,v) = \tilde{p}^\nu$ is called the defect of the extension $(L|K,v)$. If $\tilde{p}^\nu > 1$, then $L|K$ is called a defect extension. If $d(L|K,v) = 1$, then we call $L|K$ a defectless extension. Nontrivial defect only appears when char $Kv = p > 0$, in which case $\tilde{p} = p$.

Throughout this paper, when we talk of a defect extension $(L|K,v)$ of prime degree, we will always tacitly assume that it is a uv-extension. Then it follows from (1) that $[L:K] = p = \text{char } Kv$ and that $(vL : vK) = 1 = [Lv : Kv]$; the latter means that $(L|K,v)$ is an immediate extension, i.e., the canonical embeddings $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lv$ are onto.

Via ramification theory, the study of defect extensions can be reduced to the study of purely inseparable extensions and of Galois extensions of degree $p = \text{char } Kv$. To this end, we fix an extension of $v$ from $K$ to its algebraic closure $\bar{K}$ of $K$. We denote the separable-algebraic closure of $K$ by $K^{\text{sep}}$. The absolute ramification field of $(K,v)$ (with respect to the chosen extension of $v$), denoted by $(K^r,v)$, is the ramification field of the normal extension $(K^{\text{sep}}|K,v)$. If $(K(a)|K,v)$ is a defect extension, then $(K^r(a)|K^r,v)$ is a defect extension with the same defect (see Proposition 2.13). On the other hand, $K^{\text{sep}}|K^r$ is a $p$-extension, so $K^r(a)|K^r$ is a tower of purely inseparable extensions and Galois extensions of degree $p$.

Galois defect extensions of degree $p$ of valued fields of characteristic $p > 0$ have been classified by the second author in [14]. Note that such an extension $(L|K,v)$ is an Artin-Schreier extension, that is, generated by an Artin-Schreier generator $\tilde{\varphi}$ which is the root of an Artin-Schreier polynomial $X^p - X - c$ with $c \in K$. The Artin-Schreier defect extension is called dependent if it can be obtained by a transformation from a purely inseparable extension, and independent otherwise. Note that for the transformation to render a dependent Artin-Schreier defect extension it is necessary to start from a purely inseparable defect extension that does not lie in the completion of $(K,v)$. The existence of such an extension implies that $(K,v)$ does not lie dense in its perfect hull, or equivalently, that its completion is not perfect.

The classification of defect extensions is important because work by M. Temkin (see e.g. [24]) and by the second author indicates that dependent defect appears to be more harmful to the above cited problems than independent defect. In the paper [4], S. D. Cutkosky and O. Piltant give an example of an extension of valued function fields consisting of a tower of two Artin-Schreier defect extensions where strong monomialization fails. As the valuation on this extension is defined by use of generating sequences, it is hard to determine whether the Artin-Schreier defect extensions are dependent or independent. However, work of Cutkosky, L. Ghezzi and S. ElHitti shows that both of them are dependent (see e.g. [5]); this again lends credibility to the hypothesis that dependent defect is the more harmful one.

An analogous classification of Galois defect extensions of degree $p$ of valued fields of characteristic 0 with residue fields of characteristic $p > 0$ (valued fields of mixed characteristic) has so far not been given. But such a classification is important for instance for the study of infinite algebraic extensions of the field $\mathbb{Q}_p$ of $p$-adic numbers, which in contrast to $\mathbb{Q}_p$ itself may well admit defect extensions. Indeed, $\mathbb{Q}_p^{ab}$, the maximal abelian extension of $\mathbb{Q}_p$, is such a field. Other examples will be
given in Section 7. Moreover, we wish to study the valuation theory of deeply ramified fields (such as $\mathbb{Q}_p^{ab}$), which will be introduced below, in full generality without restriction to the equal characteristic case. For these fields in particular it is important to work out the similarities between the equal and the mixed characteristic cases.

The obvious problem is that a field of characteristic 0 has no nontrivial inseparable extensions. However, in [14] the dependent and independent Artin-Schreier defect extensions have been characterized via the distance of their Artin-Schreier generators; for the definition of the distance see Section 2.1. In short, the extension is independent if and only if the distance is idempotent (see Sections 2.1 and 3.2 for details).

If in the mixed characteristic case the field $K$ contains a primitive $p$-th root of unity, then every Galois extension $L|K$ of degree $p$ is a Kummer extension, that is, generated by a Kummer generator $\eta$ which satisfies $\eta^p \in K$. On the other hand, it can also be generated by a root of a polynomial of the form $f(X) = X^p + g(X) - X - a$ with $g(X) \in \mathcal{M}_K[X]$. As this is, modulo $\mathcal{M}_K[X]$, equal to an Artin-Schreier polynomial, it suggests itself to say that $(L|K,v)$ is independent if a root of $f(X)$ has an idempotent distance (cf. Section 3.3). This definition enables us to prove that independent defect extensions in mixed characteristic have some of the same properties as independent defect extensions in equal characteristic, where $K$ and its residue field have the same (positive) characteristic. Moreover, for both cases we will generalize the classification to all defect extensions of degree $p$ by reducing the general case to the case of Galois extensions in Section 3.4.

That our definition of “independent” in mixed characteristic is the right one is supported by the following observation. Take a valued field of positive characteristic. If it lies dense in its perfect hull, then by what we have said before, all of its Artin-Schreier defect extensions must be independent. If the field is complete and of rank 1 (meaning that its value group can be seen as a subgroup of $\mathbb{R}$), then it is a perfectoid field. Such fields correspond via the so-called tilting construction to perfectoid field in mixed characteristic, and many important properties are preserved under the correspondence. So we expect that also perfectoid fields in mixed characteristic only admit independent defect extensions. This indeed holds with our definition.

For our purposes, the properties of completeness and rank 1 are irrelevant, and we prefer to work with a more flexible (and first order axiomatizable) notion. In fact, all perfectoid fields are deeply ramified, in the sense of [8]. Take a valued field $(K,v)$ with valuation ring $\mathcal{O}_K$. Choose any extension of $v$ to $K^{\text{sep}}$ and denote the valuation ring of $K^{\text{sep}}$ with respect to this extension by $\mathcal{O}_{K^{\text{sep}}}$. Then $(K,v)$ is a deeply ramified field if

\begin{equation}
\Omega_{\mathcal{O}_{K^{\text{sep}}}|\mathcal{O}_K} = 0,
\end{equation}

where $\Omega_{B|A}$ denotes the module of relative differentials when $A$ is a ring and $B$ is an $A$-algebra. This definition does not depend on the chosen extension of the valuation from $K$ to $K^{\text{sep}}$.

According to [8, Theorem 6.6.12 (vi)], a nontrivially valued field $(K,v)$ is deeply ramified if and only if the following conditions hold:
(DRvg) whenever $\Gamma_1 \subset \Gamma_2$ are convex subgroups of the value group $vK$, then $\Gamma_2/\Gamma_1$ is not isomorphic to $\mathbb{Z}$ (in other words, no archimedean component of $vK$ is discrete);

(DFvr) if $\text{char } Kv = p > 0$, then the homomorphism

$$O_K/pO_K \ni x \mapsto x^p \in O_K/pO_K$$

is surjective, where $O_K$ denotes the valuation ring of the completion of $(K,v)$.

Axiom (DRvr) means that modulo $pO_K$ every element in $O_K$ is a $p$-th power.

By altering axiom (DRvg) we will now introduce new classes of valued fields, one of them containing the class of deeply ramified fields, and one contained in it in the case of positive residue characteristic. Note that axiom (DRvg) means that no archimedean component of $vK$ is isomorphic to $\mathbb{Z}$. We will call $(K,v)$ a **generalized deeply ramified field**, or in short a **gdr field**, if it satisfies axiom (DRvr) together with:

(DFvp) if $\text{char } Kv = p > 0$, then $vp$ is not the smallest positive element in the value group $vK$.

If $\text{char } Kv = p > 0$, then (DRvg) certainly holds whenever $vK$ is divisible by $p$.

We will call $(K,v)$ a **semitame field** if it satisfies axiom (DRvr) together with:

(DFst) if $\text{char } Kv = p > 0$, then the value group $vK$ is $p$-divisible.

We note:

**Proposition 1.1.** The properties (DRvg), (DFvp) and (DFst) are first order axiomatizable in the language of valued fields, and so are the classes of semitame, deeply ramified and gdr fields of fixed characteristic.

We will give the proof at the end of Section 6.

The notion of “semitame field” is reminiscent of that of “tame field”. Let us recall the definition of “tame”. For the purpose of this paper we will slightly generalize the notion of “tame extension” as defined in [17] (there, tame extensions were only defined over henselian fields). An algebraic uv-extension $(L|K,v)$ will be called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) The ramification index $(vE : vK)$ is not divisible by $\text{char } Kv$.

(TE2) The residue field extension $E/vKv$ is separable.

(TE3) The extension $(E|K,v)$ is **defectless**, i.e., $[E : K] = (vE : vK)[Ev : Kv]$.

A henselian field $(K,v)$ is called a **tame field** if its algebraic closure with the unique extension of the valuation is a tame extension, and a **separably tame field** if its separable-algebraic closure is a tame extension. The absolute ramification field $(K^\ast,v)$ is the unique maximal tame extension of the henselian field $(K,v)$ by [6, Theorem (22.7)] (see also [21, Proposition 4.1]). Hence a henselian field is tame if and only if its absolute ramification field is already algebraically closed; in particular, every tame field is perfect.

In contrast to tame and separably tame fields, we do not require semitame fields to be henselian; in this way they become closer to deeply ramified fields. The other fundamental difference to tame fields is that semitame fields may admit defect extensions, but as we will see in Theorem 1.5 below, only those with independent defect. This justifies the hope that many of the results that have been proved for tame fields and applied to the problems we have cited in the beginning (see [17, 18]) can be generalized to the case of (henselian) semitame fields.
All valued fields of residue characteristic 0 are semitame and gdr fields, and they
are deeply ramified fields if and only if (DRvg) holds. Likewise, all henselian valued
fields of residue characteristic 0 are tame fields. In the present paper, we are not
interested in the case of residue characteristic 0, so we will always assume that
\( \text{char } K_v = p > 0 \). We will now summarize the basic facts about the connections
between the properties we have introduced. The proofs will be provided in Section 6.

**Theorem 1.2.** 1) If \((K,v)\) is a nontrivially valued field with \( \text{char } K_v = p > 0 \),
then the following logical relations between its properties hold:

- tame field \( \Rightarrow \) separably tame field \( \Rightarrow \) semitame field \( \Rightarrow \)
  deeply ramified field \( \Rightarrow \) gdr field.

2) For a valued field \((K,v)\) of rank 1 with \( \text{char } K_v = p > 0 \), the three properties
“semitame field”, “deeply ramified field” and “gdr field” are equivalent.

3) For a nontrivially valued field \((K,v)\) of characteristic \( p > 0 \), the following prop-
erties are equivalent:
   a) \((K,v)\) is a semitame field,
   b) \((K,v)\) is a deeply ramified field,
   c) \((K,v)\) is a gdr field,
   d) \((K,v)\) satisfies (DRvrg),
   e) the completion of \((K,v)\) is perfect,
   f) \((K,v)\) is dense in its perfect hull,
   g) \((K_p,v)\) is dense in \((K,v)\).

4) Every perfect valued field of positive characteristic is a semitame field.

Take a valued field \((K,v)\) of characteristic 0 with residue characteristic \( p > 0 \).
Decompose \( v = v_0 \circ v_p \circ \tau \), where \( v_0 \) is the finest coarsening of \( v \) that has residue
characteristic 0, \( v_p \) is a rank 1 valuation on \( K_{v_0} \), and \( \tau \) is the valuation induced
by \( v \) on the residue field of \( v_p \) (which is of characteristic \( p > 0 \)). The valuations \( v_0 \)
and \( \tau \) may be trivial. With this notation, we have:

**Proposition 1.3.** The valued field \((K,v)\) is a gdr field if and only if \((K_{v_0},v_p)\) is.

Note that by part 2) of Theorem 1.2, \((K_{v_0},v_p)\) is already a semitame field once it
is a gdr field.

The next theorem will show that we can reduce the study of several questions
about semitame fields to considering their absolute ramification field.

**Theorem 1.4.** Take a valued field \((K,v)\), fix any extension of \( v \) to \( \tilde{K} \), and let
\((K',v)\) be the respective absolute ramification field of \((K,v)\). Then \((K',v)\) is a gdr
field if and only if \((K,v)\) is, and \((K',v)\) is a semitame field if and only if \((K,v)\) is.

Note that even without the assumptions of the theorem, if \((K',v)\) is a gdr field,
then it is already a deeply ramified field because \( vK' \) is divisible by every prime
distinct from the residue characteristic. Hence if \((K,v)\) is a gdr field, then \((K',v)\)
is a deeply ramified field. The converse is not true in general, since (DRvg) always
holds in \((K',v)\) (as long as \( v \) is nontrivial), while it may not hold in \((K,v)\).

The next theorem addresses the connection of the properties we have defined
with the classification of the defect. We denote by \((vK)_v\) the smallest convex
subgroup of \( vK \) that contains \( v \) if \( \text{char } K = 0 \), and set \((vK)_v = vK \) otherwise.

We call a valued field an **independent defect field** if all of its separable defect
extensions of degree \( p = \text{char } K_v \) have independent defect.
Theorem 1.5. 1) Take a valued field \((K, v)\) with \(\text{char } Kv = p > 0\). Then \((K, v)\) is a gdr field if and only if \((vK)_p\) is \(p\)-divisible, \(Kv\) is perfect, and \((K, v)\) is an independent defect field.

2) A nontrivially valued field \((K, v)\) is semitame if and only if every separable \(uv\)-extension of prime degree is either tame or an independent defect extension.

The classification of Artin-Schreier defect extensions is also an important tool in the proof of Theorem 1.2 in [14], which we will state now. A valued field is called algebraically maximal (or separable-algebraically maximal) if it admits no nontrivial immediate algebraic (or separable-algebraic, respectively) extensions.

Theorem 1.6. A valued field of positive characteristic is a henselian and defectless field if and only if it is separable-algebraically maximal and each finite purely inseparable extension is defectless.

This theorem in turn is used in [13] for the construction of an example showing that a certain natural axiom system for the elementary theory of \(\mathbb{F}_p((t))\) (“henselian defectless valued field of characteristic \(p\) with residue field \(\mathbb{F}_p\) and value group a \(\mathbb{Z}\)-group”) is not complete.

A full analogue of Theorem 1.6 in mixed characteristic is not presently known. But we are able to show in Section 4 that the independent defect extensions in mixed characteristic have the same properties as the ones in equal characteristic that have been used in [14] for the proof of Theorem 1.6. As a consequence, we are able to prove:

Theorem 1.7. Every algebraically maximal gdr field is a perfect, henselian and defectless field.

For the proof of this theorem we will need the following result which for the case of deeply ramified fields can be found in [8, Corollary 6.6.16 (i)]:

Theorem 1.8. Every algebraic extension of a deeply ramified field is again deeply ramified. The same holds for semitame fields and for gdr fields.

We will give the easy proof for the equal characteristic case in Section 6. For the mixed case we hope that eventually a direct valuation theoretical proof can be found. In view of Theorem 1.4 it suffices to prove that if \((K^r, v)\) is a gdr field and \((L|K^r, v)\) is an independent defect extension of degree \(p\), then also \((L, v)\) is a gdr field. Understanding this implication without referring to the methods used in [8] would be important for the study of the more general class of independent defect fields. At this point, we are able to prove:

Proposition 1.9. 1) If \((K^r, v)\) is an independent defect field, then so is \((K, v)\).

2) Take a valued field \((K, v)\) of equal positive characteristic. If \((K, v)\) is an independent defect field, then every immediate purely inseparable extension of \((K, v)\) lies in its completion.

Conjectures: 1) If \((K, v)\) is an independent defect field, then also \((K^r, v)\) is an independent defect field.

2) If \((K, v)\) is a valued field of equal positive characteristic such that every immediate purely inseparable extension of \((K, v)\) lies in its completion, then \((K, v)\) is an independent defect field.
3) A valued field \((K, v)\) of mixed characteristic with residue characteristic \(p\) is an independent defect field if and only if for every \(a \in \mathcal{O}_K\) for which the set \(\{v(a - cp) \mid c \in K\}\) has no maximal element there is some \(c \in K\) such that \(v(a - cp) \geq vp\).

Open problem: Which of the results in Theorems 1.2, 1.4, 1.7, and 1.8 can be generalized to independent defect fields?

Continuing the work presented in [4], the idea is presently investigated to employ higher ramification groups for the study of the ramification theory of 2-dimensional valued function fields. When working over valued fields with arbitrary value groups, the classical ramification numbers have to be replaced by **ramification jumps** which can be understood as cuts in the value group (cf. Section 2.4).

While we are dealing with defect extensions of prime degree, we will compute in Section 3.1 the ramification jumps of the higher ramification groups for such extensions. Theorem 3.5 shows that, given a generator \(z\) of the extension, they can easily be computed from the set

\[
v(z - K) := \{v(z - c) \mid c \in K\}.
\]

In the present case where we consider a defect extension of prime degree, which consequently is immediate, this set is an initial segment of the value group. The distance we mentioned earlier is defined as the cut induced by the convex hull of this initial segment in the divisible hull of the value group. Because of this connection, the type of the defect can in fact be read off from the ramification jump of the extension. If the extension is Galois, then Theorems 3.9 and 3.11 show that the ramification jumps are just the cuts in the value group with upper cut set \(-v(z - K)\) when \(z\) is a suitable generator, namely, an Artin-Schreier generator in the case of an Artin-Schreier extension, and an element derived from the Kummer generator in a canonical way (see (21)) in the case of a Kummer extension.

Moreover, for Galois defect extensions \((K(a)|K, v)\) of prime degree we will compute in Section 5 the image of \(M_L\) under the trace, see Theorem 5.2. This allows us to characterize the independent defect extensions in yet another way:

**Theorem 1.10.** Take a defect extension \((K(a)|K, v)\) of prime degree \(p\). In the mixed characteristic case, assume that \(K\) contains an element \(C\) such that \(C^p = -pC\) (see (12) below). Then \((K(a)|K, v)\) has independent defect if and only if for some proper convex subgroup \(H\) of \(vK\),

\[
(4) \quad \text{Tr}_{K(a)|K} (M_{K(a)}) = \{d \in K \mid vd > \alpha \text{ for all } \alpha \in H\} = M_{vH},
\]

where \(M_{vH}\) is the valuation ring of the coarsening \(v_H\) of \(v\) whose value group is \(vK/(H \cap vK)\). In particular, if \(H = \{0\}\), this means that

\[
\text{Tr}_{K(a)|K} (M_{K(a)}) = M_K.
\]

In the mixed characteristic case, \(M_{vH}\) will always contain \(p\), so that \(\text{char } Kv_H = p\).
2. Preliminaries

2.1. Cuts, distances and defect. We recall basic notions and facts connected with cuts in ordered abelian groups and distances of elements of valued field extensions. For the details and proofs see Section 2.3 of [14] and Section 3 of [22].

Take a totally ordered set \((T, \prec)\). For a nonempty subset \(S\) of \(T\) and an element \(t \in T\) we will write \(S < t\) if \(s < t\) for every \(s \in S\). A set \(S \subseteq T\) is called an initial segment of \(T\) if for each \(s \in S\) every \(t < s\) also lies in \(S\). Similarly, \(S \subseteq T\) is called a final segment of \(T\) if for each \(s \in S\) every \(t > s\) also lies in \(S\). A pair \((\Lambda^L, \Lambda^R)\) of subsets of \(T\) is called a cut in \(T\) if \(\Lambda^L\) is an initial segment of \(T\) and \(\Lambda^R = T \setminus \Lambda^L\); it then follows that \(\Lambda^R\) is a final segment of \(T\). To compare cuts in \((T, \prec)\) we will use the lower cut sets comparison. That is, for two cuts \(\Lambda_1 = (\Lambda^L_1, \Lambda^R_1)\), \(\Lambda_2 = (\Lambda^L_2, \Lambda^R_2)\) in \(T\) we will write \(\Lambda_1 \vartriangleleft \Lambda_2\) if \(\Lambda^L_1 \subseteq \Lambda^L_2\), and \(\Lambda_1 \triangleleft \Lambda_2\) if \(\Lambda^L_1 \subseteq \Lambda^L_2\).

For any \(s \in T\) define the following principal cuts:
\[
\begin{align*}
\Lambda^+ &:= \{\{t \in T \mid t < s\}, \{t \in T \mid t \geq s\}\}, \\
\Lambda^- &:= \{\{t \in T \mid t \leq s\}, \{t \in T \mid t > s\}\}.
\end{align*}
\]
We identify the element \(s\) with \(s^+\). Therefore, for a cut \(\Lambda = (\Lambda^L, \Lambda^R)\) in \(T\) and an element \(s \in T\) the inequality \(\Lambda < s\) means that for every element \(t \in \Lambda^L\) we have \(t < s\). Similarly, for any subset \(M\) of \(T\) we define \(M^+\) to be a cut \((\Lambda^L, \Lambda^R)\) in \(T\) such that \(\Lambda^L\) is the least initial segment containing \(M\), that is, \(M^+ = \{\{t \in T \mid \exists m \in M \ t \leq m\}, \{t \in T \mid t > M\}\}\).

Likewise, we denote by \(M^-\) the cut \((\Lambda^L, \Lambda^R)\) in \(T\) such that \(\Lambda^L\) is the largest initial segment disjoint from \(M\), i.e.,
\[
M^- = \{\{t \in T \mid t < M\}, \{t \in T \mid \exists m \in M \ t \geq m\}\}.
\]

For every extension \((L|K, v)\) of valued fields and \(z \in L\) define
\[
v(z - K) := \{v(z - c) \mid c \in K\}.
\]
The set \(v(z - K) \cap vK\) is an initial segment of \(vK\) and thus the lower cut set of a cut in \(vK\). However, it is more convenient to work with the cut
\[
dist(z, K) := (v(z - K) \cap vK)^+ = \text{the divisible hull } vK \text{ of } vK.
\]
We call this cut the distance of \(z\) from \(K\). The lower cut set of \(\dist(z, K)\) is the smallest initial segment of \(vK\) containing \(v(z - K) \cap vK\). If \((F|K, v)\) is an algebraic subextension of \((L|K, v)\) then \(vF = vK\). Thus \(\dist(z, K)\) and \(\dist(z, F)\) are cuts in the same group and we can compare these cuts by set inclusion of the lower cut sets. Since \(v(z - K) \subseteq v(z - F)\) we deduce that
\[
\dist(z, K) \leq \dist(z, F) \text{.}
\]
If \(\text{char } K = p > 0\) and \(z \in K\), then \(K^p\) is a subfield of \(K\), and the expressions
\[
v(z - K^p) \text{ and } \dist(z, K^p)
\]
are covered by our above definitions. We generalize this to the case where \(\text{char } K = 0\) with the same definitions but note that \(v(z - K^p) \cap vK\) is not necessarily an initial segment of \(vK\).

If \(S\) is any subset of an abelian group, then for every \(n \in \mathbb{Z}\) we set
\[
nS := \{ns \mid s \in S\}.
\]
in particular, \(-S = \{-s \mid s \in S\}\). If \(\Lambda = (\Lambda^L, \Lambda^R)\) is a cut in a divisible ordered abelian group \(\Gamma\) and \(n > 0\), then \(n\Lambda^L\) is again an initial segment of \(\Gamma\); we denote by \(n\Lambda\) the cut in \(\Gamma\) with the lower cut set \(n\Lambda^L\). Further, we define \(-\Lambda\) to be the cut \((-\Lambda^R, -\Lambda^L)\).

We say that the distance \(\text{dist}(z, K)\) is idempotent if
\[
n \cdot \text{dist}(z, K) = \text{dist}(z, K)
\]
for some natural number \(n \geq 2\) (and hence for all \(n \in \mathbb{N}\)). The following characterization of idempotent distances is a consequence of [14, Lemma 2.14]:

**Lemma 2.1.** The distance \(\text{dist}(z, K)\) is idempotent if and only if it is equal to \(H^-\) or \(H^+\) for some convex subgroup \(H\) of \(\tilde{v}K\).

If \(y\) is another element of \(L\) then we define \(z \sim_K y\) to mean that
\[
v(z - y) > \text{dist}(z, K).
\]
If this holds, then \(v(z - c) = v(y - c)\) for all \(c \in K\) such that \(v(z - c) \in vK\) and thus, \(\text{dist}(z, K) = \text{dist}(y, K)\). The next lemma was proven in [14]. It shows that the converse holds under an additional assumption.

**Lemma 2.2.** Take a valued field extension \((L|K, v)\) and elements \(z, y \in L\). If \(v(z - K) \cap vK\) has no maximal element, then \(z \sim_K y\) if and only if \(v(z - c) = v(y - c)\) for every \(c \in K\) such that \(v(z - c) \in K\).

For any \(\alpha \in vK\) and each cut \(\Lambda\) in \(vK\) we set \(\alpha + \Lambda := (\alpha + \Lambda^L, \alpha + \Lambda^R)\). An immediate consequence of the above definitions is the following lemma:

**Lemma 2.3.** Take an extension \((L|K, v)\) of valued fields. Then for every element \(c \in K\) and \(y, z \in L\),
\[
a) \text{dist}(z + c, K) = \text{dist}(z, K), \\
b) \text{dist}(cz, K) = vc + \text{dist}(z, K), \\
c) \text{if } z \sim_K y, \text{ then } z + c \sim_K y + c, \\
d) \text{if } c \neq 0 \text{ and } z \sim_K y, \text{ then } cz \sim_K cy.
\]

The next two facts are important properties of distances of elements in valued field extensions. For the proof of the next lemma see [2, Lemma 7] and [14, Lemma 2.5].

**Lemma 2.4.** Take an arbitrary immediate extension \((F|K, v)\) and a finite defectless \(w\)-extension \((L|K, v)\). Then the extension of \(v\) from \(F\) to \(F.L\) is unique, \((F.L|F, v)\) is defectless, \((F.L|L, v)\) is immediate, and for every \(a \in F \setminus K\) we have that
\[
\text{dist}(a, K) = \text{dist}(a, L).
\]
Moreover,
\[
[F.L : F] = [L : K],
\]
i.e., \(F|K\) and \(L|K\) are linearly disjoint.

For the proof of the following results see [2, Lemmas 5 and 9].
Lemma 2.5. Take a uv-extension \((F|K,v)\) and an extension of \(v\) to the algebraic closure of \(F\). Take \(K^h\) to be the henselization of \(K\) with respect to this fixed extension of \(v\). Then for every \(a \in F\) we have that \([K(a) : K] = [K^h(a) : K^h]\) as well as
\[
d(K(a)|K,v) = d(K^h(a)|K^h,v) \quad \text{and} \quad \text{dist}(a,K) = \text{dist}(a,K^h).
\]

A valued field \((K,v)\) is said to be **separably defectless** if every finite separable extension is defectless, and **inseparably defectless** if every finite purely inseparable extension is defectless. The following is Lemma 4.15 of [14].

Lemma 2.6. Every finite extension of an inseparably defectless field is again an inseparably defectless field.

For the proof of the next proposition, see [14], Proposition 2.8.

Proposition 2.7. Take a henselian field \((K,v)\) and a tame extension \(N\) of \(K\).
Then for any finite extension \(L|K\),
\[
d(L|K,v) = d(L.N|N,v).
\]
In particular, \((K,v)\) is defectless (separably defectless, inseparably defectless) if and only \((K^f,v)\) is defectless (separably defectless, inseparably defectless).

For the following theorem, see [9, Theorem 1] and [14, Theorem 2.19]).

Theorem 2.8. If \((L|K,v)\) is an immediate extension of valued fields, then for every element \(a \in L \setminus K\) we have that \(v(a − K) \subseteq vK\) and that \(v(a − K)\) has no maximal element. In particular, \(va < \text{dist}(a,K)\).

The following partial converse of this theorem also holds (cf. [14, Lemma 2.21]):

Lemma 2.9. Assume that \((K(a)|K,v)\) is a uv-extension of prime degree such that \(v(a − K)\) has no maximal element. Then the extension \((K(a)|K,v)\) is immediate and hence a defect extension.

The property that the set \(v(a − K)\) has no maximal element does not in general imply that \((K(a)|K,v)\) is immediate. However, the next lemma (cf. e.g. [22, Lemma 2.1]) shows that if in addition \((K,v)\) is henselian and \(a\) is algebraic over \(K\), then \((K(a)|K,v)\) is a defect extension.

Lemma 2.10. If \((L|K,v)\) is a defectless uv-extension, then for every element \(a \in K\) the set \(v(a − K)\) admits a maximal element.

The next lemma follows from [9, Lemma 8] and [22, Lemma 5.2]. We use the Taylor expansion
\[
f(X) = \sum_{i=0}^n f_i(c)(X-c)^i
\]
where \(f_i\) denotes the \(i\)-th formal derivative (also called Hasse-Schmidt derivative) of \(f\).

Lemma 2.11. Take a nontrivial extension \((K(a)|K,v)\) of degree \(p^k\). Assume that \(v(a − K)\) has no maximal element and in addition, for every polynomial \(g \in K[X]\) of degree \(< [K(a) : K]\) there is \(a \in v(a − K)\) such that for all \(c \in K\) with \(v(a − c) \geq a\), the value \(v_g(c)\) is fixed. Then for every nonconstant polynomial \(f \in K[X]\) of degree
\[ < p^k \text{ there are } \gamma \in v(a - K) \text{ and } h = p^\ell \text{ with } 0 \leq \ell < k \text{ such that for all } c \in K \text{ with } v(a - c) \geq \gamma \text{ and all } i \text{ with } 1 \leq i \leq \deg f, \text{ we have:} \]
\[ \text{the value } v f_i(c) \text{ is fixed, equal to } f_i(a), \]
\[ (6) \quad v f_h(c) + h \cdot v(x - c) < v f_i(c) + i \cdot v(x - c) \]
whenever \( i \neq h \),
\[ (7) \quad v(f(a) - f(c)) = v f_h(c) + h \cdot v(a - c), \]
and
\[ (8) \quad \text{dist } (f(a), K) = v f_h(c) + h \cdot \text{dist } (a, K). \]

The following is Lemma 2.4 of [14].

**Lemma 2.12.** Take a valued field \((K, v)\), a finite extension \((L|K, v)\) and a coarsening \(w\) of \(v\) on \(L\). If \((K, v)\) is henselian, then so is \((K, w)\). If \((L|K, v)\) is defectless, then so is \((L|K, w)\).

**2.2. The absolute ramification field.**

**Proposition 2.13.** Take an immediate uv-extension \((K(a)|K, v)\). Extend \(v\) to the algebraic closure of \(K\) and let \((K^h, v)\) be the henselization and \((K^r, v)\) the absolute ramification field of \((K, v)\) with respect to this extension. Then \((K^r(a)|K^r, v)\) is an immediate extension with
\[ (9) \quad [K^r(a) : K^r] = [K^h(a) : K^h] = [K(a) : K], \]
\[ (10) \quad d(K^r(a)|K^r, v) = d(K^h(a)|K^h, v) = d(K(a)|K, v), \]
\[ (11) \quad \text{dist } (a, K^r) = \text{dist } (a, K^h) = \text{dist } (a, K). \]

**Proof.** Since \((K(a)|K, v)\) is a uv-extension, we know from Lemma 2.5 that \([K^h(a) : K^h] = [K(a) : K]\) as well as \(d(K^h(a)|K^h, v) = d(K(a)|K, v)\) and \(\text{dist } (a, K^h) = \text{dist } (a, K)\). Since \((K(a)|K, v)\) is an immediate uv-extension by assumption,
\[ [K^h(a) : K^h] = [K(a) : K] = d(K(a)|K, v) = d(K^h(a)|K^h, v), \]
showing that also \([K^h(a)|K^h, v]\) is immediate.

Further, \((K^r)|K^h, v)\) is a tame and hence defectless extension. Thus by Proposition 2.4, \((K^r(a)|K^r, v)\) is immediate with \([K^r(a) : K^r] = [K^h(a) : K^h]\) and \(\text{dist } (a, K^r) = \text{dist } (a, K^h)\). By Proposition 2.7, \(d(K^r(a)|K^r, v) = d(K^h(a)|K^h, v)\).

For the proof of the following results, see Lemma 2.9 of [14].

**Lemma 2.14.** Take any valued field \((K, v)\) and let \(K^h\) and \(K^r\) be its henselization and its absolute ramification field with respect to any extension of \(v\) to the algebraic closure of \(K\). If \(\text{char } K^h = 0\), then \(K^r\) is algebraically closed. If \(\text{char } K^h = p > 0\), then every finite extension of \(K^r\) is a tower of normal extensions of degree \(p\).

Further, if \(L|K\) is a finite extension, then there is already a finite tame extension \(N\) of \(K^h\) such that \(L.N|N\) is such a tower.

The proof of this lemma uses the fact that if \(\text{char } K^h = p > 0\), then \(K^{\text{sep}}|K^r\) is a \(p\)-extension. From this we can also conclude that \(K^r\) contains all \(p\)-th roots of unity. The following is Lemma 14 of [16].
Lemma 2.15. A henselian field of characteristic 0 and residue characteristic $p > 0$ contains an element $C$ such that $C^{p-1} = -p$ if and only if it contains a primitive $p$-th root $\zeta_p$ of unity.

We therefore know that in the case of mixed characteristic, the henselian field $K^r$ contains such an element $C$. It satisfies:

\begin{equation}
C^p = -pC \quad \text{and} \quad vC = \frac{vp}{p-1}.
\end{equation}

Further, it is well known that

\begin{equation}
v(\zeta_p - 1) = \frac{vp}{p-1}
\end{equation}

(see e.g. the proof of Lemma 14 of [16]).

2.3. 1-units and $p$-th roots in valued fields of mixed characteristic. Throughout this section, $(K, v)$ will be a valued field of characteristic zero and residue characteristic $p > 0$, with valuation ring $O$ and valuation ideal $M$. Throughout this section we assume that $v$ is extended to the algebraic closure $\bar{K}$ of $K$.

A 1-unit in $(K, v)$ is an element of the form $u = 1 + b$ with $b \in M$; in other words, $u$ is a unit in $O$ with residue 1. We will call the value $v(u - 1)$ the level of the 1-unit $u$.

Lemma 2.16. 1) If $b_1, \ldots, b_n \in O$, then

\[(b_1 + \ldots + b_n)^p \equiv b_1^p + \ldots + b_n^p \mod pO.\]

2) Take elements $b_1, \ldots, b_n \in K$ of values $\geq -\frac{vp}{p}$. Then

\[(b_1 + \cdots + b_n)^p \equiv b_1^p + \cdots + b_n^p \mod O_K.\]

3) Take $\eta \in \bar{K}$ such that $\eta^p = a \in O_K$. Then for every $c \in K$ such that $v(\eta - c) \geq \frac{vp}{p}$ we have that $a \equiv c^p \mod pO$.

Proof. 1): We have:

\begin{equation}
(b_1 + b_2)^p = b_1^p + \sum_{i=1}^{p-1} \binom{p}{i} b_1^{p-i} b_2^i + b_2^p.
\end{equation}

Since the binomial coefficients under the sum are all divisible by $p$ and since $b_1, b_2 \in O$, all summands on the right hand side for $1 \leq i \leq p-1$ lie in $pO$, which proves our assertion in the case of $n = 2$. The general case follows by induction on $n$.

2): If $vb_1 \geq -\frac{vp}{p}$ and $vb_2 \geq -\frac{vp}{p}$, then $vb_1^{p-i} b_2^i \geq -vp$ for $1 \leq i \leq p-1$, so all summands in the sum on the right hand side of (14) have non-negative value. As for part 1), the assertion now follows by induction on $n$.

3): For $c \in K$ with $v(\eta - c) > 0$ we have that $vc \geq 0$ and, by part 1):

\begin{equation}
(\eta - c)^p \equiv \eta^p - c^p \equiv a - c^p \mod pO_{K(\eta)}.
\end{equation}

If $v(\eta - c) \geq \frac{vp}{p}$, then $v(\eta - c)^p \geq vp$, i.e., $a - c^p \equiv (\eta - c)^p \equiv 0 \mod pO_{K(\eta)}$. \hfill $\square$

Lemma 2.17. Take $\eta \in \bar{K}$ such that $\eta^p \in K$ and $v\eta = 0$. Then for $c \in K$ such that $v(\eta - c) > 0$, $v(\eta - c) < \frac{1}{p-1} vp$ holds if and only if $v(\eta^p - c^p) < \frac{p}{p-1}vp$, and if this is the case, then $v(\eta^p - c^p) = p\frac{v(\eta - c)}{p-1}$. If $v(\eta - c) > \frac{1}{p-1} vp$, then $v(\eta^p - c^p) = vp + v(\eta - c)$. 

\[\]
Proof. Take any $c \in K$ such that $0 < v(\eta - c)$. Then $vc = v\eta = \frac{vp}{p} = 0$. We have that
\[
\eta^p = (\eta - c + c)^p = (\eta - c)^p + \sum_{i=1}^{p-1} \binom{p}{i} (\eta - c)^i c^{p-i} + c^p.
\]
Since $vc = 0$ and the binomial coefficients under the sum all have value $vp$, the unique summand with the smallest value is $p(\eta - c)c^{p-1}$. Therefore,
\[(16) \quad v(\eta^p - c^p) \geq \min\{v(\eta - c)^p, vp(\eta - c)\} = \min\{pv(\eta - c), vp + v(\eta - c)\},
\]
with equality holding if $pv(\eta - c) \neq vp + v(\eta - c)$. We observe that
\[(17) \quad v(\eta - c) < \frac{1}{p-1}vp \iff pv(\eta - c) < vp + v(\eta - c),
\]
and the same holds for “$$\geq$$” in place of “$$<$$”. Assume that $v(\eta - c) < \frac{1}{p-1}vp$. Then by (17) and (16),
\[
v(\eta^p - c^p) = pv(\eta - c) < \frac{p}{p-1}vp.
\]
Now assume that $v(\eta - c) \geq \frac{1}{p-1}vp$. Then by (17), $pv(\eta - c) \geq vp + v(\eta - c)$, and (16) yields that
\[
v(\eta^p - c^p) \geq vp + v(\eta - c) \geq vp + \frac{1}{p-1}vp = \frac{p}{p-1}vp.
\]
Finally, if $v(\eta - c) > \frac{1}{p-1}vp$, then from (16) and the subsequent remark we conclude that
\[
v(\eta^p - c^p) = vp + v(\eta - c).
\]
\[\square\]

Taking $\eta$ to be a 1-unit $u$, we obtain:

**Corollary 2.18.** Assume that $u$ is a 1-unit. Then the level of $u$ is smaller than $\frac{1}{p-1}vp$ if and only if the level of $u^p$ is smaller than $\frac{p}{p-1}vp$, and if this is the case, then $v(u^p - 1) = pv(u-1)$.

**Lemma 2.19.** Take $\eta \in \tilde{K}$ such that $\eta^p \in K$. If there is some $c \in K$ such that
\[(18) \quad v(\eta - c) > v\eta + \frac{1}{p-1}vp,
\]
then $\eta$ lies in the henselization of $(K,v)$ within $(\tilde{K},v)$.

*Proof.* If $\eta \in K$, then there is nothing to show, so let us assume that $\eta \notin K$. Every root of $X^p - \eta^p$ is of the form $\eta\zeta_p^i$ with $0 \leq i \leq p-1$. For $0 \leq i \neq j \leq p-1$ we have that
\[
v(\eta\zeta_p^i - \eta\zeta_p^j) = v\eta + jv\zeta_p + v(\zeta_p^{i-j} - 1) = v\eta + \frac{1}{p-1}vp,
\]
where the last equality holds since $v\zeta_p = 0$ and $v(\zeta - 1) = \frac{1}{p-1}vp$ for every primitive $p$th root of unity $\zeta$. Hence if (18) holds, then it follows from Krasner’s Lemma that $\eta \in K(c)^h = K^h$, where $K^h$ denotes the henselization of $(K,v)$ within $(\tilde{K},v)$. \[\square\]
The following construction will play an important role in Section 3.3. Take a 1-unit \( \eta \in \bar{K} \) such that \( \eta^p \in K \). Then also \( \eta^p \) is a 1-unit. Assume that \( K \) contains an element \( C \) as in Lemma 2.15. Consider the substitution \( X = CY + 1 \) for the polynomial \( X^p - \eta^p \). We then obtain the polynomial \( (CY + 1)^p - \eta^p \). Dividing this polynomial by \( C^p \) and using the fact that \( C^p = -pC \), we obtain the polynomial

\[
(19) \quad f_\eta(Y) = Y^p + g(Y) - Y - \frac{\eta^p - 1}{C^p},
\]

where

\[
(20) \quad g(Y) = \sum_{i=2}^{p-1} \binom{p}{i} C^{i-p} Y^i.
\]

Note that \( g(Y) \in \mathcal{M}_K[Y] \) since \( C \in K \) and \( vC = \frac{vp}{p-1} \). We see that an element \( \tilde{\eta} \) is a root of \( X^p - \eta^p \) if and only if the element \( \tilde{\eta} - 1 \) is a root of \( f_\eta \). Thus the roots of \( f_\eta \) are of the form \( \zeta^i \eta - 1 \) with \( 0 \leq i \leq p - 1 \).

Set

\[
(21) \quad \vartheta_\eta := \frac{\eta - 1}{C}.
\]

Then \( K(\eta) = K(\vartheta_\eta) \), with \( f_\eta \) the minimal polynomial of \( \vartheta_\eta \) over \( K \).

**Lemma 2.20.** In a henselian field \((K,v)\) of mixed characteristic with residue characteristic \( p \) which contains a primitive \( p \)-th root of unity, every 1-unit of level greater than \( \frac{p}{p-1} vp \) is a \( p \)-th power.

**Proof.** By Lemma 2.15, \( K \) contains an element \( C \) as in that lemma. Take a 1-unit \( u \in K \) of level greater than \( \frac{p}{p-1} vp \). Apply the above transformation to the polynomial \( X^p - u \) with \( \eta^p = u \). By our assumption on \( u \) we have that \( \frac{\eta^p - 1}{C^p} \in \mathcal{M}_K \). Hence \( f_\eta(Y) \) is equivalent modulo \( \mathcal{M}_K[Y] \) to \( Y^p - Y \), which splits in the henselian field \( K \). Therefore, \( \eta \in K \). \( \square \)

### 2.4. Higher ramification groups and traces

Take a henselian field \((K,v)\). Assume that \( L/K \) is a Galois extension, and let \( G = \text{Gal}(L|K) \) denote its Galois group. For ideals \( I \) of \( \mathcal{O}_L \) we consider the (upper series of) **higher ramification groups**

\[
(22) \quad G_I := \left\{ \sigma \in G \mid \frac{\sigma f - f}{f} \in I \text{ for all } f \in L^\times \right\}
\]

(see [26], §12). For every ideal \( I \) of \( \mathcal{O}_L \), \( G_I \) is a normal subgroup of \( G \) ([26] (d) on p.79). The function

\[
(23) \quad \varphi : I \mapsto G_I
\]

preserves \( \subseteq \), that is, if \( I \subseteq J \), then \( G_I \subseteq G_J \). As \( \mathcal{O}_L \) is a valuation ring, the set of its ideals is linearly ordered by inclusion. This shows that also the higher ramification groups are linearly ordered by inclusion. Note that in general, \( \varphi \) will neither be injective, nor surjective.

We define functions from the set of all subgroups of \( G \) to the set of all ideals of \( \mathcal{O}_L \) in the following way. Given a subgroup \( H \subseteq G \), we define

\[
(24) \quad I_{-}(H) := \bigcap_{H \subseteq G_I} I \quad \text{and} \quad I_{+}(H) := \bigcup_{G_I \subseteq H} I.
\]
Note that $H' \subseteq H''$ implies that $I_-(H') \subseteq I_-(H'')$ and $I_+(H') \subseteq I_+(H'').$

For every ideal $I \subseteq \mathcal{O}_L$, $I_-(G_I)$ is the smallest ideal $J \subseteq I$ such that $G_J = G_I$. Any ideal of the form $I_-(G_I)$ will be called a **ramification ideal**. But note that in general, $I_+(G_I)$ may not be the largest ideal $J \supseteq I$ such that $G_J = G_I$.

The function

$$v : I \mapsto \Sigma_I := \{ vf \mid 0 \neq f \in I \}$$

is an order preserving bijection from the set of all ideals of $\mathcal{O}_L$ onto the set of all final segments of the value group $vL$ (contained in its nonnegative part $(vL)^{\geq 0}$). The set of final segments of $T$ is again linearly ordered by inclusion. The inverse of the above function is the order preserving function

$$\Sigma \mapsto I_\Sigma := \{ a \in L \mid va \in \Sigma \} \cup \{0\}.$$

We will write $G_\Sigma := G_{I_\Sigma} = \left\{ \sigma \in G \mid v \frac{\sigma f - f}{f} \in \Sigma \cup \{\infty\} \text{ for all } f \in L^\times \right\}.$

Given any subgroup $H$ of $G$, we define

$$\Sigma_-(H) := \bigcap_{H \subseteq G_\Sigma} \Sigma \quad \text{and} \quad \Sigma_+(H) := \bigcup_{G_\Sigma \subseteq H} \Sigma.$$  

As intersections and unions of final segments, $\Sigma_-(H)$ and $\Sigma_+(H)$ are themselves final segments. Note that $H' \subseteq H''$ implies that $\Sigma_-(H') \subseteq \Sigma_-(H'')$ and $\Sigma_+(H') \subseteq \Sigma_+(H'')$. Further, we observe that for every ramification group $G'$,

$$\Sigma_-(G') \subseteq \Sigma_+(G'),$$

since there is some $\Sigma$ such that $G' = G_\Sigma$. If $G' \nsubseteq G''$ are two distinct ramification groups, then

$$\emptyset = \Sigma_-(1) \subseteq \Sigma_-(G') \subseteq \Sigma_+(G') \subseteq \Sigma_-(G'') \subseteq \Sigma_+(G'') \subseteq \Sigma_+(G) = (vL)^{\geq 0}.$$

We have that

$$I_-(H) = I_{\Sigma_-(H)}, \quad I_+(H) = I_{\Sigma_+(H)}.$$

The collection of these ideals and final segments reveals information on the valuation theoretical structure of the extension $(L|K,v)$.

3. **Defect extensions of prime degree**

We will investigate defect extensions $(K(a)|K,v)$ of prime degree $p$. By what we have already stated in the Introduction, such extensions are immediate uv-extensions; moreover, $p = \text{char } K v > 0$. By Theorem 2.8, $v(a-K)$ is an initial segment of $vK$ without maximal element, and $\text{dist } (a,K) > va$.

In the following, we distinguish two cases:

- the equal characteristic case where $\text{char } K = p$,
- the mixed characteristic case where $\text{char } K = 0$ and $\text{char } K v = p$.

We fix an extension of $v$ from $K(a)$ to the algebraic closure $\overline{K}$ of $K$.

In a first section, we prove useful results about the distances of elements in separable defect extensions of prime degree, leading up to a general theorem that gives information about the ramification jumps in these extensions when they are Galois. The three sections thereafter are devoted to the definition of “dependent”
and “independent” defect extensions of prime degree, starting with two special cases.

Take a separable defect extension \((K(a)|K,v)\) of prime degree \(p\). By (10) of Proposition 2.13, \((K^r(a)|K^r,v)\) is also a defect extension of degree \(p\). By Lemma 2.14, the separable extension \(K^r(a)|K^r\) is normal and hence a Galois extension. If \(\text{char} \ K = p\), it is an Artin-Schreier extension, according to [23, VI, §6, Theorem 6.4]. In Section 3.2 we will consider the case of Artin-Schreier defect extensions.

Assume now that \((K,v)\) is of mixed characteristic with \(\text{char} \ K = p\). As noted already before Lemma 2.15, \(K^r\) contains a primitive root \(\zeta_r\) of unity. It follows from [23, VI, §6, Theorem 6.2] that \((K^r(a)|K^r,v)\) is a Kummer extension. In Section 3.3 we will consider the case of Kummer defect extensions.

### 3.1. Distances of elements in defect extensions of prime degree

We start with the following two easy but helpful observations.

**Lemma 3.1.** Let \((K(a)|K,v)\) be an algebraic extension of valued fields. If \(\sigma \in \text{Gal}(K)\) is such that \(\sigma a \neq a\), then

\[
\left\{ v \frac{\sigma(a-c)-(a-c)}{a-c} \mid c \in K \right\} = \left\{ v \frac{\sigma a-a}{a-c} \mid c \in K \right\} = -v(a-K)+v(\sigma a-a).
\]

**Proof.** The first equality holds since \(\sigma c = c\), and the second holds since

\[
v \frac{\sigma a-a}{a-c} = -v(a-c)+v(\sigma a-a).
\]

\[\square\]

**Lemma 3.2.** Take a nontrivial immediate \(uv\)-extension \((K(a)|K,v)\). Then for each \(\sigma \in \text{Gal}(K)\) and \(c \in K\),

\[
v(a-c) < v(\sigma a-a).
\]

**Proof.** Since the extension is immediate and \(a \notin K\), the set \(v(a-K)\) has no maximal element. Thus it suffices to prove that \(v(a-c) \leq v(\sigma a-a)\). If this were not true, then for some \(\sigma \in \text{Gal}(K(a)|K)\) and \(c \in K\), \(v(a-c) > v(\sigma a-a)\). But this implies that

\[
v\sigma(a-c) = v(\sigma a-c) = \min\{v(\sigma a-a), v(a-c)\} = v(\sigma a-a) < v(a-c),
\]

which contradicts our assumption that \(K(a)|K\) is a \(uv\)-extension, as \(v\sigma\) is also an extension of \(v\) from \(K\) to \(K(a)\).

An immediate consequence of Lemma 3.2 is the following.

**Corollary 3.3.** Take a defect extension \((K(a)|K,v)\) of prime degree and \(\sigma \in \text{Gal}(K)\) such that \(\sigma a \neq a\). Then

\[
dist(a,K) \leq v(\sigma a-a)^-.
\]

With the help of Lemma 2.11, we prove:

**Lemma 3.4.** Take a Galois defect extension \((K(a)|K,v)\) of prime degree and any \(f \in K(a)\). Then for all \(\sigma \in \text{Gal}(K)\) there is some \(c \in K\) such that

\[
(29) \quad \frac{v \sigma f - f}{f} > -v(a-c) + v(\sigma a-a).
\]
Proof. As stated in the Introduction, $(K(a)|K,v)$ is immediate with $[K(a) : K] = p = \text{char } Kv$. The element $f \in K(a)^	imes$ can be written as $f(a)$ for $f(X) \in K[X]$ of degree smaller than $p$. By Theorem 2.8, $v(a - K)$ has no maximal element. Hence by [2, Lemma 11], we can choose $\gamma \in v(a - K)$ so large that for all $c \in K$ with $v(a - c) \geq \gamma$, all values $v f_i(c)$ are fixed and equal to $v f_i(a)$ whenever $0 \leq i < p$, and that (6) and (7) hold by Lemma 2.11. Since $\deg f < p = [L : K]$, we have that $h = 1$. It suffices to restrict our attention to those $c \in K$ for which $v(a - c) \geq \gamma$. Then we have that

$$(30) \quad v f_1(a)(a - c) = v f_1(c)(a - c) < v f_i(c)(a - c)^i = v f_i(a)(a - c)^i$$

for all $i > 1$. From Lemma 3.2 we infer that

$$0 < v \left(\frac{\sigma a - a}{a - c}\right) < v \left(\frac{\sigma a - a}{a - c}\right)^i$$

for all $i > 1$. Using this together with (30), we obtain:

$$v f_1(a)(\sigma a - a) = v f_1(a)(a - c) \left(\frac{\sigma a - a}{a - c}\right)$$

$$< v f_i(a)(a - c)^i \left(\frac{\sigma a - a}{a - c}\right)^i = v f_i(a)(\sigma a - a)^i .$$

It follows that

$$v(\sigma f(a) - f(a)) = v(f(\sigma a) - f(a)) = v \left(\sum_{i=1}^{\deg f} f_i(a)(\sigma a - a)^i\right)$$

$$= v f_1(a)(\sigma a - a) = v f_1(c) + v(\sigma a - a) .$$

Since $h = 1$, (7) shows that

$$v f_1(c) + v(a - c) = v(f(a) - f(c)) \geq \min\{v f(a), v f(c)\} .$$

The value on the right hand side is fixed, but the value of the left hand side increases with $v(a - c)$. Since $v(a - K)$ has no maximal element, we can choose $\gamma$ so large that the value on the left hand side is larger than the one on the right hand side, which can only be the case if $v f(a) = v f(c)$, whence $v f(a) = v f_1(c) + v(a - c)$. Consequently,

$$v \frac{\sigma f(a) - f(a)}{f(a)} = v f_1(c) + v(\sigma a - a) - v f(a) > -v(a - c) + v(\sigma a - a) .$$

\qed

Theorem 3.5. Take a defect extension $(K(a)|K,v)$ of prime degree. Then for every $\sigma \in \text{Gal}(K) \setminus \{\text{id}\}$ we have:

$$(31) \quad \left\{ v \left(\frac{\sigma f - f}{f}\right) \bigg| f \in K(a)^	imes \right\} = -V(a - K) + v(\sigma a - a) .$$

If in addition $K(a)|K$ is a Galois extension, then

$$\Sigma_+(1) = \Sigma_-(G) = -V(a - K) + v(\sigma a - a) .$$

Proof. The inclusion "$\supseteq$" in (31) follows from Lemma 3.1. To show the reverse inclusion, we use Lemma 3.4. The element $f \in K(a)^	imes$ can be written as $f(a)$ for
$f(X) \in K[X]$ of degree smaller than $p$. Since $v(a - K)$ is an initial segment of $vK$, $-v(a - K)$ is a final segment of $vK$. Thus we can infer from (29) that

$$v\frac{\sigma f - f}{f} \in -v(a - K) + v(\sigma a - a).$$

This proves the inclusion “$\subseteq$”.

Now assume that in addition $K(a)|K$ is a Galois extension. Equation (31) shows that $G_{\Sigma} = 1$ if $\Sigma \not\subseteq -v(a - K) + v(\sigma a - a)$ and $G_{\Sigma} = G$ if $-v(a - K) + v(\sigma a - a) \subseteq \Sigma$. Since $-v(a - K)$ has no smallest element, it is the union of all final segments properly contained in it, whence

$$-v(a - K) + v(\sigma a - a) = \bigcup_{G_{\Sigma} = 1} \Sigma = \bigcup_{G_{\Sigma} \subseteq 1} \Sigma = \Sigma_{+}(1).$$

Trivially, $-(a - K)$ is the intersection of all final segments that contain it, so

$$-v(a - K) + v(\sigma a - a) = \bigcap_{G = G_{\Sigma}} \Sigma = \bigcap_{G_{\Sigma} \subseteq G} \Sigma = \Sigma_{-}(G).$$

3.2. Artin-Schreier defect extensions. We consider now the case of a valued field $(K, v)$ of positive characteristic $p$ and an Artin-Schreier defect extension $(K(\vartheta)|K, v)$ with Artin-Schreier generator $\vartheta$, that is, $\vartheta^p - \vartheta \in K$. The following result appeared in [14] with a different proof:

**Lemma 3.6.** Under the above assumptions, dist $(\vartheta, K) \leq 0^−$.

**Proof.** Take an automorphism $\sigma \in \text{Gal} (K(\vartheta)|K) \setminus \{\text{id}\}$. Then $\sigma(\vartheta) = \vartheta + i$ for some $i \in \mathbb{F}_p$. Therefore, $v(\sigma \vartheta - \vartheta) = vi = 0$. Now the assertion follows from Corollary 3.3. $\square$

Take $\vartheta' \in K(\vartheta)$ to be another Artin-Schreier generator of the extension $K(\vartheta)|K$. Then $\vartheta'$ is of the form $i\vartheta + c$ for some $i \in \mathbb{F}_p^×$ and $c \in K$ (cf. Lemma 2.26 of [14]). Hence from Lemma 2.3 it follows that $\delta := \text{dist}(\vartheta, K)$ does not depend on the choice of the Artin-Schreier generator. This follows also from Theorem 3.5, as for every $\sigma \in \text{Gal} (K(\vartheta)|K) \setminus \{\text{id}\}$ we have that $\sigma(\vartheta) = \vartheta + i \in \mathbb{F}_p$, and thus $v(\sigma(\vartheta) - \vartheta) = 0$ and the left hand side of equation (31) does not depend on the choice of the Artin-Schreier generator. We call $\delta$ the distance of the Artin-Schreier extension $(K(\vartheta)|K, v)$. Lemma 3.6 implies that $\delta \leq 0^−$.

The following classification was introduced in [14]. Assume that $(K(\vartheta)|K, v)$ is an Artin-Schreier defect extension. If there is an immediate purely inseparable extension $K(\eta)|K$ of degree $p$ such that

$$\eta \sim_K \vartheta,$$

then the Artin-Schreier defect extension $(K(\vartheta)|K, v)$ is called dependent; otherwise it is called independent. The following characterization of independent Artin-Schreier defect extensions by idempotent cuts was given in Proposition 4.2. of [14].

**Proposition 3.7.** An Artin-Schreier defect extension is independent if and only if its distance is idempotent.

In view of Lemma 2.1, we obtain the following characterization:
Proposition 3.8. An Artin-Schreier defect extension \((K(\vartheta)|K,v)\) is independent if and only if
\[
\text{dist} (\vartheta, K) = H^-
\]
for some proper convex subgroup \(H\) of \(\tilde{v}K\). In particular, if the value group of \((K,v)\)
is archimedean, then the Artin-Schreier defect extension \((K(\vartheta)|K,v)\) is independent if and only if
\[
\text{dist} (\vartheta, K) = 0^-.
\]

Note that by Lemma 3.6, \(\text{dist} (\vartheta, K) = H^+\) is not possible.

Since \(v(\sigma\vartheta - \vartheta) = 0\), we obtain as a corollary to Theorem 3.5:

Theorem 3.9. Take an Artin–Schreier defect extension \((K(\vartheta)|K,v)\) with Artin–Schreier generator \(\vartheta\). Then
\[
\left\{ \frac{v(\sigma f - f)}{f} \bigg| f \in K(\vartheta)^\times \right\} = -v(\vartheta - K)
\]
for \(id \neq \sigma \in G\), and
\[
\Sigma_+(1) = \Sigma_-(G) = -v(\vartheta - K).
\]

3.3. Defect extensions by \(p\)-th roots of 1-units. In this section we will study the case of a valued field \((K,v)\) of characteristic 0 with char \(Kv = p > 0\) and a defect extension \((K(\eta)|K,v)\) of degree \(p\), where \(\eta^p \in K\). We can assume that \(\eta\) is a 1-unit. Indeed, since \((K(\eta)|K,v)\) is immediate, we have that \(v(\eta) \in vK(\eta) = vK\), so there is \(c \in K\) such that \(vc = -v\eta\). Then \(v(\eta)c = 0\), and since \(\eta cv \in K(\eta)v = Kv\), there is \(d \in K\) such that \(dv = (\eta cv)^{-1}\). Then \(v(\eta cd) = 0\) and \((\eta cd)v = 1\). Hence \(\eta cd\) is a 1-unit. Furthermore, \(K(\eta cd) = K(\eta)\) and \((\eta cd)^p = \eta^pc^pd^p \in K\). Thus we can replace \(\eta\) by \(\eta cd\) and assume from the start that \(\eta\) is a 1-unit. It follows that also \(\eta^p \in K\) is a 1-unit.

From now on we assume that \(K\) contains an element \(C\) as in (12). We do not need that the extension \(K(\eta)|K\) is Galois, but if \((K,v)\) is henselian then by Lemma 2.15 it contains a primitive \(p\)th root of unity as it contains \(C\), which then yields that the extension is indeed Galois.

We will now use the construction from Section 2.3 that associates to \(\eta\) an element \(\vartheta_\eta\) whose minimal polynomial \(f_\eta\) given in (19) bears some resemblance with an Artin-Schreier polynomial. A comparison with the equal characteristic case will then help us to determine when the defect extension \((K(\eta)|K,v)\) should be called independent.

Proposition 3.10. The distances \(\text{dist} (\eta, K)\) and \(\text{dist} (\vartheta_\eta, K)\) do not depend on the choice of the generator \(\eta\) of the extension \((K(\eta)|K,v)\) as long as it is a 1-unit and satisfies \(\eta^p \in K\). Moreover,
\[
0 < \text{dist} (\eta, K) \leq \left( \frac{vp}{p - 1} \right)^-
\]
and
\[
- \frac{vp}{p - 1} < \text{dist} (\vartheta_\eta, K) = - \frac{vp}{p - 1} + \text{dist} (\eta, K) \leq 0^-.
\]
Proof. Take $\sigma \in \text{Gal} K$ such that $\sigma(\eta) = \zeta_p \eta$. Then
\begin{equation}
(36) \quad v(\sigma(\eta) - \eta) = v(\eta) + v(\zeta_p - 1) = \frac{vp}{p-1},
\end{equation}
where the last equality follows from (13) together with the fact that $\eta$ is 1-unit. Therefore, equation (31) yields that
\begin{equation}
(37) \quad v(\eta - K) = -\left\{ v\left( \frac{\sigma f - f}{f} \right) \mid f \in K^\times \right\} + \frac{vp}{p-1}.
\end{equation}
Hence the set $v(\eta - K)$, and consequently also the cut $\text{dist}(\eta, K)$, do not depend on the choice of $\eta$. Moreover, by Lemma 2.3,
\begin{equation}
\text{dist}(\vartheta_\eta, K) = \text{dist}\left( \frac{\eta - 1}{C}, K \right) = -vC + \text{dist}(\eta - 1, K) = -vC + \text{dist}(\eta, K).
\end{equation}
Since $vC = \frac{vp}{p-1}$ we obtain that
\begin{equation}
(38) \quad \text{dist}(\vartheta_\eta, K) = -\frac{vp}{p-1} + \text{dist}(\eta, K).
\end{equation}
Thus also $\text{dist}(\vartheta_\eta, K)$ does not depend on the choice of $\eta$.

From Theorem 2.8, Corollary 3.3 and (36) we deduce that
\begin{equation*}
0 = v\eta < \text{dist}(\eta, K) \leq (v(\sigma(\eta) - \eta))^\sim = \left( \frac{vp}{p-1} \right)^\sim.
\end{equation*}
Together with equation (38) this yields that
\begin{equation*}
\text{dist}(\vartheta_\eta, K) = -\frac{vp}{p-1} + \text{dist}(\eta, K) \leq -\frac{vp}{p-1} + \left( \frac{vp}{p-1} \right)^\sim = 0^\sim.
\end{equation*}
\[\square\]

The above proposition allows us to call $\text{dist}(\vartheta_\eta, K)$ the \textbf{distance of the defect extension} $(K(\eta) \vert K, v)$. The inequality $\text{dist}(\vartheta_\eta, K) < 0^\sim$ is the same as in the case of Artin-Schreier defect extensions in equal positive characteristic. As explained in the Introduction, this leads us to take over the definition of independent Artin-Schreier defect extensions to our mixed characteristic case. We call the defect extension $(K(\eta) \vert K, v)$ \textbf{independent} if
\begin{equation}
(39) \quad \text{dist}(\vartheta_\eta, K) = H^{-} \text{ or equivalently, dist}(\eta, K) = \frac{vp}{p-1} + H^{-}
\end{equation}
for some proper convex subgroup $H$ of $\hat{v}K$, and \textbf{dependent} otherwise. Note that if (39) holds, then
\begin{equation}
(40) \quad vp \notin H,
\end{equation}
as follows from (35).

Since
\begin{equation*}
v(\sigma\vartheta_\eta - \vartheta_\eta) = v\frac{\sigma\eta - \eta}{C} = \frac{vp}{p-1} - \frac{vp}{p-1} = 0
\end{equation*}
by (36) and (12), we obtain as a corollary to Theorem 3.5:
Theorem 3.11. Take a valued field \((K,v)\) of mixed characteristic containing an element \(C\) as in (12), and a Kummer defect extension \((K(\eta)|K,v)\) of prime degree \(p\) with \(\eta\) a 1-unit such that \(\eta^p \in K\). Define \(\vartheta_\eta\) by (21). Then

\[
\left\{ v_\sigma f - f \right\}_{f \in K(\vartheta)^*} = -v(\vartheta_\eta - K)
\]

for \(\text{id} \neq \sigma \in G\), and

\[
\Sigma_+(1) = \Sigma_-(G) = -v(\vartheta_\eta - K).
\]

3.4. Defect extensions of prime degree: the general case. We are now going to generalize the definition of “dependent” and “independent” to any given separable defect extension \((K(a)|K,v)\) of degree \(p\).

We choose any extension of \(v\) from \(K(a)\) to \(\tilde{K}\) and take \((K_r,a,v)\) to be the absolute ramification field of \((K,a,v)\). As we have pointed out in the beginning of Section 3, \((K^r(a)|K^r,v)\) is again a defect extension of degree \(p\) and a Galois extension. Thus it is either an Artin-Schreier or a Kummer extension, depending on the characteristic of \(K\). Using the definitions already given in Sections 3.2 and 3.3, respectively, we now define \((K(a)|K,v)\) to be a dependent defect extension if \((K^r(a)|K^r,v)\) is, and to be an independent defect extension otherwise.

We have to show that this definition is consistent with the already given definitions in the case of \((K(a)|K,v)\) itself being an extension that has already been considered in the previous two sections. If \(a\) is an Artin-Schreier generator of \(K(a)|K\), then it is also an Artin-Schreier generator of \(K^r(a)|K^r\). Likewise, if \(a = \eta\) is a 1-unit with \(\eta^p \in K\), then trivially, \(\eta^p \in K^r\). Hence (11) shows that by the definitions given in the previous sections, \((K(a)|K,v)\) is a dependent defect extension if and only if \((K^r(a)|K^r,v)\) is.

Further, the only arbitrary choice we made in the above construction was the choice of the extension of the valuation to the algebraic closure and consequently, the choice of the henselization; but equation (11) shows that the distance \(\text{dist}(a,K^r)\) does not depend on this choice.

From the above, we obtain:

Corollary 3.12. Take any valued field \((K,v)\). If \((K^r,v)\) is an independent defect field, then so is \((K,v)\).

4. Properties of independent defect extensions

Throughout this section we will assume that \((K,v)\) is a valued field of residue characteristic \(p > 0\). Except in Proposition 4.7, we also assume that \(K\) contains a primitive \(p\)-th root of unity if \(\text{char}\ K = 0\).

The following is Lemma 4.9 of [14]:

Proposition 4.1. Assume that \(\text{char}\ K = p\) and \((K(\vartheta)|K,v)\) is an independent Artin-Schreier defect extension with Artin-Schreier generator \(\vartheta\) of distance \(0^-\). Then every algebraically maximal immediate extension (and in particular, every maximal immediate extension) of \((K,v)\) contains an independent Artin-Schreier defect extension \((K(\vartheta')|K,v)\) of distance \(0^-\) and such that \(\vartheta \sim_K \vartheta'\).

Here is the analogue of this result in the case of mixed characteristic:
Proposition 4.2. Assume that char $K = 0$ and that $(K(\eta)|K, v)$ is an independent defect extension of distance $0^-$, generated by a 1-unit $\eta$ with $\eta^p \in K$. Then every algebraically maximal immediate extension of $(K, v)$ contains an independent defect extension $(K(\eta')|K, v)$ of prime degree and distance $0^-$, where $\eta'$ is also a $p$-th root of a 1-unit in $K$ and $\eta \sim_K \eta'$.

Proof. Take an algebraically maximal immediate extension $(M, v)$ of $(K, v)$. We note that $(M, v)$ is henselian. If $\eta \in M$, then the assertion is trivial.

Assume that $\eta \notin M$. Then $(M(\eta)|M, v)$ is an extension of degree $p$ with $\eta^p \in M$. Since $M$ is algebraically maximal, $(M(\eta)|M, v)$ is defectless. Indeed, otherwise $(M(\eta)|M, v)$ would be a defect extension of degree $p$, hence a nontrivial immediate extension, a contradiction to the maximality of $(M, v)$. Therefore by Lemma 2.10, the set $v(\eta - M)$ admits a maximal element. Since by Theorem 2.8 the set $v(\eta - K)$ has no maximal element, we have that $v(\eta - K) \subseteq v(\eta - M)$. Hence there is an element $b \in M$ such that $v(\eta - b) > \text{dist}(\eta, K)$. Since equation (38) yields that

$$\text{dist}(\eta, K) = \left(\frac{vp}{p^2}\right)^-$$

we may deduce that

$$v(a - b) \geq \frac{vp}{p - 1}.$$

If $b^p \in K$, we set $\eta' = b$.

Now assume that $b^p \notin K$. Since $\eta$ is a 1-unit, so is $b$ and thus,

$$v(\frac{\eta}{b} - 1) = v(\eta - b) \geq \frac{vp}{p - 1}.$$

The element $\frac{\eta}{b}$ is 1-unit of level $\geq \frac{1}{p^2} vp$, hence by Corollary 2.18, $\eta^p_{b^p}$ is 1-unit of level $\geq \frac{vp}{p - 1}$. As $(M|K, v)$ is immediate, there is some $c \in K$ such that

$$v\left(\frac{\eta^p}{b^p} - c\right) > v\left(\frac{\eta^p}{b^p} - 1\right) \geq \frac{p}{p - 1} v p.$$

Then $c$ is also a 1-unit, and we have that

$$v\left(\frac{\eta^p}{b^p} - 1\right) = v\left(\frac{\eta^p}{b^p} - c\right) > \frac{p}{p - 1} v p.$$

Therefore, by Lemma 2.20 the 1-unit $\frac{\eta^p}{b^p}$ admits a $p$-th root $u$ in the henselian field $M$. Then $bu \in M$ with

$$(bu)^p = b^p \frac{\eta^p}{b^p c} = \frac{\eta^p}{c} \in K.$$

Since $\frac{\eta^p}{b^p}$ is 1-unit of level $\geq \frac{p}{p - 1} v p$, (44) yields that the same holds for $c$. Since

$$c = \frac{\eta^p}{(bu)^p},$$

Corollary 2.18 shows that the level of the 1-unit $\frac{\eta}{bu}$ is $\geq \frac{1}{p^2} v p$. We obtain that

$$v(\eta - bu) = v\left(\frac{\eta}{bu} - 1\right) \geq \frac{vp}{p - 1},$$

and we set $\eta' = bu$.

In both cases we have now achieved that $\eta'$ is a 1-unit which is a $p$-th root of an element in $K$ such that $v(\eta - \eta') \geq \frac{vp}{p^2}$. By Proposition 3.10 we obtain that

$$v(\eta - \eta') > \text{dist}(\eta, K)$$
Corollary 4.3. Assume that there is a maximal immediate extension of \((K,v)\) in which \(K\) is relatively algebraically closed. Then \((K,v)\) admits no independent Galois defect extension of prime degree and distance 0\(^{-}\).

We wish to generalize the previous result to the case of independent defect extensions with arbitrary distance.

Lemma 4.4. Assume that for every coarsening \(w\) of \(v\) (including the valuation \(v\) itself) such that \(Kw\) is of positive characteristic there is a maximal immediate extension \((M_w,v)\) of \((K,w)\) in which \(K\) is relatively algebraically closed. Then \((K,v)\) admits no independent Galois defect extension of prime degree.

Proof. The case of equal positive characteristic has been settled in Lemma 4.11 of [14]. Hence we assume now that \((K,v)\) is of characteristic 0 with residue characteristic \(p > 0\) and containing a primitive \(p\)-th root of unity.

Suppose that \((L|K,v)\) is an independent Galois defect extension of prime degree. By Corollary 4.3, its distance cannot be 0\(^{-}\). Hence it is equal to \(H^{-}\) for some nontrivial proper convex subgroup \(H\) of \(\bar{v}K\). Denote by \(v_H\) the coarsening of \(v\) with respect to \(H\), and by \(M_{v_H}\) its valuation ideal. From (40) we know that \(vp \notin H\), so we have that \(p \in M_{v_H}\) and therefore, \(\text{char} K_{v_H} = p\). By Lemma 2.12, a coarsening of a henselian valuation is again henselian, so \((K,v_H)\) is henselian.

By our assumption, we can write \(L = K(\vartheta_\eta)\) with \(\vartheta_\eta\) as in Section 3.3. Then \(\text{dist}(\vartheta_\eta,K) = H^{-}\), which means that \(v(\vartheta_\eta - K)\) is cofinal in \((\bar{v}K)^{<0} \setminus H\). It follows that \(v_H(\vartheta_\eta - K)\) is cofinal in \((\bar{v}K)^{<0}/H = (\tilde{v}H\bar{K})^{<0}\). Since \((\tilde{v}H\bar{K})^{<0}\) is divisible, \((\tilde{v}H\bar{K})^{<0}\) has no largest element. Thus in particular, \(v_H(\vartheta_\eta - K)\) has no maximal element. Together with Lemma 2.9, this shows that \((L|K,v_H)\) is an immediate extension of henselian fields. Hence, \((L|K,v_H)\) is a Galois defect extension of prime degree and distance 0\(^{-}\). On the other hand, by assumption \((K,v_H)\) admits a maximal immediate extension in which \(K\) is relatively algebraically closed. Therefore, Corollary 4.3 shows that \((K,v_H)\) admits no Galois defect extension of prime degree and distance 0\(^{-}\), a contradiction. \(\square\)

Lemma 4.5. Take a coarsening \(w\) of \(v\) (possibly the valuation \(v\) itself) such that \((K,w)\) admits a maximal immediate extension \((M_w,w)\) in which \(K\) is relatively algebraically closed. If \((L|K,v)\) is a finite separable and defectless extension, then \((M_w,L,w)\) is a maximal immediate extension of \((L,w)\) such that \(L\) is relatively algebraically closed in \(M_w\).L.

Proof. Since \((L|K,v)\) is defectless by assumption, the same is true for the extension \((L|K,w)\) by Lemma 2.12. We note that \((K,w)\) is henselian since it is assumed to be relatively algebraically closed in the henselian field \((M_w,w)\). Hence we may apply Lemma 2.4: since \((M_w|K,w)\) is immediate and \((L|K,w)\) is defectless, \((M_w.L|L,w)\)
is immediate and $M_w|K$ and $L|K$ are linearly disjoint. The latter implies that $L$ is relatively algebraically closed in $M_w.L$ (for the proof of this fact, see [19] or [20, Chapter 24]). On the other hand, [25, Theorem 31.22] shows that $(M_w.L,w)$ is a maximal field, being a finite extension of a maximal field, and it is therefore a maximal immediate extension of $(L,w)$. \hfill \Box

**Proposition 4.6.** If $(K,v)$ is algebraically maximal and $(L|K,v)$ is a finite separable and defectless extension, then $(L,v)$ admits no independent Galois defect extension of prime degree.

*Proof.* Take a coarsening $w$ of $v$ such that $Kw$ is of positive characteristic. Note that every immediate extension of $(K,w)$ is also immediate under the finer valuation $v$. Since $(K,v)$ is algebraically maximal, this yields that also $(K,w)$ is algebraically maximal.

Take $(M_w,w)$ to be a maximal immediate extension of $(K,w)$. Then $K$ is relatively algebraically closed in $M_w$. Lemma 4.5 yields that $(M_w,L,w)$ is a maximal immediate extension of $(L,w)$ such that $L$ is relatively algebraically closed in $M_w.L$

This shows that for every coarsening $w$ of $v$ such that $Lw$ is of positive characteristic there is a maximal immediate extension of $(L,w)$ in which $L$ is relatively algebraically closed. By Lemma 4.4 this proves that $(L,v)$ admits no independent Galois defect extension of prime degree. \hfill \Box

**Proposition 4.7.** Assume that $(K,v)$ is a valued field of positive residue characteristic $p$. Then the following are equivalent

a) $(K,v)$ is henselian and defectless,

b) $(K,v)$ is algebraically maximal and in every finite tower of extensions of degree $p$ over $K^r$ every defect extension of degree $p$ is separable and independent.

*Proof.* Assume first that a) holds. Since $K$ is henselian and defectless, it admits in particular no immediate algebraic extension, that is, $(K,v)$ is algebraically maximal.

Take now a finite tower $L$ of extensions of degree $p$ over $K^r$. Choose generators $a_1, \ldots, a_s$ of the extension $L|K^r$ and set $K' = K(a_1, \ldots, a_s)$. Then $(K'|K,v)$ is finite, hence by assumption a defectless extension. Since the extension $(K'|K,v)$ is tame, Proposition 2.7 yields that

$$1 = d(K'|K,v) = d(K'.K^r|K^r,v) = d(L|K^r,v).$$

Hence $L|K^r$ is a defectless extension, and so is every extension of degree $p$ in the tower $L|K^r$. This shows that condition b) holds.

Suppose now that $(K,v)$ satisfies condition b). Since $(K,v)$ is algebraically maximal, it is henselian. Take a finite extension $(L|K,v)$. We wish to show that the extension is defectless. Take $K'$ to be the relative separable-algebraic closure of $K$ in $L$. By Lemma 2.14, there is a finite tame extension $N$ of $K$ such that $K'.N|N$ is a tower $N = N_0 \subsetneq N_1 \subsetneq \ldots \subsetneq N_m = K'.N$ of Galois extensions $N_i|N_{i-1}$ of degree $p$. If char $K = 0$, we can assume in addition that $N$ contains a primitive $p$-th root of unity, replacing $N$ by $N(\zeta_p)$ if necessary; since $p$ does not divide $[N(\zeta_p) : N]$, this is also a tame extension of $(K,v)$.

We first show that the extension $(K'|K,v)$ is defectless. Proposition 2.7 shows that $d(K'.N|N,v) = d(K'|K,v)$, so it suffices to show that $(K'.N|N,v)$ is defectless. We observe that also $K^r = N_0.K^r \subsetneq N_1.K^r \subsetneq \ldots N_m.K^r = K'.K^r$ is a tower of
Galois extensions $N_i.K^r|N_{i-1}.K^r$ of degree $p$. Assume that $(N_i|N, v)$ is a defectless extension for some $i \leq m$ and consider the extension $(N_i|N_{i-1}, v)$. Condition b) implies that the extension $(N_i.K^r|N_{i-1}.K^r, v)$ is either defectless or an independent Galois defect extension. Since $(K, v)$ is algebraically maximal and $(N_{i-1}|K, v)$ is a finite separable defectless extension, Proposition 4.6 shows that $(N_i|N_{i-1}, v)$ cannot be an independent defect extension. Therefore, also $(N_i.K^r|N_{i-1}.K^r, v)$ cannot be an independent defect extension. Hence by assumption, this extension is defectless. From Proposition 2.7 it thus follows that $(N_i|N_{i-1}, v)$ is defectless. This shows that also $(N_i|N, v)$ is a defectless extension. By induction on $i$ we obtain that $(K'.N|N, v)$ is a defectless extension.

The above conclusion together with Proposition 2.7 yields that

\begin{equation}
(45) \quad d(L|K, v) = d(L.K^r|K^r, v) = d(L.K^r|K'.K^r, v).
\end{equation}

Since $L|K'$ is purely inseparable, $L.K^r|K'.K^r$ is a tower of purely inseparable extensions of degree $p$. On the other hand, assumption b) implies that every defect extension of degree $p$ in the tower $L.K^r|K^r$ is separable. Thus every extension in the tower $L.K^r|K'.K^r$ is defectless. This shows that $d(L.K^r|K'.K^r, v) = 1$ and together with equation (45) yields that $(L|K, v)$ is a defectless extension.

Note that if char $K = p > 0$, then condition b) holds if and only if $(K, v)$ is separable-algebraically maximal and inseparably defectless. Indeed, assume that $(K, v)$ satisfies b). Then it is separable-algebraically maximal. If $(K, v)$ would admit a purely inseparable defect extension $(L, v)$, then Proposition 2.7 would yield that $(L.K^r|K^r, v)$ were also a purely inseparable defect extension, which contradicts our assumption that every defect extension of degree $p$ in the tower $L.K^r|K^r$ is separable.

Suppose now that $(K, v)$ is separable-algebraically maximal and inseparably defectless. Then $(K, v)$ is algebraically maximal, and by Proposition 2.7, $(K', v)$ is inseparably defectless. Take a finite extension $(L|K^r, v)$. By Lemma 2.14, $L|K^r$ is a finite tower of normal extensions of degree $p$. As $(K^r, v)$ is inseparably defectless, Lemma 2.6 yields that every purely inseparable extension of degree $p$ in this tower is defectless. Moreover, since every finite extension of $K^r$ does not admit purely inseparable defect extensions, it also admits no dependent Artin-Schreier defect extensions. This yields that every defect extension of degree $p$ in the tower $L|K^r$ is independent.

We have now shown that in the case of valued fields of positive characteristic, our above characterization of henselian defectless fields is equivalent to Theorem 1.2 of [14].

5. THE TRACE OF DEFECT EXTENSIONS OF PRIME DEGREE

In this section we will consider the trace on separable defect extensions of prime degree. The proof of the following fact can be found in [10, Section 6.3].

**Lemma 5.1.** Take a separable field extension $K(a)|K$ of degree $n$ and let $f(X) \in K[X]$ be the minimal polynomial of $a$ over $K$. Then

\begin{equation}
(46) \quad \text{Tr}_{K(a)|K} \left( \frac{a^m}{f'(a)} \right) = \begin{cases} 
0 & \text{if } 1 \leq m \leq n-2 \\
1 & \text{if } m = n-1.
\end{cases}
\end{equation}
Throughout this section, we let \((K(a)|K,v)\) be a defect extension of prime degree \(p\), where
- \(a = \vartheta\) with \(\vartheta^p - \vartheta \in K\) if \(\text{char } K = p\) (equal characteristic case),
- \(a = \eta\) with \(\eta^p \in K\) if \(\text{char } K = 0\) and \(\text{char } Kv = p\) (mixed characteristic case).

For arbitrary \(d \in K\), we note:
\[
\text{(47)} \quad d(a - c)^{p-1} \in M_{K(a)} \iff vd > -(p - 1)v(a - c).
\]
Take \(\Lambda\) to be the smallest final segment of \(\vec{v}K\) containing \(-(p - 1)v(a - K)\). Then the above equation yields that
\[
\text{(48)} \quad vd \in \Lambda \iff \exists c \in K : d(a - c)^{p-1} \in M_{K(a)}.
\]

First we consider the equal characteristic case. By Lemma 5.1,
\[
\text{(49)} \quad \text{Tr}_{K(\vartheta)|K}(\vartheta^i) = \begin{cases} 0 & \text{if } 1 \leq i \leq p - 2 \\ -1 & \text{if } i = p - 1. \end{cases}
\]
This also holds for \(\vartheta - c\) for arbitrary \(c \in K\) in place of \(\vartheta\) since it is also an Artin-Schreier generator. In particular,
\[
\text{Tr}_{K(\vartheta)|K}(d(\vartheta - c)^{p-1}) = -d.
\]
By (48) it follows that
\[
\text{(50)} \quad \text{Tr}_{K(\vartheta)|K}(M_{K(\vartheta)}) \supseteq \{d \mid d \in K \text{ and } vd > -(p - 1)\text{dist}(\vartheta, K)\}.
\]

Now we consider the mixed characteristic case. Since \(\eta^p \in K\), we have that
\[
\text{Tr}_{K(\eta)|K}(\eta^i) = 0 \quad \text{for } 1 \leq i \leq p - 1.
\]
For \(c \in K\) and \(0 \leq j \leq p - 1\), we compute:
\[
(\eta - c)^j = \sum_{i=1}^{j} \binom{j}{i} \eta^i(-c)^{j-i} + (-c)^j.
\]
Thus for every \(d \in K\),
\[
\text{Tr}_{K(\eta)|K}(d(\eta - c)^j) = pd(-c)^j.
\]
If \(vd > -(p - 1)\text{dist}(\eta, K)\), then we may choose \(c \in K\) with \(vd > -(p - 1)v(\eta - c)\); this remains true if we make \(v(\eta - c)\) even larger. Since \(\eta\) is a 1-unit, there is \(c \in K\) such that \(v(\eta - c) > 0\), which implies that \(vc = 0\). Hence we may choose \(c \in K\) with \(vd > -(p - 1)v(\eta - c)\) and \(vc = 0\). Applying (51) with \(j = p - 1\), we find that \(\text{Tr}_{K(\eta)|K}(d(-c)^{(p-1)(\eta - c)^{p-1}}) = pd\). It follows that
\[
\text{(52)} \quad \text{Tr}_{K(\eta)|K}(M_{K(\eta)}) \supseteq \{pd \mid d \in K \text{ and } vd > -(p - 1)\text{dist}(\eta, K)\}.
\]

In order to prove the opposite inclusions in (50) and (52), we have to find out enough information about the elements \(g(a) \in K(a)\) that lie in \(M_{K(a)}\). Using the Taylor expansion, we write
\[
g(a) = \sum_{i=0}^{p-1} g_i(c)(a - c)^i.
\]
By Lemma 2.11 there is \(c \in K\) such that among the values \(v g_i(c)(a - c)^i, 0 \leq i \leq p - 1\), there is precisely one of minimal value, and the same holds for all \(c' \in K\).
with \( v(a-c') \geq v(a-c) \). In particular, we may assume that \( v(a-c) > va \). For all such \( c \), we have:
\[
v g(a) = \min_{0 \leq i \leq p-1} v g_i(c)(a-c)^i.
\]
Hence for \( g(a) \) to lie in \( \mathcal{M}_{K(a)} \) it is necessary that \( v g_i(c)(a-c)^i > 0 \), or equivalently,
\[
(53) \quad v g_i(c) > -iv(a-c)
\]
for \( 0 \leq i \leq p-1 \) and \( c \in K \) as above.

In the equal characteristic case, for \( g(\vartheta) \in \mathcal{M}_{K(\vartheta)} \) and \( c \in K \) as above, we find:
\[
\text{Tr}_{K(\vartheta)/K}(g(\vartheta)) = \sum_{i=0}^{p-1} \text{Tr}_{K(\vartheta)/K}(g_i(c)(\vartheta-c)^i) = -g_{p-1}(c).
\]
Since \( g_{p-1}(c) > -(p-1)v(\vartheta-c) \) by (53), this proves the desired equality in (50).

In the mixed characteristic case, for \( g(\eta) \in \mathcal{M}_{K(\eta)} \) and \( c \in K \) as above, we find:
\[
\text{Tr}_{K(\eta)/K}(g(\eta)) = \sum_{j=0}^{p-1} \text{Tr}_{K(\eta)/K}(g_j(c)(\eta-c)^j) = p \sum_{j=0}^{p-1} g_j(c)(-c)^j
\]
As we assume that \( v(\eta-c) > 0 \), we have that \( \eta c = 0 \) and \( -iv(\eta-c) \geq -(p-1)v(\eta-c) \) for \( 0 \leq i \leq p-1 \). Hence by (53),
\[
(55) \quad v \sum_{j=0}^{p-1} g_j(c)(-c)^j \geq -(p-1)v(\eta-c).
\]
This proves the desired equality in (52). Therefore, \( d' \in \text{Tr}_{K(\eta)/K}(\mathcal{M}_{K(\eta)}) \) if and only if \( vd' > vp - (p-1)\text{dist}(\eta, K) \). From equation (35) we know that
\[
\text{dist}(\eta, K) = \frac{vp}{p-1} + \text{dist}(\vartheta, K).
\]
Hence \( d' \in \text{Tr}_{K(\eta)/K}(\mathcal{M}_{K(\eta)}) \) if and only if \( vd' > -(p-1)\text{dist}(\vartheta, K) \). We have now proved:

**Theorem 5.2.** Take a defect extension \( (K(a)|K, v) \) of prime degree \( p \), where the generator \( a \) is as specified after Lemma 5.1.1 in the beginning of this section. Then in the equal characteristic case,
\[
(54) \quad \text{Tr}_{K(\vartheta)/K}(\mathcal{M}_{K(\vartheta)}) = \{ d \ | \ d \in K \text{ and } vd > -(p-1)\text{dist}(\vartheta, K) \},
\]
and in the mixed characteristic case,
\[
(55) \quad \text{Tr}_{K(\eta)/K}(\mathcal{M}_{K(\eta)}) = \{ pd \ | \ d \in K \text{ and } vd > -(p-1)\text{dist}(\eta, K) \};
\]
if in addition \( K \) contains an element \( C \) as in (12), then
\[
(56) \quad \text{Tr}_{K(\eta)/K}(\mathcal{M}_{K(\eta)}) = \{ d \ | \ d \in K \text{ and } vd > -(p-1)\text{dist}(\vartheta, K) \}.
\]

**Proof of Theorem 1.10.** Assume that the defect extension \( (K(a)|K, v) \) is independent. In the equal characteristic case, we then have that \( \text{dist}(\vartheta, K) = H^- \) for some proper convex subgroup \( H \) of \( vK \) by Proposition 3.8, so \( \text{dist}(\vartheta, K) \) is idempotent, whence \( (p-1)\text{dist}(\vartheta, K) = \text{dist}(\vartheta, K) \). Then \( vd > -(p-1)\text{dist}(\vartheta, K) \) means that \( vd > -H^- = H^+ \), or in other words,
\[
vd \geq \alpha \text{ for all } \alpha \in H.
\]
This in fact means that \( d \) is an element of the valuation ideal \( \mathcal{M}_{v_H} \) of the coarsening \( v_H \) of \( v \) whose value group has divisible hull \( vK/H \). Hence (4) holds.

In the mixed characteristic case, when we assume in addition that \( K \) contains the element \( C \), then a similar argument as in the equal characteristic case shows...
again that $\text{dist}(\vartheta_{\eta}, K) = H^−$ for some proper convex subgroup $H$ of $\tilde{vK}$ by (39). As before, this yields (4). By (40), $vp \notin H$, which means that $p \in \mathcal{M}_{vH}$ and consequently, $\text{char} K_{vH} = p$.

Conversely, if the defect extension $(K(a)|K,v)$ is dependent, then $\text{dist}(\vartheta, K)$ and $\text{dist}(\vartheta_{\eta}, K)$, respectively, are not idempotent, and neither are $- (p−1)\text{dist}(\vartheta, K)$ and $-(p−1)\text{dist}(\vartheta_{\eta}, K)$, respectively; in this case, there is no convex subgroup $H$ such that $\text{Tr}_{K(a)|K} (\mathcal{M}_{K(a)}) = \{a | vd > H\}$. □

6. SEMITAME, DEEPLY RAMIFIED AND GDR FIELDS

Throughout this section, we will consider a valued field $(K,v)$ of residue characteristic $p > 0$, if not stated otherwise. To start with, we state a few simple observations.

Lemma 6.1. 1) If $\text{char} K = p > 0$, then

$\mathcal{O}_K/p\mathcal{O}_K \ni x \mapsto x^p \in \mathcal{O}_K/p\mathcal{O}_K$

is surjective if and only if $K$ is perfect; in particular, (DRvr) holds if and only if $K$ is perfect.

2) If (57) is surjective, then (DRvr) holds.

3) Assume that $\text{char} K = 0$. Then the following assertions are equivalent:
   a) (57) is surjective,
   b) for every $\hat{a} \in \mathcal{O}_{\tilde{K}}$ there is $c \in \mathcal{O}_K$ such that $\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$,
   c) (DRvr) holds.

4) If $(K,v)$ satisfies (DRvr), then so does every extension of $(K,v)$ within its completion.

Proof. 1): From $\text{char} K = p > 0$ it follows that $p\mathcal{O}_K = \{0\}$, hence the surjectivity of the homomorphism in (3) means that every element in $\mathcal{O}_K$ is a $p$-th power. Hence the same is true for every element in $K$, i.e., $K$ is perfect. Replacing $K$ by $\tilde{K}$ in (57), we thus obtain that $\tilde{K}$ is perfect.

2): Assume first that $\text{char} K = p > 0$. Then by part 1) the surjectivity of (57) implies that $K$ is perfect. Since the completion of a perfect field is again perfect, it follows that $\tilde{K}$ is perfect. Hence again by part 1), (DRvr) holds.

Now assume that $\text{char} K = 0$. Take $\hat{a} \in \mathcal{O}_{\tilde{K}}$. Then there exists $a \in K$ such that $\hat{a} \equiv a \mod p\mathcal{O}_{\tilde{K}}$. By assumption, there is some $c \in \mathcal{O}_K$ such that $a \equiv c^p \mod p\mathcal{O}_K$. It follows that $\hat{a} \equiv a \equiv c^p \mod p\mathcal{O}_K$, showing that (DRvr) also holds in this case.

3): Assume that $\text{char} K = 0$. Trivially, b) implies a), and part 2) of our lemma shows that a) implies c). To show that c) implies b), take $\hat{a} \in \mathcal{O}_{\tilde{K}}$. Then by (DRvr) there is $\hat{c} \in \mathcal{O}_{\tilde{K}}$ such that $\hat{a} \equiv \hat{c}^p \mod p\mathcal{O}_{\tilde{K}}$. We take $c \in \mathcal{O}_{\tilde{K}}$ such that $c \equiv \hat{c} \mod p\mathcal{O}_{\tilde{K}}$. Then $\hat{a} \equiv \hat{c}^p \equiv c^p \mod p\mathcal{O}_{\tilde{K}}$, whence $\hat{a} \equiv c^p \mod p\mathcal{O}_{K(\hat{a})}$.

4): Take $(L|K,v)$ to be a subextension of $(\tilde{K}|K,v)$. Then $\tilde{L} = \tilde{K}$, and in the case of $\text{char} K = p > 0$ our assertion follows from part 1).

Now assume that $(K,v)$ is of mixed characteristic and satisfies (DRvr). Then by the implication c)$\Rightarrow$b) of part 3), for every $\hat{a} \in \mathcal{O}_{\tilde{K}} = \mathcal{O}_L$ there is $c \in \mathcal{O}_K \subseteq \mathcal{O}_L$ such
Lemma 6.2. 1) If $(K,v)$ satisfies (DRvr), then the following assertions hold:

a) The residue field $K^v$ is perfect.

b) If $\chi = p > 0$, then $vK$ is $p$-divisible and $(K,v)$ is a semitame field.

2) Assume that $(K,v)$ is a gdr field of mixed characteristic. Then the convex hull $(vK)^{vp}$ of $Zp$ in $vK$ is $p$-divisible. If in addition $(vK)^{vp} = vK$, then $(K,v)$ is a semitame field.

3) Assume that $(K,v)$ is a gdr field of mixed characteristic and that $a \in K$ with $va \in (vK)^{vp}$. Then there is $c \in K$ such that

\begin{equation}
va \geq va + vp.
\end{equation}

Proof. 1): To prove part a), take any $a \in \mathcal{O}$. By assumption, there is $\hat{c} \in O_K$ such that $a \equiv \hat{c}^p \mod pO_K$. From this we obtain that $av = \hat{c}^p v = (\hat{e}v)^p \in \hat{K}v = K^v$. Hence $K^v$ is perfect.

To prove part b), assume that $\chi = p > 0$. Then by part 1) of Lemma 6.1, (DRvr) implies that $\hat{K}$ is perfect, so $vK = v\hat{K}$ is $p$-divisible and (DRst) holds, showing that $(K,v)$ is a semitame field.

2): First, let us show that every $\alpha \in vK$ with $0 \leq \alpha < vp$ is divisible by $p$. Take $a \in \mathcal{O}$ such that $va = \alpha$. From (DRvr) we obtain that there is $\hat{c} \in O_K$ such that $a \equiv \hat{c}^p \mod pO_K$. Since $va < vp$, this yields that $va = v\hat{c}^p = pv\hat{c}$, showing that $\alpha = va$ is divisible by $p$ in $\hat{K}$.

By assumption, $vp$ is not the smallest positive element in $vK$, hence there is $\alpha \in vK$ such that $0 < \alpha < vp$, and we know that $\alpha$ is divisible by $p$. We may assume that $2\alpha \geq vp$ since otherwise we replace $\alpha$ by $vp - \alpha$. In this way we make sure that $(vK)^{vp}$ is equal to the smallest convex subgroup containing $\alpha$. This implies that for every $\beta \in (vK)^{vp}$ there is some $n \in \mathbb{Z}$ such that $0 \leq \beta - n\alpha < vp$. Then by what we have already shown, $\beta - n\alpha$ is divisible by $p$. Since also $\alpha$ is divisible by $p$, the same is consequently true for $\beta$.

If in addition $(vK)^{vp} = vK$, then $vK$ is $p$-divisible, and since (DRvr) holds by assumption, $(K,v)$ is a semitame field.

3): Since $va \in (vK)^{vp}$, part 2) shows that there is $b \in K$ such that $pvb = va$. Hence $vb^{-p}a = 0$ and since $(K,v)$ is a gdr field, there is $d \in K$ such that $v(b^{-p}a - d^p) \geq vp$, whence

\begin{equation}
v(a - (bd)^p) = pvb + v(b^{-p}a - d^p) \geq va + vp.
\end{equation}

With $c := bd$, this yields (58). 

Lemma 6.3. If $(vK)^{vp}$ is $p$-divisible and $K^v$ is perfect, then $v(\eta - K)$ does not admit a maximal element smaller than $\frac{vp}{p}$.

Proof. Take $c \in K$ such that $0 < v(\eta - c) < \frac{vp}{p}$. Then $v(\eta - c)^p < vp$ and from (15) it follows that $v(\eta - c)^p = v(\eta - c)^p < vp$. Since $(vK)^{vp}$ is $p$-divisible, there is some $d_1 \in K$ such that $vd_1 = -v(\eta - c)^p$. Then $vd_1^p(\eta - c) = 0$, and since $K^v$ is perfect, there is some $d_2 \in K$ such that $d_2^p v = (d_2^p(\eta - c)^p) v^{-1}$. Then $v(d_2^p(\eta - c)^p) = 0$ and $(d_2^p(\eta - c)^p) v = 1$. With $d = (d_1 d_2)^{-1}$ it follows that $v(d^{-p} (\eta - c)^p - 1) > 0$, whence $v(\eta - c)^p > v(\eta - c)^p$. Again by (15), we obtain that $v(\eta - c - d)^p \equiv \eta - c^p - d^p \mod pO$, and it follows that $v(\eta - c - d) > v(\eta - c)$. 

\[\Box\]
Proposition 6.4. 1) Assume that $(vK)_{vp}$ is $p$-divisible, $Kv$ is perfect, and $(K,v)$ is an independent defect field. Then $(K,v)$ is a gdr field.

2) If every separable uv-extension of degree $p$ of $(K,v)$ is either tame or an independent defect extension, then $(K,v)$ is a semitame field.

Proof. 1): From our assumption that $(vK)_{vp}$ is $p$-divisible it follows that $(DRvp)$ holds. It remains to show that $(K,v)$ satisfies $(DRvr)$.

Assume first that char $K > 0$. Then by assumption, $vK$ is $p$-divisible and $Kv$ is perfect, hence the perfect hull of $K$ is an immediate extension of $(K,v)$. Our assumption that $(K,v)$ is an independent defect field implies that $(K,v)$ has no dependent Artin-Schreier defect extension. This yields that the perfect hull of $K$ lies in its completion (cf. Corollary 4.6 of [14]). It follows that the completion is perfect and hence $(K,v)$ satisfies $(DRvr)$ by part 1) of Lemma 6.1.

Now assume that char $K = 0$. Assume further that $b \in K$ is not a $p$-th power, and take $\eta \in K$ with $v^p = b$. Then by Lemma 6.3, $v(\eta - K)$ has a maximal element $\frac{\eta}{p}$, or it has no maximal element at all. In the first case, part 3) of Lemma 2.16 shows the existence of $c \in K$ such that $b \equiv c^p \mod pO_K$. In the second case, we know from Lemma 2.10 that $(K(\eta)|K,v)$ is a defect extension. By assumption, it is independent, so dist $(\eta, K) = v^p \frac{p}{p-1} + H^-$ for some proper convex subgroup $H$ of $vK$ with $vp \notin H$. Hence again there is some $c \in K$ such that $v(\eta - c) \geq \frac{vp}{p}$, which by part 3) of Lemma 2.16 gives us $b \equiv c^p \mod pO_K$. This shows that (57) is surjective. Hence by part 2) of Lemma 6.1, $(DRvr)$ holds.

2): Our assumptions yield that $vK$ is $p$-divisible (so $(DRst)$ holds), and $Kv$ is perfect. Indeed, if $\alpha \in vK$ is not divisible by $p$ and $a \in K$ with $va = a$, then taking a $p$-th root of $a$ induces an extension that is neither tame nor immediate. The same holds if $a \in K$ is such that $av$ does not have a $p$-th root in $Kv$. Since defect extensions of degree $p$ are not tame, our assumption yields that every separable defect extension of degree $p$ is independent. Hence we obtain from part 1) that $(DRvr)$ holds.

Proof of Theorem 1.2. 1): Assume that $(K,v)$ is nontrivially valued. The implication tame field $\Rightarrow$ separably tame field is obvious, and so is the implication semitame field $\Rightarrow$ deeply ramified field. To prove the implication deeply ramified field $\Rightarrow$ gdr field, we first observe that if char $K = p > 0$, then $vp = \infty$ which is not the smallest positive element of $vK$. If char $K = 0$, then we take $\Gamma_1$ to be the largest convex subgroup of $vK$ not containing $vp$, and $\Gamma_2$ to be the smallest convex subgroup of $vK$ containing $vp$. If $vp$ were the smallest positive element of $vK$, then we would have that $\Gamma_1 = \{0\}$ and $\Gamma_2 = Zvp$, whence $\Gamma_2/\Gamma_1 \simeq Z$ in contradiction to $(DRvg)$.

Now assume that $(K,v)$ is a separably tame field. If char $K > 0$, then by [17, Corollary 3.12], $(K,v)$ is dense in its perfect hull. Then the completion of the perfect hull is also the completion of $(K,v)$. Since the completion of a perfect valued field is again perfect, we obtain that the completion of $(K,v)$ is perfect. Now part 1) of Lemma 6.1 shows that $(K,v)$ is a semitame field. If char $K = 0$, then the separably tame field $(K,v)$ is a tame field. Hence every finite extension of $(K,v)$ is a tame extension. Thus by part 2) of Proposition 6.4, $(K,v)$ is a semitame field.

2): Assume that $(K,v)$ is a gdr field of rank 1 and mixed characteristic. Since the rank is 1, we have that $(vK)_{vp} = vK$. Hence by part 2) of Lemma 6.2, $(K,v)$ is
a semitame field. This together with part 1) of our theorem shows the required equivalence in the case of mixed characteristic. For the case of equal characteristic, it will be shown in the proof of part 3).

3): Assume that \((K, v)\) is a nontrivially valued field of characteristic \(p > 0\). The implications a)\(\Rightarrow\)b)\(\Rightarrow\)c) have already been shown in part 1).

c)\(\Rightarrow\)d): This holds by definition.

d)\(\Rightarrow\)e): This holds by part 1) of Lemma 6.1.

e)\(\Rightarrow\)f): If \((K, v)\) is dense in its perfect hull, then it contains the perfect hull of \(K\); since \((K, v)\) is dense in its completion, it is then also dense in its perfect hull.

f)\(\Rightarrow\)g): If \((K, v)\) is dense in its perfect hull, then in particular it is dense in \(K^{1/p} = \{a^{1/p} \mid a \in K\}\). Since \(x \mapsto x^p\) is an isomorphism which preserves valuation divisibility, the latter holds if and only if \((K^p, v)\) is dense in \((K, v)\).

g)\(\Rightarrow\)f): Assume that \((K^p, v)\) is dense in \((K, v)\). Since for each \(i \in \mathbb{N}\), \(x \mapsto x^{p^i}\) is an isomorphism which preserves valuation divisibility, it follows that \((K^{1/p^{i-1}}, v)\) is dense in \((K^{1/p^i}, v)\). By transitivity of density we obtain that \((K, v)\) is dense in \((K^{1/p^i}, v)\) for each \(i \in \mathbb{N}\), and hence also in its perfect hull.

f)\(\Rightarrow\)c): This implication was already shown in the proof of part 1) of our theorem.

e)\(\Rightarrow\)b): Assume that \(\hat{K}\) is perfect. The extension \((\hat{K}|K, v)\) is immediate, so \(vK = v\hat{K}\), which is \(p\)-divisible. Hence \((\text{DRst})\) holds. By part 1) of Lemma 6.1, also \((\text{DRvr})\) holds.

4): The assertion follows from the implication f)\(\Rightarrow\)a) of part 3) as a perfect field is equal to its perfect hull.

Our next goal is the proof of Proposition 1.3, for which we need some preparation.

**Lemma 6.5.** Assume that \((K, v)\) is of mixed characteristic, and let \(v_0\) be the coarsening of \(v\) with respect to \((vK)_{vp}\), that is, the finest coarsening that has a residue field of characteristic 0. Further, denote by \(w\) the valuation induced by \(v\) on \(Kv_0\). Then \((K, v)\) is a gdr field if and only if \((Kv_0, w)\) is a gdr field.

**Proof.** First assume that \((K, v)\) is a gdr field. Then \(vp\) is not the smallest positive element in \(vK\), which implies that \(wp\) is not the smallest element in \(w(Kv_0)\). Take any \(b \in O_{Kv_0}\). Then choose \(a \in O_K\) such that \(av_0 = b\). Since \((K, v)\) is a gdr field, there is some \(c \in O_K\) such that \(a - c^p \in \mathfrak{p}O_K\). It follows that \(cv_0 \in O_{Kv_0}\) with \(b - (cv_0)^p = (a - c^p)v_0 \in pO_{Kv_0}\), showing that \((Kv_0, w)\) satisfies \((\text{DRvr})\) by part 3) of Lemma 6.1. Hence \((Kv_0, w)\) is a gdr field.

Now assume that \((Kv_0, w)\) is a gdr field. Then \(wp\) is not the smallest element in \(w(Kv_0)\), which implies that \(vp\) is not the smallest positive element in \(vK\). Take any \(a \in O_K\). Then \(av_0 \in O_{Kv_0}\) and there is some \(d \in O_{Kv_0}\) such that \(av_0 - d^p \in \mathfrak{p}O_{Kv_0}\). Choose \(c \in O_K\) such that \(cv_0 = d\). It follows that \(a - c^p \in \mathfrak{p}O_K\). We have now shown that \((K, v)\) is a gdr field.

**Proposition 6.6.** Assume that \((K, v)\) is a gdr field of mixed characteristic, and take \(a \in O_K\) such that \(va \in (vK)_{vp}\).

1) Assume that \(va = 0\). Then for every \(c \in O_K\) with \(0 < v(a - c^p) \in (vK)_p\) there is \(d \in O_K\) such that

\[v(a - d^p) = vp + \frac{1}{p}v(a - c^p).\]
2) Assume that \( va \in (vK)_p \) and that \( \text{dist} (a, K^p) < va + \frac{p}{p-1} vp. \) Then

\[
va + vp < \text{dist} (a, K^p) = va + \frac{p}{p-1} vp + H^-
\]

where \( H \) is a convex subgroup of \( \tilde{v}K \) not containing \( vp \).

Proof. 1) Set \( \alpha := v(a - c^p) > 0. \) Since \((K, v)\) is a gdr field, part 3) of Lemma 6.2 shows that there is \( d \in K \) such that:

\[
(59) \quad v(a - c^p - d^p) \geq vp + \alpha.
\]

It follows that \( vd^p = \alpha. \) Since \( vc = va = 0, \)

\[
(60) \quad v((c + d)^p - c^p - d^p) = v \sum_{i=1}^{p-1} \binom{p}{i} c^{p-i} d^i = vp + vd = vp + \frac{\alpha}{p}.
\]

From (59) and (60), we obtain for \( d := c + d: \)

\[
v(a - d^p) = \min \{ vp + \alpha, vp + \frac{\alpha}{p} \} = vp + \frac{\alpha}{p}.
\]

2) First we prove the assertion in the case of \( va = 0. \) Since \((K, v)\) is a gdr field, there is some \( c \in K \) such that \( v(a - c^p) \geq vp, \) so \( \text{dist} (a, K^p) \geq vp. \)

We will use the following observation. If \((vK)_p \ni v(a - c^p) \geq \frac{p}{p-1}vp - \varepsilon > 0\) for some \( c \in K \) and positive \( \varepsilon \in vK, \) then by part 1) there is \( d \in \mathcal{O}_K \) such that

\[
v(a - d^p) = vp + \frac{v(a - c^p)}{p} \geq vp + \frac{1}{p-1}vp - \frac{1}{p} \varepsilon = \frac{p}{p-1}vp - \frac{1}{p} \varepsilon.
\]

By assumption, \( \text{dist} (a, K^p) < \frac{p}{p-1}vp. \) Hence the set of all convex subgroups \( H' \) of \( \tilde{v}K \) such that \( v(a - K^p) \cap (\frac{p}{p-1}vp + H') = \emptyset \) is nonempty as it contains \{0\}. The set is closed under arbitrary unions, so it contains a maximal subgroup \( H, \) the union of all subgroups in the set. Since \( vp \in v(a - K^p), \) we see that \( H \) cannot contain \( vp. \)

Take any positive \( \delta \notin H. \) Then by the definition of \( H, \) there is some \( n \in \mathbb{N} \) such that \( v(a - K^p) \) contains a value \( \geq \frac{p}{p-1}vp - n\delta. \) We set \( \varepsilon := \min \{ \frac{p}{p-1}vp - vp, n\delta \} \) and observe that there is \( c \in K \) such that

\[
v(a - c^p) \geq \frac{p}{p-1}vp - \varepsilon \geq vp > 0.
\]

Note that \( v(a - c^p) \in (vK)_p \) since \( \text{dist} (a, K^p) < \frac{p}{p-1}vp. \) Using our above observation, by induction starting from \( c_0 = c \) we find \( c_i \in K \) such that

\[
v(a - c_i^p) \geq \frac{p}{p-1}vp - \frac{1}{p^i} \varepsilon.
\]

We choose some \( j \in \mathbb{N} \) such that \( \frac{1}{p^j} \leq 1. \) Then

\[
\frac{1}{p^j} \varepsilon \leq \frac{n}{p^j} \delta \leq \delta
\]

and consequently,

\[
v(a - c_j^p) > \frac{p}{p-1}vp - \delta.
\]

This together with the definition of \( H \) shows that

\[
(61) \quad vp < \text{dist} (a, K^p) = \frac{p}{p-1}vp + H^-.
\]
If $0 \neq va \in (vK)_p$, then since $(K,v)$ is a gdr field, part 2) of Lemma 6.2 shows that there is $b \in K$ such that $vb^p = va$. By what we have already shown, (61) holds for $b^{-p}a$ in place of $a$. We have that

$$v(a - (bc)^p) = vb^p + v(b^{-p}a - c^p) = va + v(b^{-p}a - c^p),$$

whence

$$\text{dist} (a, K^p) = va + \text{dist} (b^{-p}a, K^p),$$

which together with (61) for $b^{-p}a$ in place of $a$ proves assertion 2) of our lemma. \hfill \Box

Proof of Proposition 1.3.
In view of Lemma 6.5, where we take $w = v_p \circ \sigma$, it suffices to prove the proposition under the additional assumption that $v_0$ is trivial, that is, $vK = (vK)_p$. Then the assertion is trivial if $\sigma$ is trivial, so we assume that it is not. This implies that $vp$ is not the smallest positive element in $vK$.

Let us first assume that $(K,v)$ is a gdr field. Then $\frac{vp}{p} \in vK$ by part 2) of Lemma 6.2, so $\frac{vp}{p} \in v_p K$, showing that $v_p p$ is not the smallest positive element in $v_p K$. It remains to show that $(K,v_p)$ satisfies (DRvr); by part 3) of Lemma 6.1 it suffices to prove that $(57)$ is surjective in $(K,v_p)$. Take any $a \in O_{v_p}$. Since $(K,v)$ is a gdr field, by part 3) of Lemma 6.2 there is $c \in K$ such that $v(a - c^p) \geq va + vp$, whence $v_p (a - c^p) \geq v_p a + v_p p \geq v_p p$.

Now assume that $(\hat{K},v_p)$ is a gdr field. We know already that (DRvp) holds in $(K,v)$, so it remains to show that $(57)$ holds. Take $a \in O_v \subseteq O_{v_p}$. Since $(K,v_p)$ is a gdr field, part 2) of Proposition 6.6 implies that there is some $c \in K$ such that $v_p (a - c^p) > v_p p$, whence $v(a - c^p) > vp$. \hfill \Box

We will now prepare the proof of Theorems 1.4 and 1.5.

Lemma 6.7. Every algebraic extension of a deeply ramified field of positive characteristic is again a deeply ramified field.

Proof. By part 3) of Theorem 1.2, a valued field $(K,v)$ of positive characteristic is a deeply ramified field if and only if its completion $(\hat{K},v)$ is perfect. Take any algebraic extension $(L,K,v)$. Then the completion $(\hat{L},v)$ of $(L,v)$ contains $(\hat{K},v)$. Since $\hat{K}$ is perfect, so is $L.\hat{K}$. Since $(\hat{L},v)$ is also the completion of $(L.\hat{K},v)$, it is perfect too. \hfill \Box

Lemma 6.8. Assume that $(K,v)$ is a gdr field of mixed characteristic. Further, take a defect extension $(K(\eta)|K,v)$ with $\eta^p \in K$ such that $v\eta = 0$. Then

$$(62) \quad \text{dist} (\eta, K) = \frac{1}{p-1} vp + H^-,$$

where $H$ is a convex subgroup of $vK$ not containing $vp$.

If in addition $K$ contains an element $C$ with properties (12) and $\eta$ is a 1-unit, then

$$(63) \quad \text{dist} (\vartheta, K) = H^-.$$

Proof. Suppose that there is some $c \in K$ such that $v(\eta - c) \geq \frac{1}{p-1} vp$. Since the defect extension $(K(\eta)|K,v)$ is immediate, $v(\eta - c)$ has no maximal element, and so there will also be some element $c \in K$ such that $v(\eta - c) > \frac{1}{p-1} vp$.

Then by Lemma 2.19, $\eta$ lies in some henselization $K^h$. But this is impossible
since by Lemma 2.5, the uv-extension \((K(\eta)|K,v)\) is linearly disjoint from \(K^h|K\). We conclude that \(\text{dist} (\eta,K) < \frac{1}{p^r} vp\). By Lemma 2.17, this is equivalent to \(\text{dist} (\eta^p, K^p) < \frac{p}{p-1} vp\). Therefore, we can apply Proposition 6.6 to \(a = \eta^p\). We find that

\[
\text{dist} (\eta^p, K^p) = \frac{p}{p-1} vp + H^-
\]

where \(H\) is a convex subgroup of \(\bar{v}K\) not containing \(vp\). As \(\bar{v}K\) is \(p\)-divisible, we can again apply Lemma 2.17 to obtain that (64) is equivalent to (62).

Now assume in addition that \(K\) contains an element \(C\) with properties (12) and \(\eta\) is a 1-unit. Then \(K(\eta) = K(\vartheta_v)\) and from (62) together with (38) we obtain (63).

**Proposition 6.9.** Every gdr field is an independent defect field.

**Proof.** In the case of residue characteristic 0, the assertion is trivial. So we assume that \((K,v)\) is a gdr field of positive residue characteristic.

Take a separable defect extension \((L|K,v)\) of prime degree. Let \((K^r,v)\) be an absolute ramification field of \((K,v)\). By Theorem 1.4 also \((K^r,v)\) is a gdr field. From Lemma 2.14 we know that \(L.K^r|K^r\) is a Galois extension, and Proposition 2.7 shows that it is again a defect extension of prime degree.

Assume first that \(\text{char } \)\(K > 0\), so \(L.K^r|K^r\) is an Artin-Schreier extension. Then by part 3) of Theorem 1.2, the perfect hull of \((K^r,v)\) lies in its completion; consequently, there are no dependent Artin-Schreier defect extensions. Therefore, \((L.K^r|K^r,v)\) is an independent defect extension, and the same holds by definition for \((L|K,v)\).

Now assume that \(\text{char } \)\(K = 0\), so \(L.K^r|K^r\) is a Kummer extension. As shown in the beginning of Section 3.3 we can assume that the Kummer generator \(\eta\) is a 1-unit. Further, \(K^r\) contains an element \(C\) with properties (12) because otherwise, it would generate an extension of degree at most \(p-1\) which consequently would be tame, contradicting the fact that \((K^r,v)\) is the maximal tame extension of \((K,v)\).

Now Lemma 6.8 shows that the extension \((L.K^r|K^r,v)\) is independent. It follows that also \((L|K,v)\) is independent.

**Lemma 6.10.** Fix any extension of \(v\) from \(K\) to \(\bar{K}\), and let \((K^r,v)\) be the respective absolute ramification field of \((K,v)\). If \((K^r,v)\) is a gdr field, then so is \((K,v)\), and if \((K^r,v)\) is a semitame field, then so is \((K,v)\).

**Proof.** Assume that \((K^r,v)\) is a gdr field and hence an independent defect field by Proposition 6.9. By parts 1) and 2) of Proposition 6.2, \((vK^r)_vp\) is \(p\)-divisible and \(K^r/vK\) is perfect. Since \(vK^r/vK\) has no \(p\)-torsion and \(K^r/vK\) is separable, it follows that \((vK^r)_vp\) is \(p\)-divisible and \(K^r/vK\) is perfect. From Corollary 3.12 we know that \((K,v)\) is an independent defect field. Part 1) of Proposition 6.4 now shows that \((K,v)\) is a gdr field.

Now assume that \((K^r,v)\) is a semitame field. Then by part 1) of Theorem 1.2, \((K^r,v)\) is a gdr field, hence so is \((K,v)\). Since \(vK^r\) is \(p\)-divisible and the order of every element in \(vK^r/vK\) is coprime to \(p\), also \(vK\) is \(p\)-divisible. Hence by definition, \((K^r,v)\) is a semitame field.

**Lemma 6.11.** Assume that \((K,v)\) is a henselian gdr field of mixed characteristic and \((L|K,v)\) is a finite extension. Then the following assertions hold.
1) If \([L : K] = [Lv : Kv]\), then also \((L, v)\) is a gdr field.

2) Take a prime \(q\) different from \(p\). Assume that \(L = K(a)\) with \(a^q \in K\), \(va \notin vK\) and \(q = (vL : vK)\). Then also \((L, v)\) is a gdr field.

Proof. We assume that \((K, v)\) is a gdr field of residue characteristic \(p > 0\). In order to prove part 1), we take a finite extension \((L/K, v)\) such that \([L : K] = [Lv : Kv]\).

Since \(Kv\) is perfect by part 1) of Lemma 6.2, \(LvKV\) is separable and we write \(Lv = Kv(\xi)\) with \(\xi \in Lv\). Since also \(Lv\) is perfect, there are \(\xi_0, \ldots, \xi_n \in Kv\) with \(n = [Lv : Kv] - 1\) such that \(\xi = (\xi_n\xi^n + \ldots + \xi_1\xi + \xi_0)^p\). Let \(F\) be the extension of \(\mathbb{F}_p\) generated by the coefficients of the minimal polynomial of \(\xi\) over \(Kv\) and the elements \(\xi_0, \ldots, \xi_n\). As a finitely generated extension of the perfect field \(\mathbb{F}_p\), \(F\) is separably generated, that is, it admits a transcendence basis \(t_1, \ldots, t_k\) such that \(F/\mathbb{F}_p(t_1, \ldots, t_k)\) is separable-algebraic. We have that \(F \subseteq Kv\), so we may choose \(x_1, \ldots, x_k \in K\) such that \(x_iv = t_i\). Then \(v(\xi_1, \ldots, x_k) = vK = Zv\) and \(Q(x_1, \ldots, x_k) = Fp(t_1, \ldots, t_k)\) (cf. [3, chapter VI, §10.3, Theorem 1]). Using Hensel’s Lemma, we find an extension \(K_0\) of \(Q(x_1, \ldots, x_k)\) within the henselian field \(K\) such that \(K_0v = F\) and \(vK_0 = vQ(x_1, \ldots, x_k) = Zv\).

Using Hensel’s Lemma again, we find \(a \in L\) such that \(av = \xi, [K_0(a) : K_0] = [F(\xi) : F]\) and \(vK_0(a) = vK_0 = Zv\). By construction, \(\xi^1/p \in F(\xi)\), so we can choose \(b \in K_0(a)\) such that \(bv = \xi^1/p\). Then \(av = (bv)^p = b^pv\), so \(v(a-b^p) > 0\) and thus \(v(a-b^p) \geq v\).

We observe that since \(F\) contains all coefficients of the minimal polynomial of \(\xi\) over \(Kv\),

\[ [Kv(\xi) : Kv] = [F(\xi) : F] = [K_0(a) : K_0] \geq [K(a) : K] \geq [Kv(\xi) : Kv]. \]

Hence equality holds everywhere; in particular, \(K(a) = L\). Also, we obtain that \(1, a, \ldots, a^n\) is a basis of \(K(a)/K\) with the residues \(1, av, \ldots, a^nv\) linearly independent over \(Kv\). Hence if we write an arbitrary element of \(K(a)\) as \(\sum_{i=0}^n c_ia^i\) with \(c_i \in K\), then

\[ v \sum_{i=0}^n c_ia^i = \min_{0 \leq i \leq n} vci. \]

Thus, for the sum to have non-negative value, all \(c_i\) must have non-negative value. Since \((K, v)\) is a gdr field, we then have \(d_i \in K\) such that \(c_i \equiv d_i^p \mod p\mathcal{O}_K\). So we obtain from Lemma 2.16 that

\[ \sum_{i=0}^n c_ia^i \equiv \sum_{i=0}^n d_i^p(b^p)^i \equiv \left(\sum_{i=0}^n d_i^p\right)^p \mod p\mathcal{O}_L, \]

where the last equivalence holds by part 1) of Lemma 2.16. This shows that \((L, v)\) is a gdr field.

In order to prove part 2), we take a prime \(q\) different from \(p\) and a finite extension \((L/K, v)\) such that \(L = K(a)\) with \(a^q \in K\), \(\alpha := va \notin vK\) and \(q = (vL : vK)\). We obtain that \([K(a) : K] = q = (vK(a) : vK)\). As \(p\) and \(q\) are coprime, also \(pva = va^p\) generates \(vK(a)\) over \(vK\), and \(K(a) = K(a^p)\). So \(1, a^p, \ldots, a^{p(q-1)}\) is a basis of \(K(a)/K\) with \(v1, va^p, \ldots, va^p^{p(q-1)}\) belonging to distinct cosets of \(vK\). Hence if we write an arbitrary element \(b\) of \(K(a)\) as \(b = \sum_{i=0}^{q-1} c_ia^{pi}\) with \(c_i \in K\), then

\[ v \sum_{i=0}^{q-1} c_ia^{pi} = \min_{0 \leq i \leq q} vci + ivc_p. \]
Assume that the sum has non-negative value. Then all $c_i a^{p^i}$ must have non-negative value. But for $i > 0$, this does not imply that $vc_i \geq 0$; we only know that $vc_i a^{p^i} > 0$ since $iva^p \notin vK$, whence $va^p > -vc_i$.

Suppose that $va$ is not equivalent to an element in $vK$ modulo $(vL)_{vp}$. Then the same holds for $vc_i + piva$ in place of $va$, for $1 \leq i < q$, so that $vc_i a^{p^i} \notin (vL)_{vp}$. In this case, $b$ is equivalent to $c_0$ modulo $pO_L$. Since $(K, v)$ is a gdr field, there is $d_0 \in K$ such that $b \equiv c_0 \equiv d_0^p \mod pO_K$. Hence we may now assume that $va$ is equivalent to an element $\delta \in vK$ modulo $(vL)_{vp}$. We choose $d \in K$ with $vd = \delta$ and replace $a$ by $a/d$, so from now on we can assume that $va \in (vL)_{vp}$.

As $(K, v)$ is a gdr field, $(vK)_{vp}$ is $p$-divisible by part 2) of Lemma 6.2. It follows that $p(vK)_{vp}$ lies dense in $(vL)_{vp}$ and thus there is $b_i \in K$ such that $−vc_i ≤ p vb_i ≤ va^{p^i}$, whence $vc_i b_i^{p^i} ≥ 0$ and $vb_i^{−p} a^{p^i} ≥ 0$. Again since $(K, v)$ is a gdr field, there are $d_i \in K$ such that $c_i b_i^{p^i} \equiv d_i^p \mod pO_K$. So we obtain that

$$\sum_{i=0}^{q-1} c_i a^{p^i} \equiv \sum_{i=0}^{q-1} (c_i b_i^{p^i}) (b_i^{−p} a^{p^i}) = \sum_{i=0}^{q-1} d_i^p b_i^{−p} a^{p^i} \equiv \left(\sum_{i=0}^{q-1} d_i b_ia^i\right)^p \mod pO_L,$$

where the last equivalence holds by part 1) of Lemma 2.16. □

**Proof of Theorem 1.4:**

The case of residue characteristic 0 is trivial, so we assume that char $Kv = p > 0$. It has been proven already in Lemma 6.10 that if $(K^r, v)$ is a gdr field, then so is $(K, v)$, and if $(K^r, v)$ is a semitame field, then so is $(K, v)$. Let us now assume that $(K, v)$ is a gdr field; we aim to show that so is $(K^r, v)$.

First we consider the case of equal characteristic $p > 0$. Then by part 3) of Theorem 1.2, $(K, v)$ is a deeply ramified field. Hence by Lemma 6.7 also $(K^r, v)$ is a deeply ramified field and thus a gdr field.

Now we consider the case of a gdr field $(K, v)$ of mixed characteristic with char $Kv = p > 0$. In this part of the proof we will freely make use of facts from ramification theory; for details, see [6, 7, 17].

We let $L$ be a maximal extension of $K$ inside of $K^r$ that is again a gdr field; since the union over an ascending chain of gdr fields is again a gdr field, $L$ exists by Zorn’s Lemma.

First we will show that $(L, v)$ is henselian. As in Proposition 1.3, we decompose $v = v_0 \circ v_p \circ \tau$, where $v_0$ is the finest coarsening of $v$ that has residue characteristic 0, $v_p$ is a rank 1 valuation on $Kv_0$, and $\tau$ is the valuation induced by $v$ on the residue field of $v_p$ (which is of characteristic $p$). The valuations $v_0$ and $\tau$ may be trivial. As $K^r|K$ is algebraic, the restrictions of the respective valuations to any intermediate field of $K^r|K$ and the respective residue fields have the same properties. On any of these intermediate fields, $v$ is henselian if and only if $v_0$, $v_p$ and $\tau$ are.

Suppose that $v_0$ is not henselian on $L$. As $(K^r, v)$ is henselian, so is $(K^r, v_0)$ which therefore contains a henselization $L^{h(v_0)}$ of $L$ with respect to $v_0$. As henselizations are immediate extensions, we know that $L^{h(v_0)v_0} = Lv_0$; by Proposition 1.3, $(Lv_0, v_p)$ is a gdr field. Using the same proposition again, we find that also $(L^{h(v_0)}, v_0)$ is a gdr field. By the maximality of $L$ we conclude that $L^{h(v_0)} = L$, so $v_0$ is henselian on $L$.

Next, suppose that $v_p$ is not henselian on $Lv_0$. As $(K^r, v)$ is henselian, so is $(K^r v_0, v_p)$ which therefore contains a henselization $L^{h(v_p)}_{v_0}$ of $Lv_0$ with respect to
v_p. From Proposition 1.3 we infer that (Lv_0, v_p) is a gdr field. As its rank is 1, its henselization lies in its completion. Hence by part 4) of Lemma 6.1, (Lv_0^{h(v_p)}, v_p) satisfies (DRv_r). Since (DRv_p) holds in (Lv_0, v_p), it also holds in (Lv_0^{h(v_p)}, v_p), so the latter is a gdr field. The extension Lv_0^{h(v_p)}Lv_0 is separable-algebraic, so we can use Hensel’s Lemma to find an extension L' of L within K' such that L'v_0 = Lv_0^{h(v_p)}.

Using Proposition 1.3 again, we find that (L', v) is a gdr field. Hence L' = L by the maximality of L, that is, Lv_0 = L_0^{h(v_r)} showing that (Lv_0, v_p) is henselian.

Finally, suppose that σ is not henselian on Lv_0v_p. As (K', v) is henselian, so is (K'v_0v_p, v) which therefore contains a henselization Lv_0v_p^{h(σ)} of Lv_0v_p with respect to σ. Suppose that Lv_0v_p^{h(σ)}Lv_0v_p is nontrivial, so it contains a finite separable subextension. Using Hensel’s Lemma, we lift it to a subextension F[L of K'|L such that [F : L] = [Fv_0v_p : Lv_0v_p]. By what we have shown already, (L, v_Lv_p) is henselian, and by definition it is of mixed characteristic. Therefore, we can employ part 1) of Lemma 6.11 to deduce that (F, v_Lv_p) is a gdr field. By Proposition 1.3, also (F, v) is a gdr field. This contradiction to the maximality of L shows that Lv_0v_p^{h(σ)} = Lv_0v_p, that is, (Lv_0v_p, v) is henselian. Altogether, we have now shown that (L, v) is henselian.

The residue field of K' is the separable-algebraic closure of Kv. Suppose that Lv is not separable-algebraically closed, so it admits a finite separable-algebraic extension. Using Hensel’s Lemma, we lift it to a subextension F[L of K'|L such that [F : L] = [Fv : Lv]. Again by part 1) of Lemma 6.11, (F, v) is a gdr field. This contradiction to the maximality of L shows that Lv is separable-algebraically closed.

The value group of K' is the closure of vK under division by all primes other than p. Suppose that vL ≠ vK'. Then there is some prime q ≠ p and a ∈ K' with va ∉ vK and q = (vL : vK). By part 1) of Lemma 6.11, also (L(a), v) is a gdr field, which again contradicts the maximality of (L, v). We conclude that vL = vK'.

By what we have shown, Lv = K'v and vL = vK'. Since K ⊆ L ⊆ K', we know that K' = L', so the fact that (L'|L, v) is a tame extension together with the equality of the value groups and residue fields implies that L = L' = K'. We have proved that (K', v) is a gdr field.

Assume now that (K, v) is a semitame field. Then by part 1) of Theorem 1.2, (K, v) is a gdr field. As we have shown above, it follows that the same is true for (K', v). Since vK is p-divisible, vK' is p-divisible too. Hence by definition, (K', v) is a semitame field.

Proof of Theorem 1.5:
1) Assume that (K, v) is a gdr field. The assertions on vK and Kv have been proven in Lemma 6.2. Further, by Proposition 6.9, (K, v) is an independent defect field.

For the converse, we may assume that char Kv > 0 since every valued field with residue characteristic 0 is a semitame field. Now our assertion is the content of part 1) of Proposition 6.4.

2) The assertion is trivial if char Kv = 0, so we may assume that char Kv > 0.

First, we assume that (K, v) is a semitame field. Then by part 1) of Lemma 6.2, Kv is perfect. Since also vK is p-divisible by assumption, equation (1) shows that
every uv-extension \((L|K,v)\) of degree \(p\) of \((K,v)\) satisfies \(vL : vK = 1\), so it either has defect \(p\), or \([Lv : Kv] = p\) with \(Lv|Kv\) a separable extension. In the latter case, the extension has no defect and is tame. Otherwise, it is a defect extension of degree \(p\). Then, as \((K,v)\) is a gdr field by Theorem 1.2, part 1) of our theorem shows that it must be an independent defect extension.

The converse is the content of part 2) of Proposition 6.4.

\[\square\]

**Proof of Theorem 1.8.** By [8, Corollary 6.6.16 (i)], every algebraic extension of a deeply ramified field is again a deeply ramified field. For the convenience of the reader, we gave the easy proof for the case of deeply ramified fields of positive characteristic in Lemma 6.7, and for extensions within the absolute ramification field, it can be deduced from Theorem 1.4 as follows. If \((L|K,v)\) is an extension within \(K^r\), then \(L^r = K^r\); if \((K,v)\) is a deeply ramified field, then it is a gdr field and Theorem 1.4 shows that also \((L,v)\) is a gdr field. On the other hand, condition (DRvg) is preserved under algebraic extensions, so \((L,v)\) is a deeply ramified field.

It remains to deal with semitame fields and with gdr fields. For semitame fields the proof is immediate as they are just the deeply ramified fields with \(p\)-divisible value groups. Both properties are preserved under algebraic extensions.

Now take a gdr field \((K,v)\). Every valued field or residue characteristic 0 is a gdr field, so we may assume that \(\text{char } K^v = p > 0\). If \((K,v)\) is of equal positive characteristic, then it is a deeply ramified field by part 3) of Theorem 1.2 and has already been dealt with above. Thus we assume that \((K,v)\) is of mixed characteristic. With \(v_0\) and \(w\) as in Lemma 6.5 we know from that lemma that \((Kv_0,w)\) is a gdr field. Hence by part 2) of Theorem 1.2, it is a semitame field. Now take any algebraic extension \((L|K,v)\). Then also \((Lv_0|Kv_0,w)\) is an algebraic extension, and by what we have shown already, \((Lv_0,w)\) is again a semitame field, and thus again by part 2) of Theorem 1.2 a gdr field. Hence by Lemma 6.5, \((L,v)\) is a gdr field.

\[\square\]

**Proof of Theorem 1.7.** Every algebraically maximal field with residue characteristic 0 is henselian and defectless. Therefore, we may assume that \((K,v)\) is an algebraically maximal gdr field of positive residue characteristic \(p\). If \(\text{char } K = p\), then by part 3) of Theorem 1.2, \((K,v)\) is dense in its perfect hull. But as it is algebraically maximal, this extension must be trivial, i.e., \(K\) is perfect.

Take an absolute ramification field \((K^r,v)\) of \((K,v)\) and a finite tower \(K^r = L_0 \subset L_1 \subset \ldots \subset L_n\) of extensions of degree \(p\) over \(K^r\). By Theorem 1.8, every \((L_i,v)\) is a gdr field. Hence Theorem 1.5 yields that among the extensions \((L_i|L_{i-1},v)\), \(1 \leq i \leq n\), every separable defect extension is independent. Now Proposition 4.7 shows that \((K,v)\) is henselian and defectless.

\[\square\]

**Proof of Proposition 1.9.** Part 1) is the content of Corollary 3.12. Part 2) is shown in [14, Corollary 4.6].

**Proof of Proposition 1.1.** It is well known that first order properties of the value group \(vK\) of a valued field \((K,v)\) can be encoded in \((K,v)\) in the language of valued fields. The axiomatization for (DRvp) and (DRst) is straightforward. Further, (DRvg) holds in an ordered abelian group \((G,\prec)\) if and only if for each positive \(\alpha \in G\) there is \(\beta \in G\) such that \(2\beta \leq \alpha \leq 3\beta\).

If \((K,v)\) is of mixed characteristic, then (DRvr) is equivalent to the surjectivity of (57), and this in turn holds if and only if for each \(a \in K\) with \(va \geq 0\) there is
$b \in K$ such that $v(a - b^p) \geq vp$. Hence the classes of semitame, deeply ramified and gdr fields of mixed characteristic are first order axiomatizable.

If $(K, v)$ is of equal positive characteristic, then part 3) of Theorem 1.2 shows that semitame, deeply ramified and gdr fields form the same class, which can be axiomatized by saying that $(K^p, v)$ is dense in $(K, v)$, or in other words, for every $a \in vK$ and every $a \in K$ there is $b \in K$ such that $v(a - b^p) \geq \alpha$.

In the case of equal characteristic 0, $(DRvp)$, $(DRvr)$ and $(DRst)$ are trivial and all valued fields are semitame and gdr fields, while the class of deeply ramified fields consists of those which satisfy $(DRvg)$. 

\[ \square \]

7. TWO CONSTRUCTIONS

In this section we give constructions for independent and dependent defect extensions in mixed characteristic. First, we show how to construct a semitame field with an independent defect extension of degree $p$.

**Theorem 7.1.** Consider the field $\mathbb{Q}_p$ of $p$-adic numbers together with the $p$-adic valuation $v_p$. Set $a_0 := p$ and by induction, choose $a_i \in \mathbb{Q}_p$ such that $a_i^p = a_{i-1}$ for $i \in \mathbb{N}$. Then $K := \mathbb{Q}_p(a_i | i \in \mathbb{N})$ together with the unique extension of $v$ is a semitame field and hence a deeply ramified field.

Further, take $\vartheta \in \mathbb{Q}_p$ such that

$$\vartheta^p - \vartheta = \frac{1}{p}.$$ 

Then $(K(\vartheta)|K, v)$ is an independent defect extension of degree $p$.

**Proof.** By choice of the $a_i$, $\frac{v_p}{p} = va_i \in \mathbb{Q}_p(a_i)$. Therefore,

$$p^i \leq (vQ_p(a_i) : vQ_p) \leq (vQ_p(a_i) : vQ_p)[Q_p(a_i)v : Q_pv] \leq [Q_p(a_i) : Q_p] \leq p^i.$$ 

Hence equality holds everywhere, and $[Q_p(a_i)v : Q_pv] = 1$. We thus obtain that $vQ_p(a_i) = \frac{1}{p^i}vQ_p$ and $Q_p(a_i)v = Q_pv$. Consequently,

$$vK = \bigcup_{i \in \mathbb{N}} vQ_p(a_i) = \frac{1}{p^\infty} \mathbb{Z} \quad \text{and} \quad Kv = \mathbb{Q}_p v.$$ 

This shows that $vK$ is $p$-divisible and that its only proper convex subgroup is $H = \{0\}$. In order to show that $(K, v)$ is a semitame field it remains to show that it satisfies $(DRvr)$.

Take $b \in \mathcal{O}_K$. Then $b \in \mathbb{Q}_p(a_i)$ for some $i \in \mathbb{N}$ and we can write:

$$b \equiv \sum_{j=0}^{n} c_j a_i^j \mod p\mathcal{O}_{\mathbb{Q}_p(a_i)}$$

with $n < [Q_p(a_i) : Q_p] = p^i$ and $c_j \in \{0, \ldots, p-1\}$. Since $c_j^p \equiv c_j \mod p\mathcal{O}_{\mathbb{Q}_p}$ and $a_{i+1}^p = a_i$, we can compute:

$$\left(\sum_{j=0}^{n} c_j a_i^j\right)^p \equiv \sum_{j=0}^{n} c_j^p(a_i^{pj})^j \equiv \sum_{j=0}^{n} c_j a_i^j \equiv b \mod p\mathcal{O}_{\mathbb{Q}_p(a_i)}.$$ 

In view of part 2) of Lemma 6.1, this proves that $(K, v)$ satisfies $(DRvr)$ and is therefore a semitame field.
Now we take $\vartheta$ as in the assertion of our theorem. Our first aim is to show that the extension $(K(\vartheta)|K,v)$ is nontrivial and immediate. For each $i \in \mathbb{N}$, we set

$$b_i = \sum_{j=1}^{i} \frac{1}{a_j} \in K(a_i)$$

and compute, using part 2) of Lemma 2.16:

$$(\vartheta - b_i)^p - (\vartheta - b_i) \equiv \vartheta^p - \sum_{j=1}^{i} \frac{1}{a_j^p} - \vartheta + \sum_{j=1}^{i} \frac{1}{a_j} \mod \mathcal{O}_{\mathbb{Q}_p(a_i)}.$$  

It follows that $v(\vartheta - b_i) < 0$ and

$$\frac{-vp}{p^i} = v\frac{1}{a_i} = \min\{pv(\vartheta - b_i), v(\vartheta - b_i)\} = pv(\vartheta - b_i),$$

whence

$$(65) \quad v(\vartheta - b_i) = -\frac{vp}{p^i+1}.$$ 

We have that

$$p \leq (v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i)) \leq (v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i))|\mathbb{Q}_p(a_i)\mathbb{Q}_p(a_i)v| \leq [K(a_i, \vartheta) : K] \leq p.$$ 

Thus equality holds everywhere and we have that $(v\mathbb{Q}_p(a_i, \vartheta) : v\mathbb{Q}_p(a_i)) = p$ as well as $\mathbb{Q}_p(a_i, \vartheta)v = \mathbb{Q}_p(a_i)v = \mathbb{Q}_p$. The former shows that $v\mathbb{Q}_p(a_i, \vartheta) = \frac{1}{p^{i+1}}v\mathbb{Q}_p$, which implies that for all $i \in \mathbb{N}$, $\vartheta \notin \mathbb{Q}_p(a_i)$. Hence $\vartheta \notin K$, and we have:

$$vK(\vartheta) = \bigcup_{i \in \mathbb{N}} v\mathbb{Q}_p(a_i, \vartheta) = \frac{1}{p^{i+1}}\mathbb{Z} = K'v \quad \text{and} \quad K(\vartheta)v = \mathbb{Q}_pv = vK.$$ 

This shows that $(K(\vartheta)|K,v)$ is nontrivial and immediate, as asserted. As we have already proven that $(K,v)$ is a semitame field, it follows from Theorem 1.5 that the extension has independent defect. \hfill $\square$

What we have just presented is the mixed characteristic analogue of the following example given in, e.g., [12, Example 12]. Take $K$ to be the perfect hull of $\mathbb{F}_p((t))$, that is, $K = \mathbb{F}_p((t))(t^{1/p^i} | i \in \mathbb{N})$. Take $v$ to be the $t$-adic valuation on $\mathbb{F}_p((t))$; since it is henselian, there is a unique extension to $K$ and $(K,v)$ is again henselian. The Artin-Schreier extension $(K(\vartheta)|K,v)$ generated by a root $\vartheta$ of the polynomial $X^p - X - \frac{1}{p} \in \mathbb{Z}$ is nontrivial and immediate. As $K$ is perfect, it does not admit any dependent Artin-Schreier defect extension, so the extension $(K(\vartheta)|K,v)$ has independent defect. In fact, $(K,v)$ is a semitame field.

We turn to the construction of a dependent defect extension of degree $p$. The following is an analogue of Example 3.22 of [15].

**Theorem 7.2.** Set $a_0 := -\frac{1}{p} \in \mathbb{Q}_p$ and by induction, choose $a_i \in \mathbb{Q}_p^\times$ such that $a_i^p - a_i = -a_{i-1}$ for $i \in \mathbb{N}$. Consider $K := \mathbb{Q}_p(a_i \mid i \in \mathbb{N})$ together with the unique
extension of $v$. Then $vK$ is $p$-divisible. Further, take $\eta \in \widetilde{\mathbb{Q}}_p$ such that

$$\eta^p = \frac{1}{p}.$$ 

Then $(K(\eta)|K,v)$ is a dependent defect extension of degree $p$. Consequently, $(K,v)$ does not satisfy $(DR_v)$. 

**Proof.** By induction on $i$, we again obtain that $va_i = \frac{1}{p^i}vp$. As in Theorem 7.1 we deduce that $vQ_p(a_i) = \frac{1}{p^i}vQ_p$ and for $K := Q_p(a_i \mid i \in \mathbb{N})$ we obtain that $vK = \frac{1}{p^i}vQ_p$ and $Kv = Q_p$. In particular, the only proper convex subgroup of $vK$ is $H = \{0\}$. We set

$$b_i = \sum_{j=1}^{i} a_j \in K(a_i)$$

and compute, using part 2) of Lemma 2.16:

$$(\eta - b_i)^p = \eta^p - \sum_{j=1}^{i} a_j^p = \frac{1}{p} + \sum_{j=1}^{i} (a_{j-1} - a_j)$$

$$= \frac{1}{p} + \sum_{j=0}^{i-1} a_j - \sum_{j=1}^{i} a_j = -a_i \mod Q_p(a_i).$$

It follows that

$$vp = va_i = pv(\eta - b_i),$$

whence

$$v(\eta - b_i) = \frac{vp}{p^{i+1}}.$$ 

As in the proof of Theorem 7.1 we deduce that $(K(\eta)|K,v)$ is nontrivial and immediate. It remains to show that its defect is dependent.

From (66) we see that dist $(\eta, K) \geq 0^–$. Suppose that dist $(\eta, K) > 0^–$. Then there is an element $c \in K$ such that $v(\eta - c) > v(\eta - b_i)$ for every $i \in \mathbb{N}$. Hence,

$$(67) \quad v(c - b_i) = \min\{v(\eta - c), v(\eta - b_i)\} = v(\eta - b_i) = \frac{vp}{p^{i+1}}.$$ 

Since $c \in K$, we have that $c \in Q_p(a_i)$ for some $i \in \mathbb{N}$. Then we obtain that $c - b_i \in Q_p(a_i)$, but equation (67) shows that $v(c - b_i) = \frac{vp}{p^{i+1}} \notin Q_p(a_i)$, a contradiction. Therefore, dist $(\eta, K) = 0^–$. 

Since $(K(\eta)|K,v)$ is immediate, there is $d \in K$ such that $d\eta$ is a 1-unit. We have that $vd = -v\eta = \frac{vp}{p}$ and

$$\text{dist} (d\eta, K) = vd + 0^- = \frac{vp}{p} + 0^- < \frac{vp}{p-1} + 0^-.$$ 

As $K(\eta) = K(d\eta)$, this shows that $(K(\eta)|K,v)$ is a dependent defect extension of degree $p$. 

This second example shows that in order to obtain a semitame field it is not sufficient to just make the value group $p$-divisible and the residue field perfect, not even if one starts from a discretely valued field.
References


Institute of Mathematics, University of Silesia in Katowice, Bankowa 14, 40-007 Katowice, Poland
E-mail address: anna.blaszczok@us.edu.pl

Institute of Mathematics, University of Szczecin, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: fvk@usz.edu.pl