MORI DREAM SURFACES ASSOCIATED WITH CURVES WITH ONE PLACE AT INFINITY

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Abstract. We study a class of rational surfaces associated to curves with one place at infinity and explicitly describe generators of the Cox ring and global sections of line bundles on these surfaces. In particular, we show that their Cox rings are finitely generated, i.e. they are Mori dream spaces. We also compute their *global Zariski semigroups at infinity* (consisting of line bundles which have no base points 'at infinity') and *global Enriques semigroups* (generated by closures of curves in $\mathbb{C}^2$). In particular, we show that the global Zariski semigroups at infinity and Enriques semigroups of surfaces corresponding to pencils which are *equisingular at infinity* are isomorphic, which answers a question of [CPRL02]. We also give an *effective* algorithm to determine if a (rational) surface *admits systems of numerical curvettes* in the sense of [CPR05].

1. Introduction

Pencils of curves on $\mathbb{P}^2$ form a classical topic in algebraic geometry. A particular class $\mathcal{V}_1$ of pencils that appeared in numerous works (see e.g. [Moh74], [Sat77], [ABCN96], [Suz99], [ABCN00], [CPRL02], [GM04], [Mon07]) consists of *pencils at infinity* corresponding to *curves with one place at infinity*, i.e. pencils $V := \mathbb{C}\langle F, H^d\rangle$[1], where H is the equation of a straight line L (*the line at infinity*), and F is a polynomial of degree d such that the curve $C := \{F = 0\}$ has *one place at infinity* (i.e. C intersects L only at one point P and the germ of C at P is irreducible). [CPR05] studied a more general class $\mathcal{V} \supseteq \mathcal{V}_1$ of pencils where F is of the form $F_1^{a_1} \cdots F_k^{a_k}$, where each F_j, $1 \leq j \leq k$, defines a curve with one place at infinity. In this article we continue the study of [CPRL02] and [CPR05] on line bundles on the surfaces X_V resulting from (minimal) resolutions of basepoints of pencils $V \in \mathcal{V}$ and show that they are *Mori dream spaces*, i.e. their Cox rings are finitely generated. Finite generation of Cox rings of surfaces have been extensively studied (see e.g. [BP04], [HT04], [TVAV09], [AHL10], [Ott13]), but we would like to point out, as it was noted in [CPRL02], that the surfaces X_V are unlike 'typical examples' of Mori dream spaces in the sense that in general the anti-canonical bundles on these surfaces are far from being nef, so that the Mori theory does not apply. Cox rings of X_V for a special class of $V \in \mathcal{V}$ were in fact shown to be finitely generated in [GM05]; however, our method is (different and) *explicit* in the following sense:

- we explicitly describe the generators of the Cox ring, and
- compute a (non-canonical) basis of the global sections of each line bundle on these surfaces.

This enables us to tackle the sharper problem of characterizing some interesting classes of line bundles on these surfaces:

[1] We write $\mathbb{C}\langle F_1, \ldots, F_k\rangle$ to denote the vector space over $\mathbb{C}$ spanned by $F_1, \ldots, F_k$.

The main problem studied in [CPRL02] was to achieve (for surfaces of the form X_V, $V \in \mathcal{V}_1$) global analogues of the work of Enriques and Zariski, i.e. to characterize line bundles on X_V which are generated by global sections (Zariski), and those which have global sections supported on (the strict transform of) curves that properly intersect the line at infinity and the exceptional curves (Enriques). We completely solve this problem and give an explicit description of the global Enriques and Zariski semigroups. In particular we answer the question asked in [CPRL02] about these semigroups:

Question 1.1 ([CPRL02])**.** Let $V_i := \langle F_i, H^d \rangle \in \mathcal{V}_1$, $i = 1, 2$. Identify $\mathbb{P}^2 \setminus \{H = 0\}$ with $\mathbb{C}^2$, and assume V_1 and V_2 are *equisingular at infinity*, i.e. the curves C_i defined by F_i (of the same degree d) have equisingular germs 'at infinity' i.e. at the the points $O_i := C_i \cap \{H = 0\}$, $1 \le i \le 2$. Under what conditions are the global Enriques semigroups and/or global Zariski semigroups on X_{V_i}'s isomorphic?

We show that the answer to Question 1.1 is "always" (Corollary 6.4). In fact our results extend to the larger class $\mathcal{V}$: if $V_1, V_2 \in \mathcal{V}$ are equisingular at infinity, then Corollary 6.6 states that

(1) the global Enriques semigroups of X_{V_1} and X_{V_2} are isomorphic.
(2) the 'Zariski semigroups at infinity' (consisting of divisors which have no base point on $X_V \setminus \mathbb{C}^2$) of X_{V_1} and X_{V_2} are isomorphic.

Pick $V = \mathbb{C}\langle F_1^{a_1} \cdots F_k^{a_k}, H^d \rangle \in \mathcal{V}$. Then X_V can be considered naturally as a compactification of $\mathbb{C}^2$ (by identifying $\mathbb{C}^2$ with the complement of $\{H = 0\}$ in $\mathbb{P}^2$). Set $f_i := F_i|_{\mathbb{C}^2}$, $1 \le i \le k$. As in [CPR05] and [CPRL02], we show that the divisors D_{ij} corresponding to *approximate roots* (introduced by Abhyankar and Moh [AM73]) of the f_i's play a crucial role in the structure of line bundles on X_V: they essentially generate the Cox ring of X_V (Theorem 4.5 and Remark 4.2) and *tropically generates* the global Enriques semigroup $P^{st}(X_V)$:

Definition 1.2. Given $S \subseteq \mathbb{Z}^k$, the *tropical closure* of S is the smallest semigroup $\bar{S} \subseteq \mathbb{Z}^k$ containing S which is also closed under taking (coordinatewise) maximum, i.e. if $\alpha_i := (\alpha_{i1}, \ldots, \alpha_{ik}) \in \bar{S}$, $1 \le i \le 2$, then $\max\{\alpha_1, \alpha_2\} := (\max\{\alpha_{11}, \alpha_{21}\}, \ldots, \max\{\alpha_{1k}, \alpha_{2k}\}) \in \bar{S}$. We say that S *tropically generates* $T \subseteq \mathbb{Z}^k$ iff $T = \bar{S}$.

Theorem 1.3 (follows from Corollary 5.3 and Remark 4.2)**.** *Let $\Gamma_0, \ldots, \Gamma_N$ be the irreducible components of $X_V \setminus \mathbb{C}^2$. Consider the corresponding identification of* $\operatorname{Pic} X_V$ *with $\mathbb{Z}^{N+1}$. Then $\{D_{ij}\}_{i,j}$ tropically generates $P^{st}(X_V) \subseteq \mathbb{Z}^{N+1}$.*

The global Zariski semigroup at infinity $\tilde{P}_\infty(X_V)$ can also be expressed in terms of (products of) the approximate roots of the f_i's. The statements however turn out to be a bit more technical, and we refer the reader to Theorem 5.7 for a precise description of $\tilde{P}_\infty(X)$ for surfaces X in a more general class $\mathcal{S}_{pol}$, and to Corollary 6.3 for a version customized for X_V for $V \in \mathcal{V}_1$.

Our results on Cox ring and global Enriques and Zariski semigroups are valid for a class $\mathcal{S}_{pol}$ of surfaces which (strictly) contains all X_V, $V \in \mathcal{V}$. The class $\mathcal{S}_{pol}$ appears naturally in the study of compactifications of $\mathbb{C}^2$. Indeed, if X is a compactification of $\mathbb{C}^2$ then to each irreducible curve Γ 'at infinity' on X (i.e. Γ is an irreducible component of $X_V \setminus \mathbb{C}^2$) one can associate a (finite) sequence of elements in $\mathbb{C}[x, x^{-1}, y]$ which describe the order of vanishing of rational functions along Γ; these are called *key forms* associated to C. The key forms are global variants of *key polynomials* of valuations introduced by MacLane [Mac36], and they can be used to determine *effectively* various properties of the associated (divisorial) valuation, see e.g. [Mon13a], [Mon13b], [MN13]. In particular, [Mon13b, Theorem 1.8] uses key forms to characterize a class $\mathcal{C}$ of divisorial valuations on $\mathbb{C}(x,y)$ whose *skewness* (introduced in [FJ04]) can be 'read from' the slope of an extremal ray in the cone of curves on an associated compactification of $\mathbb{C}^2$. On the other hand, [CPR05] introduced the class S_{num} of compactifications of $\mathbb{C}^2$ which *admit systems of numerical curvettes*. Theorem 3.2 shows that for a compactification X of $\mathbb{C}^2$, $X \in S_{num}$ iff the divisorial valuation associated to each curve at infinity on X is in $\mathcal{C}$; in particular it gives an *effective* algorithm to determine if $X \in S_{num}$.

The class S_{pol}, which is the central topic of this article, is the subset of S_{num} for which the key forms associated to the curvess at infinity are *polynomials*. In particular S_{pol} contains all X_V, $V \in \mathcal{V}$ (Proposition 3.8). A geometric interpretation of the class S_{pol} comes from the following observation: if X is a compactification of $\mathbb{C}^2$ which dominates $\mathbb{P}^2$, then $X \in S_{pol}$ iff X admits a system of numerical curvettes which 'essentially come from affine curves' (Theorem 3.4).

The organization of this article is follows: Section 2 and the appendix (Section 7) contains some background regarding key forms, *semidegrees* (which are negatives of divisorial valuations), and associated *degree-wise Puiseux series*. In Section 3 we introduce the classes $S^+_{pol} \subseteq S_{pol} \subseteq S_{num}$ of surfaces, characterize them in terms of associated 'systems of curvettes', and show that each X_V for $V \in \mathcal{V}$ (resp. $V \in \mathcal{V}_1$) is a member of S_{pol} (resp. S^+_{pol}). Theorem 4.5 in Section 4 gives an explicit description of Cox rings of surfaces in S_{pol} and in Section 5 we describe global Enriques semigroups and global Zariski semigroups at infinity of surfaces in S_{pol}. Finally, in Section 6 we reformulate the results of Sections 4 and 5 for surfaces of the form X_V for $V \in \mathcal{V}_1$ in terms of the approximate roots of the associated polynomial(s).

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2. Background

2.1. Semidegrees and degree-wise Puiseux series. Let X be a normal complete algebraic surface containing $U := \mathbb{C}^2$, and Γ be an irreducible curve at infinity, i.e. an irreducible components of $X \setminus U$. Let δ be the *semidegree* on $\mathbb{C}(X)$ determined by Γ, i.e. $\delta(f)$, where $f \in \mathbb{C}(X)$, is the order of pole of f along Γ (in other words, δ is the negative of the *divisorial discrete valuation* on $\mathbb{C}(X)$ associated to Γ). Choose coordinates (x,y) on U such that $\delta(x) > 0$ and let $\tilde{p} := \delta(x)$. Then there exists $\phi(x) \in \mathbb{C}[x^{1/\tilde{p}}, x^{-1/\tilde{p}}]$, and a rational number $r \in \frac{1}{\tilde{p}}\mathbb{Z}$ such that $r < \operatorname{ord}_x(\phi)$ and

$$\delta(f(x,y)) = \tilde{p}\deg_x\left(f(x,y)|_{y=\phi(x)+\xi x^r}\right) \tag{1}$$

for all $f \in \mathbb{C}(x,y)$, where ξ is an indeterminate ([Mon11, Theorem 1.2]). We call $\tilde{\phi}(x) := \phi(x) + \xi x^r$ the *generic degree-wise Puiseux series* associated to δ. The field of *degree-wise Puiseux series* in x is

$$\mathbb{C}\langle\langle x \rangle\rangle := \bigcup_{p=1}^{\infty} \mathbb{C}((x^{-1/p})) = \left\{ \sum_{j \le k} a x^{j/p} : k, p \in \mathbb{Z},\ p \ge 1 \right\},$$

where for each integer $p \geq 1$, $\mathbb{C}((x^{-1/p}))$ denotes the field of Laurent series in $x^{-1/p}$. We refer to Section 7 for some notions, e.g. *conjugacy, Puiseux pairs*, e.t.c. of degree-wise Puiseux series. The usual factorization of polynomials in terms of Puiseux series (see e.g. [CA00, Section 1.5]) implies the following

Theorem 2.1. *Let $f \in \mathbb{C}[x,y]$. Then there are unique (up to conjugacy) degree-wise Puiseux series $\phi_1, \ldots, \phi_k$, a unique non-negative integer m and $c \in \mathbb{C}^*$ such that*

$$f = cx^m \prod_{i=1}^{k} \prod_{\substack{\psi_{ij} \text{ is a con-} \\ \text{jugate of } \psi_i}} (y - \psi_{ij}(x))$$

The following proposition, which is a straightforward corollary of [Mon11, Proposition 4.2], illustrates a relation between degree-wise Puiseux factorization of polynomials and the behaviour at infinity of the curves they define:

Proposition 2.2. *Let X be a normal algebraic surface containing $\mathbb{C}^2$. Let $\Gamma_1, \ldots, \Gamma_N$ be the irreducible components of $X \setminus \mathbb{C}^2$ and δ_j, $1 \leq j \leq N$, be the semidegree on $\mathbb{C}(X)$ induced by Γ_j. Choose coordinates (x,y) on $\mathbb{C}^2$ such that $\delta_j(x) > 0$ for each j, $1 \leq j \leq N$. For each j, $1 \leq j \leq N$, let $\tilde{\phi}_j(x,\xi) := \phi_j(x) + \xi x^{r_j}$ be the generic degree-wise Puiseux series associated to δ_j. Let $f \in \mathbb{C}[x,y]$ and C be the closure in X of the curve $\{f = 0\} \subseteq \mathbb{C}^2$.*

(1) *Assume C is proper and $C \cap \Gamma_j \cap \Gamma_k = \emptyset$ for all $j \neq k$. Then the degree-wise Puiseux factorization of f is of the form*

$$f = c \prod_{i=1}^{N} \prod_{j=1}^{n_i} \prod_{k=1}^{\tilde{q}_{ij}} (y - \psi_{ijk}(x)) \tag{2}$$

where $c \in \mathbb{C}^*$, $n_i := |C \cap \Gamma_i|$, and ψ_{ijk}'s are conjugates of some ψ_{ij} of the form

$$\phi_i(x) + \xi_{ij} x^{r_i} + l.d.t.$$

where $\xi_{ij} \in \mathbb{C}$ and l.d.t. stands for 'terms with lower degree' in x.

(2) *Assume in addition that each C intersects each Γ_j transversally (at points where both Γ_j and X are nonsingular). Then for all i,j, if $\xi_{ij} \neq 0$, then $\tilde{q}_{ij} = \delta_i(x)$ and $\psi_{ij} \in \mathbb{C}((x^{-1/\delta_i(x)}))$.*

2.2. **Key forms.** Let X, Γ, δ and $\tilde{\phi}(x,\xi) = \phi(x) + \xi x^r$ be as in Section 2.1. One can associate to δ a finite sequence of elements in $\mathbb{C}[x, x^{-1}, y]$ called the *key forms* of δ (see [Mon13a, Definition 3.17]). The sequence starts with $f_0 := x$, $f_1 := y$, and for each $n \geq 1$, the $(n+1)$-th key form f_{n+1} is an element in $\mathbb{C}[f_0, f_0^{-1}, f_1, f_2, \ldots, f_n]$ whose δ-value is smaller than the 'expected' value. An algorithm and detailed example for the computation of key forms of δ from ϕ and ω appears in [Mon13a, Section 3.3].

Example 2.3.

$\phi(x) + \xi x^{\omega}$	key forms
$\xi x^{p/q}$	x, y
$cx^{p/q} + \xi x^{\omega}$, $c \in \mathbb{C}^*$, $q > 0$, p, q rel. prime integers, $\omega < \frac{p}{q}$	$x, y, y^q - c^q x^p$
$x^{5/2} + x^{-3/2} + \xi x^{-5/2}$	$x, y, y^2 - x^5, y^2 - x^5 - 2x$
$x^{5/2} + x^{-1} + x^{-3/2} + \xi x^{-5/2}$	$x, y, y^2 - x^5, y^2 - x^5 - 2x^{-1}y, y^2 - x^5 - 2x^{-1}y - 2x$

Theorem 2.4 below sums up the properties of key forms that we need. The first assertion of Theorem 2.4 follows from defining properties of key forms ([Mon13a, Proposition 3.28]), the second assertion follows from [Mon13a, Proposition 4.2] and the third assertion is a corollary of [Mon13b, Theorem 1.4].

Theorem 2.4. *Let the key forms of δ be $f_0, \ldots, f_n$.*

1. *Let p be the polydromy order (see Definition 7.1) of $\phi(x)$. Then there is $\psi(x) \in \mathbb{C}((x^{-1/p}))$ such that:*

$$\psi(x) = \phi(x) + \text{terms with } x\text{-degree} \leq r,$$

$$f_n = \prod_{j=1}^{p} (y - \psi_j(x)), \quad \text{where } \psi_j\text{'s are the conjugates of } \psi. \tag{3}$$

2. *The following are equivalent:*
   - *(a) f_k is a polynomial for all k, $0 \leq k \leq n$.*
   - *(b) f_n is a polynomial.*
   - *(c) There exists a polynomial f with degree-wise Puiseux factorization of the form:*

$$f = \prod_{i=1}^{k} \prod_{\substack{\psi_{ij} \text{ is a con-}\\ \text{jugate of } \psi_i}} (y - \psi_{ij}(x)), \quad \text{where for each } i, \tag{4}$$
$$\psi_i = \phi(x) + \xi_i x^r + l.d.t., \qquad \xi_i \in \mathbb{C}.$$

3. *The following are equivalent:*
   - *(a) $\delta(f) \geq 0$ for all $f \in \mathbb{C}[x, y] \setminus \{0\}$.*
   - *(b) $\delta(f_n) \geq 0$.*
   - *(c) $\delta(f) \geq 0$ for some $f \in \mathbb{C}[x, x^{-1}, y]$ which satisfies (4).*

Proposition 2.5 ([Mon11, Propositions 4.2 and 4.7])**.** *Let δ be as in Theorem 2.4. Assume δ is* not *the usual degree in (x, y) coordinates, Then there exists a unique compactification X' of $\mathbb{C}^2$ such that*

1. *X' is projective and normal.*
2. *$X'_\infty := X' \setminus \mathbb{C}^2$ has two irreducible components C'_1, C'_2.*
3. *The semidegree on $\mathbb{C}[x, y]$ corresponding to C'_1 and C'_2 are respectively* deg *and δ.*

All singularities of X' are rational *(which implies in particular that all Weil divisors are $\mathbb{Q}$-Cartier). Let (C'_i, C'_j), $1 \leq i, j \leq 2$, denote the intersection number of C'_i and C'_j. Then*

4. *$(C'_2, C'_2) < 0$,*
5. *$(C'_1, C'_1) = -q\delta(f_{l+1})$, where q is a positive rational number,*
6. *Let $D'_2 := C'_1 - \frac{(C'_1, C'_1)}{(C'_1, C'_2)} C'_2$. Then $(C'_1, D'_2) = 0$ and $(C'_2, D'_2) = 1$.*

3. Surfaces admitting systems of numerical (semi-affine) curvettes

Definition 3.1. [CPR05] introduced the notion of surfaces *admitting systems of numerical curvettes*. Here we extend the scope of the definition: Let X be a normal complete algebraic surface containing $U := \mathbb{C}^2$ as a dense open subset, and let $\Gamma_1, \ldots, \Gamma_N$ be the irreducible components of $X \setminus U$. An effective Weil divisor Δ on X is called a *numerical Γ_i-curvette*, $1 \leq i \leq N$, if $(\Delta, \Gamma'_i) > 0$ and $(\Delta, \Gamma'_j) = 0$ for all $j \neq i$. We say that X *admits a system of* *numerical curvettes*, or equivalently, $X \in \mathcal{S}_{num}$, iff there exists a numerical Γ_i-curvette for all i, $1 \leq i \leq N$.

Theorem 3.2. *Let X be as in Definition 3.1. For each j, $1 \leq j \leq N$, let δ_j be the semidegree associated to Γ_j and f_j be the* last key form *of δ_j. Then the following are equivalent:*
(a) X admits a system of numerical curvettes.
(b) For each j, $1 \leq j \leq N$, either $\delta_j(f_j) \geq 0$ or f_j is a polynomial.

Proof. Choose a system of coordinates (x, y) on $\mathbb{C}^2 = U$ such that no δ_j is the degree in (x, y)-coordinates, and let $X_0 \cong \mathbb{P}^2$ be the usual compactification of U given by $(x, y) \hookrightarrow [x : y : 1]$. Let $\tilde{X}$ be the minimal normal surface containing U which dominates both X and X_0 (via morphisms induced by identification of U). Then $\tilde{X} \setminus U$ has $N + 1$ irreducible components $\tilde{\Gamma}_0, \ldots, \tilde{\Gamma}_N$, where $\tilde{\Gamma}_0$ (resp. $\tilde{\Gamma}_i$, $1 \leq i \leq N$,) is the strict transform of the line at infinity on X_0 (resp. Γ_i). It follows that the semidegree associated to $\tilde{\Gamma}_0$ (resp. $\tilde{\Gamma}_i$, $1 \leq i \leq N$,) is the degree in (x, y)-coordinates (resp. δ_i). Let $\pi : \tilde{X} \to X$ be the morphism induced by identification of U.

(a) $\Rightarrow$ (b): Assume X admits a system of numerical curvettes. Fix j, $1 \leq j \leq N$. Let $\Delta_j := D_j + \sum_{i=1}^N a_i \Gamma_i$ be a numerical Γ_j-curvette on X, where D_j is an effective Weil divisor such that $\operatorname{Supp}(D_j) \not\supset \Gamma_i$ for any i, and $a_i \geq 0$ for all i, $1 \leq i \leq N$. Let $\tilde{\Delta}_j := \pi^*(\Delta_j)$. Then $\tilde{\Delta}_j$ is of the form $\tilde{D}_j + \sum_{i=0}^N a_i \Gamma_i$, where $a_0 \geq 0$ and $\tilde{D}_j$ is the strict transform of D_j. Moreover, $\tilde{\Delta}_j$ is a numerical $\tilde{\Gamma}_j$-curvette on $\tilde{X}$. Let X_j be the surface X' constructed in Proposition 2.5 with $\delta = \delta_j$. Let $\pi_j : \tilde{X} \to X_j$ the natural birational morphism, and Γ'_0 (resp. Γ'_j) be the image of $\tilde{\Gamma}_0$ (resp. $\tilde{\Gamma}_j$) on X_j (note that $\Gamma'_0 = C_1$ and $\Gamma'_j = C_2$ in the notation of Proposition 2.5). Set $\Delta'_j := {\pi_j}_*(\tilde{\Delta}_j) = D'_j + a_0\Gamma'_0 + a_j\Gamma'_j$, where $D'_j := {\pi_j}_*(\tilde{D}_j)$.

Claim 3.2.1. $\Delta'_j \neq 0$. *Moreover, $\tilde{\Delta}_j = \pi_j^*(\Delta'_j)$.*

Proof. Indeed, the only way Δ'_j can be zero is if $a_0 = a_j = 0$ and $\tilde{D}_j = 0$. But then we would have $(\tilde{\Delta}_j, \tilde{\Delta}_j) = 0$. Since the intersection matrix of $\bigcup_{1 \leq i \leq N} \tilde{\Gamma}_i$ is negative definite, this would imply that $\tilde{\Delta}_j = 0$, which is a contradiction. This proves the first assertion. The second assertion follows from the fact that the coefficients of exceptional curves in the pullback of a divisor are uniquely determined by the requirement that the intersection number of the pullback with every exceptional curve is zero. □

W.l.o.g. we may assume $\delta_j(f_j) < 0$. Then Proposition 2.5 implies that $(\Gamma'_0, \Gamma'_0) > 0$. Since $(\tilde{\Delta}_j, \tilde{\Gamma}_0) = 0$, it follows that

$$(D'_j, \Gamma'_1) = (D'_j, \Gamma'_0) + a_0(\Gamma'_0, \Gamma'_0) + a_j(\Gamma'_j, \Gamma'_0) = 0. \tag{5}$$

Since (Γ'_0, Γ'_0) and (Γ'_j, Γ'_0) are positive, identity (5) implies that $a_0 = a_j = (D'_j, \Gamma'_0) = 0$. Let f be the polynomial in $\mathbb{C}[x, y]$ that defines $D'_j \cap \mathbb{C}^2$. Since D'_j does not intersect Γ'_0, it follows that f satisfies identity (4) for $\delta := \delta_j$. Theorem 2.4 then implies that f_j is a polynomial, as required to complete the proof that (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): Fix j, $1 \leq j \leq N$. Let $\pi_j : \tilde{X} \to X_j$ be as above. We claim that there is a numerical Γ'_j-curvette D'_j on X_j. Indeed, if $\delta_j(f_j) \geq 0$, the existence of D'_j follows from Proposition 2.5. On the other hand, if f_j is a polynomial, then set D'_j to be the closure in

X_j of the curve defined by f_j on $\mathbb{C}^2$. Proposition 2.2 then implies that D'_j is a numerical Γ'_j-curvette, which proves the claim. Now note that $\pi_*(\pi_j^*(D'_j))$ is a numerical Γ_j-curvette on X, which completes the proof of the theorem. □

Definition 3.3. Let X be a normal complete algebraic surface such that

(1) X contains $U := \mathbb{C}^2$ as a dense open subset, and
(2) the identity map of $U = \mathbb{C}^2$ extends to a morphism $X \to X_0 := \mathbb{P}^2$.

Let $\Gamma_0, \Gamma_1, \ldots, \Gamma_N$ be the irreducible components of $X \setminus U$, where Γ_0 is the strict transform of the line at infinity on X_0. An effective Weil divisor Δ on X is called a *numerical semi-affine Γ_i-curvette*, $1 \leq i \leq N$, if

(1) Δ is a numerical Γ_i-curvette,
(2) $\mathrm{Supp}(\Delta_i)$ contains neither Γ_0 nor Γ_i, and
(3) $\mathrm{Supp}(\Delta_i) \cap U \neq \emptyset$.

We say that X *admits a system of numerical semi-affine curvettes* if there exists a numerical semi-affine Γ_i-curvette) for all i, $1 \leq i \leq N$.

Theorem 3.4. *Let X be as in Definition 3.3. For each j, $1 \leq j \leq N$, let δ_j be the semidegree associated to Γ_j and f_j be the* last *key form of δ_j. Then the following are equivalent:*

(a) X admits a system of numerical semi-affine curvettes which are Cartier divisors.
(b) X admits a system of numerical semi-affine curvettes.
(c) For each j, $1 \leq j \leq N$, the last key form of δ_j is a polynomial.
(d) For each j, $1 \leq j \leq N$, all the key forms of δ_j are polynomials.

Proof. (b) ⇒ (c): Fix j, $1 \leq j \leq N$. Let X_j be as in the proof of Theorem 3.2, $\pi_j : X \to X_j$ be the natural morphism, $\Gamma'_0 := \pi_j(\Gamma_0)$ and $\Gamma'_j := \pi_j(\Gamma_j)$. Let $\Delta_j := D_j + \sum_{i=0}^N a_i\Gamma_i$ be a numerical semi-affine Γ_j-curvette on X, where D_j is a non-zero effective Weil divisor such that $\mathrm{Supp}(D_j) \not\supset \Gamma_i$ for any i, $a_0 = a_j = 0$, and $a_i \geq 0$, $0 \leq i \leq N$. Set $\Delta'_j := {\pi_j}_*(\Delta_j) = D'_j$, where $D'_j := {\pi_j}_*(D_j)$. It follows that $(D'_j, \Gamma'_0) = (D_j, \Gamma_0) = 0$. Let f be the polynomial in $\mathbb{C}[x, y]$ that defines $D'_j \cap \mathbb{C}^2$. The same arguments as in the paragraph following the proof of Claim 3.2.1 then show that the last key form of δ_j is a polynomial, as required.

(c) ⇒ (a): Fix j, $1 \leq j \leq N$. Let $\pi_j : X \to X_j$ be as above. Let D'_j be the closure in X_j of the curve defined by the last key form of δ_j on $U = \mathbb{C}^2$. Theorem 2.4 implies that $D'_j \cap \Gamma'_0 = \emptyset$, so that D'_j is a numerical semi-affine Γ'_j-curvette D'_j on X_j. Moreover, it follows from Proposition 2.5 that for some $n \geq 1$, nD'_j is a Cartier divisor. Then $\pi_j^*(nD'_j))$ is a Cartier divisor on X which is also a numerical semi-affine Γ_j-curvette, as required.

The equivalence (d) ⇔ (c) follows from Theorem 2.4. Since the implication (a) ⇒ (b) is obvious, the proof of Theorem 3.4 is complete. □

We are now ready to define the central object of this article.

Definition 3.5. We denote by $\mathcal{S}_{pol}$ the collection of normal complete algebraic surfaces X which satisfy the following properties:

(1) X contains $U := \mathbb{C}^2$ as an open subset,
(2) for each irreducible curve $\Gamma \subseteq X \setminus U$, all key forms associated to (the semidegree corresponding to) Γ are polynomials (i.e. regular functions on U).

S^+_{pol} is the subcollection of S_{pol} consisting of $X \in S_{pol}$ which satisfies in addition:

(3) For each semidegree δ on $\mathbb{C}[U] = \mathbb{C}[x,y]$ associated to some irreducible curve on $X \setminus U$, $\delta(f) \geq 0$ for all $f \in \mathbb{C}[x,y] \setminus \{0\}$ (by Theorem 2.4 this is equivalent to saying that $\delta(f_\delta) \geq 0$, where f_δ is the *last* key form for δ).

Remark 3.6. If X is a complete rational surface which dominates $\mathbb{P}^2$ via a birational morphism, then it follows from Theorem 3.4 that $X \in S_{pol}$ iff X admits a system of numerical semi-affine curvettes.

Remark 3.7. It is not hard to see (e.g. follows from Theorem 4.5 below) that every surface in S_{pol} is in fact projective.

Now we describe an important subclass of S_{pol}. Recall that in the introduction we considered collections $\mathcal{V}$ of pencils $V := \mathbb{C}\langle F, H^d\rangle$ where

(1) H is the equation of a straight line L, and
(2) $F = F_1^{a_1} \cdots F_k^{a_k}$, where each F_j, $1 \leq j \leq k$, defines a curve with one place at infinity, and $\sum_{j=1}^k a_j \deg(F_j) = d$.

For a $V \in \mathcal{V}$, recall that X_V denotes the surface resulting from (minimal) resolutions of basepoints of V on $\mathbb{P}^2$. Also recall that $\mathcal{V}_1$ is the subset of $\mathcal{V}$ consisting of pencils for which F itself defines an irreducible curve with one place at infinity.

Proposition 3.8.

(1) $X_V \in S_{pol}$ for each $V \in \mathcal{V}$.
(2) If $V \in \mathcal{V}_1$, then $X_V \in S^+_{pol}$.

Proof. Identify $U := \mathbb{C}^2$ with $\mathbb{P}^2 \setminus L$. Denote the irreducible components of $X_V \setminus U$ by $\Gamma_0, \ldots, \Gamma_N$, where Γ_0 is the strict transform of L. For each i, $0 \leq i \leq N$, let δ_i be the associated semidegree on $\mathbb{C}[x,y] := \mathbb{C}[U]$. At first assume $V \in \mathcal{V}_1$, i.e. F defines a curve with one place at infinity. In that case dual graph of the irreducible components of is of the form as in Figure 1.

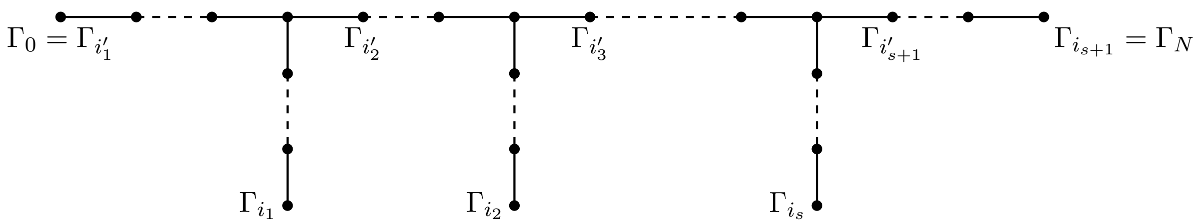

FIGURE 1. Dual graph of curves at infinity on X_V for $V \in \mathcal{V}_1$

It follows from a result of Abhyankar and Moh that there are polynomials $f_1, \ldots, f_s, f_{s+1} = F|_{\mathbb{C}^2}$, (where s is the number of 'vertical segments' in Figure 1), such that for each j, $1 \leq j \leq s+1$, the closure C_j in X_V of the curve $\{f_j = 0\} \subseteq \mathbb{C}^2$ is a numerical Γ_{i_j}-curvette. Fix j, $1 \leq j \leq s+1$, and define

$$[\Gamma_{i'_j}, \Gamma_{i_j}] := \begin{cases} \text{'L-shaped segment' between (and including) } \Gamma_{i'_j} \text{ and } \Gamma_{i_j} & \text{if } 1 \leq j \leq s, \\ \text{'horizontal segment' between (and including) } \Gamma_{i'_{s+1}} \text{ and } \Gamma_{i_{s+1}} & \text{if } j = s+1. \end{cases}$$

It follows that for each l such that $\Gamma_l \in [\Gamma_{i'_j}, \Gamma_{i_j}]$, identity (4) is satisfied with $f = f_j$ and $\delta = \delta_l$, and therefore all the key forms of δ_l are polynomials (via assertion 2 of Theorem 2.4). Moreover, it is not hard to compute that $\delta_l(f_j) \geq \delta_{i_j}(f_j) \geq 0$, which implies that δ_l is non-negative on all non-zero polynomials (via assertion 3 of Theorem 2.4). This proves that $X_V \in S^+_{pol}$.

Now consider the general case that $F = F_1^{a_1} \cdots F_k^{a_k}$ for $k \geq 1$ and positive integers $a_1, \ldots, a_k$. Then for each i, $0 \leq i \leq N$, one of the following (mutually exclusive) cases holds (see e.g. [CPR05, Proof of Theorem 2]):

Case 1. Γ_i is the strict transforms of some irreducible curve on $X_{V_j} \setminus U$, $1 \leq j \leq k$, where V_j is the pencil generated by F_j and $H^{\deg(F_j)}$.

Case 2. There exists j, $1 \leq j \leq k$, such that $-\delta_i$ is a *monomial valuation* centered at O_j, where O_j is the (unique) point at infinity on the closure C_j in X_{V_j} of the curve $F_j|_U = 0$. Moreover,
- (a) O_j is the point of intersection of C_j and the 'last exceptional curve' Γ_{j,N_j} on X_{V_j} (i.e. Γ_{j,N_j} is the exceptional divisor of the last blow up performed in the resolution of the pencil V_j).
- (b) O_j does *not* belong to any irreducible component of $X_{V_j} \setminus U$ other than Γ_{j,N_j}.

In Case 1, we have already seen that the key forms corresponding to δ_i are polynomials. So assume Case 2 holds. But then it can be shown that there exists a numerical Γ_i-curvette on X (see [CPR05, Proof of Theorem 2][2]). Let δ_{j,N_j} be the semidegree on $\mathbb{C}[x,y]$ associated to Γ_{j,N_j}. It follows from the construction of X_{V_j} that $f_j := F_j|_U$ is the last key form of δ_{j,N_j} and $\delta_{j,N_j}(f_j) = 0$. Property 2b then implies that $\delta_i(f_j) < 0$, which implies via assertion 3 of Theorem 2.4 that δ_i-value of the last key form of δ_i is *negative*. Theorem 3.2 then implies that the last key form of δ_i is a polynomial. It follows from Theorem 2.4 that all key forms of δ_i are polynomials, as required to complete the proof of the proposition. □

4. Cox ring of $X \in S_{pol}$

Notation 4.0. Throughout the rest of this article we denote by X a surface from the class S_{pol}. Let $\Gamma_1, \ldots, \Gamma_N$ be the irreducible components of $X \setminus \mathbb{C}^2$, and δ_i, $1 \leq i \leq N$, be the order of pole along Γ_i. Choose coordinates (x,y) on $\mathbb{C}^2$ such that $\delta_i(x) > 0$ for each i. Let $\tilde{\phi}_i(x,\xi) := \phi_i(x) + \xi x^{r_i}$, $1 \leq i \leq N$, be the generic degree-wise Puiseux series of δ_i. Fix i, $1 \leq i \leq N$. Let the *formal Puiseux pairs* (Definition 7.2) of $\tilde{\phi}_i$ be $(q_{i,1}, p_{i,1}), \ldots, (q_{i,l_i+1}, p_{i,l_i+1})$. Recall that the *characteristic exponents* (Definition 7.2) of $\tilde{\phi}_i$ are $r_{i,j} := \frac{q_{i,j}}{p_{i,1} \cdots p_{i,j}}$, $1 \leq j \leq l_i + 1$. For each j, $0 \leq j \leq l_i$, consider the semidegree δ_{ij} with generic degree-wise Puiseux series

$$\tilde{\phi}_{ij}(x,\xi) := [\phi_i(x)]_{>r_{i,j+1}} + \xi x^{r_{i,j+1}},$$

where $[\phi_i(x)]_{>r_{i,j+1}}$ denotes the sum of all terms of ϕ_i with degree (in x) greater than $r_{i,j+1}$.

Note that $\delta_{il_i} = \delta_i$. For each j, $0 \leq j \leq l_i$, let f_{ij} be the last key form of δ_{ij}. The following lemma records some properties of f_{ij}'s. It is an immediate corollary of Theorem 2.4.

Lemma 4.1. *Fix i, j, $1 \leq i \leq N$, $0 \leq j \leq l_i$. Then*

$1_{(ij)}$. *$f_{ij} = 0$ is a curve with one place at infinity.*

[2] Even though [CPR05, Theorem 2] assumes that $\gcd(a_1, \ldots, a_k) = 1$, the proof for existence of systems of numerical curvettes on X_V does not use this assumption.

$2_{(ij)}$. *f_{ij} has a degree-wise Puiseux root ϕ_{ij} (which is unique up to conjugacy) such that*
  (a) Puiseux pairs of ϕ_{ij} are $(q_{i,1}, p_{i,1}), \dots, (q_{i,j}, p_{i,j})$, and
  (b) $\phi_{ij} = [\phi_i(x)]_{>r_{i,j+1}} +$ terms with degree (in x) less than *or equal to $r_{i,j+1}$.* ☐

Remark 4.2. Assume that $X = X_V$ for some $V = \mathbb{C}\langle F, H^d\rangle \in \mathcal{V}$, where $F := F_1^{a_1} \cdots F_k^{a_k}$ such that $f_j := F_j|_{\mathbb{C}^2}$ defines a curve with one place at infinity for each j, $1 \leq j \leq k$. Let $\{g_{i'j'}\}_{j'}$ be the *approximate roots* [AM73] of $f_{i'}$, $1 \leq i' \leq k$. Then $\{g_{i',j'}\}_{i',j'}$ satisfies the properties of $\{f_{ij}\}_{i,j}$ of Lemma 4.1, i.e. for each i, j, $1 \leq i \leq N$, $0 \leq j \leq l_i$, there exists i', j' such that properties $1_{(ij)}$ and $2_{(ij)}$ of Lemma 4.1 hold with $f_{ij} := g_{i'j'}$.

Definition 4.3. The *Cox ring*, or the *total coordinate ring*, of X is

$$\mathcal{R}(X) := \bigoplus_{(d_1,\dots,d_N)\in\mathbb{Z}^N} \Gamma\left(X, \mathcal{O}\left(\sum d_j\Gamma_j\right)\right)$$

For $\vec{d} := (d_1, \dots, d_N) \in \mathbb{Z}^N$ and $f \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$, we denote by $(f)_{\vec{d}}$ the 'copy' of f in the $\vec{d}$'th graded component of $\mathcal{R}(X)$. Let $\vec{e}_j$ be the j-th unit vector in $\mathbb{Z}^N$. Moreover, for $f \in \mathbb{C}(x, y)$ we denote by $\vec{\delta}(f)$ the element $(\delta_1(f), \dots, \delta_N(f)) \in \mathbb{Z}^N$.

Convention 4.4. For each $\vec{d} := (d_1, \dots, d_N) \in \mathbb{Z}^N$ and $f \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$, we identify f with $f|_{\mathbb{C}^2} \in \mathbb{C}[x, y]$.

Theorem 4.5. *Pick any collection of (not necessarily pairwise distinct) polynomials f_{ij}, $1 \leq i \leq N$, $0 \leq j \leq l_i$, which satisfy properties $1_{(ij)}$ and $2_{(ij)}$ of Lemma 4.1. Let $N' := \sum_{i=0}^N (l_i+1)$. For all $\vec{\gamma} := (\gamma_{ij})_{i,j} \in \mathbb{Z}_{\geq 0}^{N'}$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$, define $g_{\alpha\beta\vec{\gamma}} := x^\alpha y^\beta \prod_{i,j} f_{ij}^{\gamma_{ij}} \in \mathbb{C}[x, y]$.*

  (1) $(f_{ij})_{\vec{\delta}(f_{ij})}$'s together with $(x)_{\vec{\delta}(x)}$, $(y)_{\vec{\delta}(y)}$, and $(1)_{\vec{e}_j}$, $1 \leq j \leq N$, generate $\mathcal{R}(X)$ as a $\mathbb{C}$-algebra.
  (2) Let $\vec{d} := (d_1, \dots, d_N) \in \mathbb{Z}^N$. Let $\mathcal{E}_{\vec{d}}$ be the collection of all $(\alpha, \beta, \vec{\gamma}) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{N'}$ such that

$$(*_k) \qquad \alpha\delta_k(x) + \beta\delta_k(y) + \sum_{i,j} \gamma_{ij}\delta_k(f_{ij}) \leq d_k$$

  is satisfied for all k, $1 \leq k \leq N$. Then $\Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$ is a generated as a vector space over $\mathbb{C}$ by $\{g_{\alpha\beta\vec{\gamma}} : (\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}}\}$.
  (3) Let $\vec{d} := (d_1, \dots, d_N) \in \mathbb{Z}^N$. Define

$$\mathcal{E}'_{\vec{d}} := \{(a, b) \in \mathbb{Z}_{\geq 0}^2 : b = \beta + \sum_{i,j} \gamma_{ij} \deg_y(f_{ij}) = \deg_y(g_{a,\beta,\vec{\gamma}})$$
$$\text{for some } \beta \in \mathbb{Z}_{\geq 0} \text{ and } \vec{\gamma} = (\gamma_{ij})_{i,j} \in \mathbb{Z}_{\geq 0}^{N'} \text{ such that } (a, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}}\}.$$

  Then $\dim \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j)) = |\mathcal{E}'_{\vec{d}}|$. For each $(a, b) \in \mathcal{E}'_{\vec{d}}$, pick any $\beta \in \mathbb{Z}_{\geq 0}$ and $\vec{\gamma} \in \mathbb{Z}_{\geq 0}^{N'}$ such that $(a, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}}$ and $b = \deg_y(g_{a,\beta,\vec{\gamma}})$, and define $g_{a,b} := g_{a,\beta,\vec{\gamma}}$. Then $\{g_{a,b} : (a, b) \in \mathcal{E}'_{\vec{d}}\}$ is a basis of $\Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$.

Proof. Note that for all $(\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}}$, we have

$$\vec{\delta}(g_{\alpha\beta\vec{\gamma}}) = \vec{\delta}\left(x^\alpha y^\beta \prod_{i,j} f_{ij}^{\gamma_{ij}}\right) = \alpha\vec{\delta}(f) + \beta\vec{\delta}(y) + \sum_{i,j} \vec{\delta}(f_{ij}) \leq \vec{d}$$

(where "$\leq$" denotes coordinate-wise inequality). In particular,

(6) $$(\alpha,\beta,\vec{\gamma}) \in \mathcal{E}_{\vec{d}} \iff g_{\alpha\beta\vec{\gamma}} \in \Gamma\left(X, \mathcal{O}(\sum d_j\Gamma_j)\right).$$

This implies that assertion 1 follows from assertion 2. Moreover, assertion 3 also is a corollary of assertion 2. Therefore it suffices to prove assertion 2.

Pick $\vec{d} \in \mathbb{Z}^N$ and $h \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$. We will show by induction on $\deg_y(h)$ that h belongs to the vector space generated by $\mathcal{G}_{\vec{d}} := \{g_{\alpha\beta\vec{\gamma}} : (\alpha,\beta,\vec{\gamma}) \in \mathcal{E}_{\vec{d}}\}$. Indeed, if $\deg_y(h) = 0$, then it is clear. So assume $\deg_y(h) \geq 1$, and consider the degree-wise Puiseux factorization of h:

$$h = cx^m \prod_{i=1}^{M} \prod_{\substack{\psi_{ij} \text{ is a con-} \\ \text{jugate of } \psi_i}} (y - \psi_{ij}(x)).$$

Fix i, $1 \leq i \leq M$. For each j, j', $1 \leq j \leq N$, $1 \leq j' \leq d_i^*$ (where d_i^* is the number of conjugates of ψ_i), define

$$s_{ij'j} := \deg_x(\psi_{ij'}(x) - \tilde{\phi}_j(x,\xi)) = \deg_x(\psi_{ij'}(x) - \phi_j(x) - \xi x^{r_j}),$$
$$s_i := \min\{s_{ij'j}\}_{j',j}.$$

Pick j'_i, j_i such that $s_i = s_{ij'_ij_i}$. We may assume w.l.o.g. that $j'_i = 1$. Let

$$\psi_i^* := [\psi_{i1}(x)]_{>s_i} = [\phi_{j_i}(x)]_{>s_i}.$$

For each k, $0 \leq k \leq l_{j_i}$, let ϕ_{j_ik} be a degree-wise Puiseux root of f_{j_ik} (recall that f_{j_ik}'s satisfy properties $1_{(ij)}$ and $2_{(ij)}$ of Lemma 4.1). Since $s_i \geq r_{j_i}$, it follows that if $\psi_i^* \neq 0$, then there is a unique k_i, $0 \leq k_i \leq l_{j_i}$, such that

(a) $\phi_{j_ik_i}$ has the same Puiseux pairs as ψ_i^*, and
(b) a conjugate of $\phi_{j_ik_i}$ is of the form $\psi_i^* +$ terms with degree less than or equal to s_i.

Define

$$h_i := \prod_{\substack{\psi_{ik} \text{ is a con-} \\ \text{jugate of } \psi_i}} (y - \psi_{ik}(x)) \in \mathbb{C}\langle\langle x \rangle\rangle[y].$$

Note that $h = \prod_{i=1}^M h_i$ and $\deg_y(h_i) =$ number of conjugates of ψ_i. Let

$$m_i := \begin{cases} \deg_y(h_i) & \text{if } \psi_i^* = 0, \\ \frac{\text{number of conjugates of } \psi_i}{\text{number of conjugates of } \psi_i^*} & \text{otherwise.} \end{cases} \qquad h_i^* := \begin{cases} y^{m_i} & \text{if } \psi_i^* = 0, \\ f_{j_ik_i}^{m_i} & \text{otherwise.} \end{cases}$$

Note that m_i is a positive integer and $h_i^* \in \mathbb{C}[x,y]$. Define

$$h^* := cx^m \prod_{i=1}^{M} h_i^* \in \mathbb{C}[x,y].$$

Claim 4.5.1. *$\delta_j(h^*) \leq \delta_j(h)$ for each j, $1 \leq j \leq N$.*

Proof. Fix i, j, $1 \leq i \leq M$, $1 \leq j \leq N$. It suffices to show that $\delta_j(h_i^*) \leq \delta_j(h_i)$. As in (1),

$$\delta_j(h_i) = \delta_j(x) \deg_x \left(h_i(x,y)|_{y=\tilde{\phi}_j(x,\xi)} \right)$$
$$= \delta_j(x) \deg_x \left(\prod_{k=1}^{d_i^*} (\tilde{\phi}_j(x,\xi) - \psi_{ik}(x)) \right) = \delta_j(x) \sum_{k=1}^{d_i^*} s_{ikj},$$

where $d_i^* := \deg_y(h_i) = \deg_y(h_i^*)$. If $\psi_i^* = 0$, then $s_{ikj} \geq \deg_x(\tilde{\phi}_j(x,\xi))$ for all k, which implies that

$$\delta_j(h_i) \geq \delta_j(x) d_i^* \deg_x(\tilde{\phi}_j(x,\xi)) = d_i^* \delta_j(y) = \delta_j(y^{d_i^*}) = \delta_j(h_i^*),$$

as required to prove the claim. So assume $\psi_i^* \neq 0$. Define

$$s_{ij} := \min\{s_{ij'j}\}_{j'}.$$
$$s'_{ij} := \min\{\deg_x(\tilde{\phi}_j(x,\xi) - \phi(x)) : \phi \text{ is a conjugate of } \phi_{j_i k_i}\}.$$

Claim 4.5.2 and Lemma 7.6 below then imply that $\delta_j(h_i) \geq \delta_j(h_i^*)$, as required to prove Claim 4.5.1. ☐

Claim 4.5.2. $s_{ij} \geq s'_{ij}$.

Proof. Indeed, replacing ϕ_j by one of its conjugates if necessary, we may assume that $s_{ij} = s_{i1j}$. Similarly, property (b) of $\phi_{j_i k_i}$ implies that replacing $\phi_{j_i k_i}$ by one of its conjugates if necessary, we may assume that $\phi_{j_i k_i}$ is of the form $[\psi_{i1}(x)]_{>s_i} +$ terms with degree less than or equal to s_i. Since $s_{ij} \geq s_i$, this implies that $\deg_x(\tilde{\phi}_j(x,\xi) - \phi_{j_i k_i}(x)) \leq s_{ij}$, as required. ☐

Claim 4.5.1 implies that $h^* \in \Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$, so that $h - h^*$ is also an element of $\Gamma(X, \mathcal{O}(\sum d_j \Gamma_j))$. Since $\deg_y(h - h^*) < \deg_y(h)$, the inductive hypothesis implies that $h - h^*$ is in the vector space generated by $\mathcal{G}_{\vec{d}}$. Now note that by construction $h^* = c g_{\alpha\beta\vec{\gamma}}$ for some $\alpha, \beta, \vec{\gamma}$. Claim 4.5.1 and (6) then imply that $g_{\alpha\beta\vec{\gamma}} \in \mathcal{G}_{\vec{d}}$, and complete the proof of the theorem. ☐

5. Global Enriques and Zariski semigroups of divisors on $X \in S_{pol}$

Global Enriques and Zariski semigroups of line bundles on surfaces of the form X_V, where V is a pencil at infinity, were introduced in [CPRL02]. In this section we adapt these notions for surfaces in S_{pol} and then compute these semigroups. We continue with the set up introduced in Notation 4.0.

Definition 5.1. Recall that the divisor class group $\mathrm{Cl}(X)$ (of Weil divisors on X modulo linear equivalence) is isomorphic to $\mathbb{Z}^N$ and generated by $\Gamma_1, \ldots, \Gamma_N$. The *global Enriques semigroup* $P^{st}(X)$ of X consists of all $D \in \mathrm{Cl}(X)$ such that D is linearly equivalent to an effective Weil divisor whose support does not contain any of the Γ_j's. The *global Zariski semigroup*, or the *characteristic semigroup* of X is the semigroup $\tilde{P}(X)$ generated by all divisors in $\mathrm{Cl}(X)$ which are base point free. The *Zariski semigroup at infinity* $\tilde{P}_\infty(X)$ is generated by all divisions in $\mathrm{Cl}(X)$ which have no base point 'at infinity', i.e. the base locus consists of finitely many points on $\mathbb{C}^2$.

Remark 5.2. If $X \in S^+_{pol}$ (e.g. if $X = X_V$ for some $V \in \mathcal{V}_1$) then $\tilde{P}_\infty(X) = \tilde{P}(X)$. Indeed, assume $X \in S^+_{pol}$ and $D = \sum_{j=1}^N a_j \Gamma_j \in \tilde{P}_\infty(X)$. Then there is $C \sim D$ such that C is the closure (in X) of a curve $\{f = 0\} \subseteq \mathbb{C}^2$. Since $X \in S^+_{pol}$, it follows that $\delta_j(f-1) = \delta_j(f)$ for each j, $1 \leq j \leq N$, and therefore the closure C' (in X) of $\{f = 1\} \subseteq \mathbb{C}^2$ is also linearly equivalent to D. Since C' and C have no common points in $\mathbb{C}^2$, it follows that $\mathcal{O}(D)$ has no 'finite' base points. Since $\mathcal{O}(D)$ has no base points at infinity by assumption, it follows that $\mathcal{O}(D)$ has no base points at all.

For $\vec{d} \in \mathbb{Z}^N$, we denote by $\Gamma_{\vec{d}}$ the divisor $\sum_j d_j \Gamma_j \in \mathrm{Cl}(X)$. Theorem 4.5 gives an immediate characterization of those $\vec{d}$ for which $\Gamma_{\vec{d}} \in P^{st}(X)$.

Corollary 5.3. *Let f_{ij}, $1 \leq i \leq N$, $0 \leq j \leq l_i$, be as in Theorem 4.5. For each $\vec{d} \in \mathbb{Z}^N$ and each j, $1 \leq j \leq N$, define:*

$$\begin{aligned} M_{\vec{d},j} &:= \max\left\{\delta_j(g_{\alpha\beta\vec{\gamma}}) : (\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}}\right\} \\ &= \max\left\{\alpha\delta_j(x) + \beta\delta_j(y) + \sum_{i,j} \gamma_{ij}\delta_j(f_{ij}) : (\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}}\right\}. \end{aligned}$$

Then $\Gamma_{\vec{d}} \in P^{st}(X)$ iff $d_j = M_{\vec{d},j}$ for all j, $1 \leq j \leq N$. □

Definition 5.4. Fix i, $1 \leq j \leq N$. Let $\tilde{\delta}_i := \delta_i/\delta_i(x)$ be the 'normalized' version of δ_i. Recall that the *formal Puiseux pairs* (Definition 7.2) of $\tilde{\phi}_i$ are $(q_{i,1}, p_{i,1}), \ldots, (q_{i,l_i+1}, p_{i,l_i+1})$. Define:

$$\begin{aligned} p_i &:= p_{i,1} \cdots p_{i,l_i}, \\ \tilde{p}_i &:= \delta_i(x) = p_i p_{i,l_i+1} = p_{i,1} \cdots p_{i,l_i+1}. \end{aligned}$$

Let f_{ij}, $1 \leq i \leq N$, $0 \leq j \leq l_i$, as in Theorem 4.5. For each i, j, let ϕ_{ij} be a degree-wise Puiseux root of f_{ij}. Fix i, $1 \leq i \leq N$, and define $\mathcal{N}_i := \{i' : \tilde{\delta}_{i'} \leq \tilde{\delta}_i\}$. Recall (Property $2_{(ij)}$ implies) that f_{il_i} has a generic degree-wise Puiseux root $\phi_{il_i}(x)$ such that

$$\phi_{il_i} = [\phi_i(x)]_{>r_i} + \text{ terms with degree (in } x\text{) } \textit{less} \text{ than or equal to } r_i$$

For each $i' \in \mathcal{N}_i$, (replacing $\phi_{i'}$ by a conjugate if necessary we may assume that) there exists $c_{ii'} \in \mathbb{C}$ such that

$$\phi_{i'l_{i'}} = [\phi_i(x)]_{>r_i} + c_{ii'}x^{r_i} + \text{l.d.t.} \tag{7}$$

Note that $c_{ii'}$ is unique only up to a multiplication by a p_{i,l_i+1}-th root of unity. For each $c \in \mathbb{C}$, define

$$\mathcal{N}_{i,c} := \left\{i' \in \mathcal{N}_i \setminus \{i\} : c_{ii'}^{p_{i,l_i+1}} = c\right\}.$$

Then $\mathcal{N}_i = \bigcup_{c\in\mathbb{C}} \mathcal{N}_{i,c} \cup \{i\}$. For convenience we record the following lemma which is a direct consequence of Lemma 4.1:

Lemma 5.5. *If $c_{ii} \neq 0$, then $p_{i,l_i+1} = 1$. In particular, identity (7) for $i := i'$ uniquely determines c_{ii}.* □

Remark-Definition 5.6. For each $f \in \mathbb{C}[x,y] \setminus \{0\}$, and each i, $1 \leq i \leq N$, we write $\mathrm{Lc}_i(f)$ for the coefficient of $x^{\tilde{\delta}_i(f)}$ in $f|_{y=\tilde{\phi}_i(x,\xi)}$. Note that $\mathrm{Lc}_i(f)$ can be factored as $\xi^r h(\xi^{p_{i,l_i+1}})$, where

(1) r is the number of degree-wise Puiseux roots $\psi(x)$ of f such that $\deg_x(\phi_i - \psi(x)) < r_i$,
(2) h is a polynomial in one variable,

(3) $p_{i,l_i+1} \deg(h)$ is the number of degree-wise Puiseux roots $\psi(x)$ of f such that $\deg_x(\phi_i - \psi(x)) = r_i$.

Theorem 5.7. *For $\vec{a} := (a_1, \ldots, a_N) \in \mathbb{Z}^N_{\geq 0}$, define*

$$\begin{aligned} d_i(\vec{a}) &:= \sum_{j=1}^{N} a_j \omega_{ij},\ 1 \leq i \leq N,\ \textit{where} \\ \omega_{ij} &:= \begin{cases} p_{j,l_j+1}\delta_i(f_{jl_j}) & \textit{if } i \notin \mathcal{N}_j \\ p_{j,l_j+1}\tilde{p}_i\tilde{\delta}_j(f_{jl_j}) & \textit{if } i \in \mathcal{N}_j. \end{cases} \\ \vec{d}_{\vec{a}} &:= (d_1(\vec{a}), \ldots, d_N(\vec{a})), \\ m_i(\vec{a}) &:= \sum_{i' \in \mathcal{N}_i} \frac{\tilde{p}_{i'} a_{i'}}{p_i},\ 1 \leq i \leq N, \\ m_{i,c}(\vec{a}) &:= \sum_{i' \in \mathcal{N}_{i,c}} \frac{\tilde{p}_{i'} a_{i'}}{p_i},\ c \in \mathbb{C},\ 1 \leq i \leq N. \end{aligned}$$

Let f_{ij}'s be as in Theorem 4.5. Then for each $\vec{d} \in \mathbb{Z}^N$, the following are equivalent:
(a) $\mathcal{O}(\sum d_j \Gamma_j)$ has no base point at infinity.
(b) $\vec{d} = \vec{d}_{\vec{a}}$ for some $\vec{a} \in \mathbb{Z}^N_{\geq 0}$ such that for each i, $1 \leq i \leq N$,

$$d_i(\vec{a}) = \max\left\{\delta_i\left(g_{\alpha\beta\vec{\gamma}}\right) : (\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}_{\vec{a}}}\right\}, \tag{8}$$

$$m_i(\vec{a}) = \max\left\{\deg_\xi\left(\mathrm{Lc}_i\left(g_{\alpha\beta\vec{\gamma}}\right)\right) : (\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}_{\vec{a}}},\ \delta_i(g_{\alpha\beta\vec{\gamma}}) = d_i(\vec{a})\right\}, \tag{9}$$

$$m_{i,c_{ii}}(\vec{a}) = \min\left\{\mathrm{ord}_{\xi - c_{ii}}\left(\mathrm{Lc}_i\left(g_{\alpha\beta\vec{\gamma}}\right)\right) : (\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}_{\vec{a}}},\ \delta_i(g_{\alpha\beta\vec{\gamma}}) = d_i(\vec{a})\right\}, \tag{10}$$

$$a_i p_{i,l_i+1} + m_{i,c_{ii}}(\vec{a}) = \max\left\{\mathrm{ord}_{\xi - c_{ii}}\left(\mathrm{Lc}_i\left(g_{\alpha\beta\vec{\gamma}}\right)\right) : (\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}_{\vec{a}}},\ \delta_i(g_{\alpha\beta\vec{\gamma}}) = d_i(\vec{a})\right\}. \tag{11}$$

Proof. (a) $\Rightarrow$ (b): Assume $\mathcal{O}(\sum d_j \Gamma_j)$ has no base point at infinity. Let C be the divisor corresponding to a generic global section of $\mathcal{O}(\sum d_j \Gamma_j)$. By Bertini's theorem C is transversal to the curve at infinity, and for each j,

(1) C intersects each C_j (transversally) at a_j distinct points for some $a_j \geq 0$,
(2) $C \cap C_i \cap C_j = \emptyset$ for each $i \neq j$,
(3) C does not pass through the 'zero point' of C_j for any j.

(note that $a_1, \ldots, a_N$ are independent of C). Pick $h \in \mathbb{C}[x, y]$ which defines $C \cap \mathbb{C}^2$. Then h has a degree-wise Puiseux factorization of the form

$$h = c \prod_{i=1}^{N} \prod_{j=1}^{a_i} \prod_{k=1}^{\tilde{p}_i} (y - \psi_{ijk}(x))$$

where $c \in \mathbb{C}^*$, and ψ_{ijk}'s are conjugates of some ψ_{ij} of the form $\phi_i(x) + \xi_{ij} x^{r_i} + \text{l.o.t.}$ for $\xi_{ij} \in \mathbb{C}^*$ which are 'base point free' in the sense that

$$\text{for every finite set } S \subseteq \mathbb{C}^*, \text{ if } h \text{ is generic enough, then } \{\xi_{ij}\}_j \cap S = \emptyset. \tag{12}$$

A straightforward computation then shows that for each i, i', j, $1 \leq i, i' \leq N$, $1 \leq j \leq a_i$,

$$\delta_{i'}\left(\prod_{k=1}^{\tilde{p}_i}(y - \psi_{ijk}(x))\right) = \omega_{i'i}, \text{ so that}$$

$$\delta_{i'}(h) = \sum_{i=1}^{N} a_i \omega_{i'i} = d_{i'}(\vec{a}). \tag{13}$$

It follows that $\vec{d} = \vec{d}_{\vec{a}}$. The first assertion of the following claim is a straightforward implication of the 'base point freeness' of ξ_{ij}'s, and the second assertion follows from the genericness of h.

Claim 5.7.1. *Fix i', $1 \leq i' \leq N$. Let*

$$\eta_{i'}(\xi) := \prod_{i \in \mathcal{N}_{i'} \setminus \{i'\}} (\xi^{p_{i',l_{i'}+1}} - c_{i'i}^{p_{i',l_{i'}+1}})^{(\tilde{p}_i a_i)/\tilde{p}_{i'}}. \tag{14}$$

Then

(1) $\mathrm{Lc}_{i'}(h) = c' \eta_{i'}(\xi) \prod_{j=1}^{a_{i'}} \left(\xi^{p_{i',l_{i'}+1}} - \xi_{i'j}^{p_{i',l_{i'}+1}}\right)$ *for some* $c' \in \mathbb{C}^*$.

(2) *Let* $h' \in \Gamma(X, \mathcal{O}(\sum_j d_j \Gamma_j))$ *such that* $\delta_{i'}(h') = d_{i'}$. *Then* $\mathrm{Lc}_{i'}(h') = \eta_{i'}(\xi)\mu_{h'}(\xi)$ *for some* $\mu_{h'} \in \mathbb{C}[\xi]$. □

Theorem 4.5, identity (13), Claim 5.7.1 and Lemma 5.5 together imply that $\vec{a}$ satisfies (8), (9) and (10).

Now we show that $\vec{a}$ satisfies (11). Recall the procedure from the proof of Theorem 4.5 to write h as a linear combination of the $g_{\alpha\beta\vec{\gamma}}$. Set $h_0 := h$, and inductively write $h_{k+1} = h_k - h_k^*$, where h_k^* is computed from h_k in the same way as h^* is computed from h in the proof of Theorem 4.5.

Claim 5.7.2. *Fix i', $1 \leq i' \leq N$. Let $k_{i'} \geq 0$ be the smallest integer such that $\delta_{i'}(h^*_{k_{i'}}) = \delta_{i'}(h)$. Then for each k, $0 \leq k \leq k_{i'}$, the degree-wise Puiseux factorization of h_k is of the form*

$$h_k = g_k(x, y) \prod_{i \in \mathcal{N}_{i'}} \prod_{j=1}^{a_i} \prod_{k=1}^{\tilde{p}_i} (y - \psi_{ijk}(x)), \tag{15}$$

for some $g_k(x, y) \in \mathbb{C}\langle\langle x \rangle\rangle[y]$.

Proof. We prove the claim by induction on k. It is clearly true if $k = 0$. Now pick k, $0 \leq k < k_{i'}$, such that (15) holds for k. We will show that (15) holds for $k + 1$. Write the degree-wise Puiseux factorization of h_k as

$$h_k = cx^m \prod_{k'=1}^{K} \prod_{\substack{\psi_{k'l'} \text{ is a con-} \\ \text{jugate of } \psi_{k'}}} (y - \theta_{k'l'}(x))$$

Recall that h_k^* is formed from (15) by replacing each $h_{kk'} := \prod_{l'}(y - \theta_{k'l'}(x))$ by $h^*_{kk'} := f_{i_{k'}j_{k'}}^{n_{k'}}$ for certain $i_{k'}, j_{k'}, n_{k'}$. Since $k < k_{i'}$, we have $\delta_{i'}(h_k) > \delta_{i'}(h_k^*)$. This implies that there exists

k' such that $\delta_{i'}(h_{kk'}) > \delta_{i'}(h^*_{kk'})$. It then follows (e.g. due to Lemma 7.6) that

$$s_{i'k'} := \min_{l'}\{\deg_x(\phi_{i'}(x) + \xi x^{r_{i'}} - \theta_{k'l'}(x)\} > \min_{l}\{\deg_x(\phi_{i'}(x) + \xi x^{r_{i'}} - \phi_{i_{k'}j_{k'}l}(x)\} \geq r_{i'},$$

where $\phi_{i_{k'}j_{k'}l}$'s on the right hand side runs over all conjugates of $\phi_{i_{k'}j_{k'}}$. Pick l'_0 (resp. l_0) for which the minimum is achieved on the left (resp. right) hand side of the preceding inequality. Then

$$\psi_{k'l'_0}(x) = \phi_{i'}(x) + bx^{s_{i'k'}} + \text{l.d.t.}$$
$$\phi_{i_{k'}j_{k'}l_0}(x) = \phi_{i'}(x) + \text{terms with degree less than } s_{i'k'}.$$

Now fix $i \in \mathcal{N}_{i'}$. Then ϕ_i has a conjugate which agrees with $\phi_{i'}$ up to degree $r_{i'}$, so that

$$\min_{l}\{\deg_x(\phi_i(x) + \xi x^{r_i} - \phi_{i_{k'}j_{k'}l}(x)\} < s_{i'k'} = \min_{l'}\{\deg_x(\phi_i(x) + \xi x^{r_i} - \theta_{k'l'}(x)\}.$$

Lemma 7.6 then implies that $\delta_i(h_{kk'}) > \delta_i(h^*_{kk'})$, so that $\delta_i(h_k) > \delta_i(h^*_k)$. By induction hypothesis the (closure in X of the) curve $h_k = 0$ intersects Γ_i transversally at a_i points. It follows that the curve defined by $h_{k+1} = h_k - h^*_k$ also intersects Γ_i transversally at each of those a_i points. This implies that (15) holds for $k+1$, as required. □

Fix i, j, $i \in \mathcal{N}_{i'}\setminus\{i'\}$, $1 \leq j \leq a_i$. Claim 5.7.2 implies that $h_{ij} := \prod_{k=1}^{\tilde{p}_i}(y - \psi_{ijk}(x))$ divides $h_{k_{i'}}$ in $\mathbb{C}\langle\langle x\rangle\rangle[y]$. Recall that ψ_{ijk}'s are conjugates of ψ_{ij} and $\psi_{ij} = \phi_i(x) + \xi_{ij}x^{r_i} + \text{l.o.t.}$. The arguments in the proof of Theorem 4.5 show that the factor h^*_{ij} of $h^*_{k_{i'}}$ corresponding to h_{ij} can be defined to be $f_{il_i}^{p_{i,l_i+1}}$. It follows that $\mathrm{Lc}_{i'}(h^*_{k_{i'}})$ has a factor of the form

$$\sigma(\xi) := \prod_{i\in\mathcal{N}_{i'}}\prod_{j=1}^{a_i}\mathrm{Lc}_{i'}(h^*_{ij}) = \prod_{i\in\mathcal{N}_{i'}}(\xi^{p_{i',l_{i'}+1}} - c_{i'i}^{p_{i',l_{i'}+1}})^{(\tilde{p}_i a_i)/\tilde{p}_{i'}} = \eta_{i'}(\xi)(\xi^{p_{i',l_{i'}+1}} - c_{i'i'}^{p_{i',l_{i'}+1}})^{a_{i'}}.$$

Since $\deg(\sigma) = m_{i'}(\vec{a})$, identity (9) implies that $\mathrm{Lc}_{i'}(h^*_{k_{i'}}) = c_{i'}\sigma(\xi)$ for some $c_{i'} \in \mathbb{C}^*$, and consequently we get, using Lemma 5.5, that

$$\mathrm{ord}_{\xi - c_{i'i'}}\left(\mathrm{Lc}_{i'}(h^*_{k_{i'}})\right) = \mathrm{ord}_{\xi - c_{i'i'}}(\sigma(\xi)) = m_{i',c_{i'i'}}(\vec{a}) + a_{i'}p_{i',l_{i'}+1}.$$

On the other hand, assertion 2 of Claim 5.7.1, coupled with identity (9) and Lemma 5.5 imply that $\mathrm{ord}_{\xi - c_{i'i'}}(h') \leq m_{i',c_{i'i'}}(\vec{a}) + a_{i'}p_{i',l_{i'}+1}$ for all $h' \in \Gamma(X, \mathcal{O}(\sum_j d_j\Gamma_j))$ such that $\delta_{i'}(h') = d_{i'}$. The proves (11), and finishes the proof that (a) ⇒ (b).

(b) ⇒ (a): Pick $\vec{a}$ that satisfies (8), (9), (10), (11), and set $\vec{d} := \vec{d}_{\vec{a}}$. At first the following claim:

Claim 5.7.3. *Fix i, $1 \leq i \leq N$. Let $\eta_i(\xi)$ be as in (14). A generic $h \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$ satisfies $\mathrm{Lc}_i(h) = c\eta_i(\xi)\prod_{j=1}^{a_i}\left(\xi^{p_{i,l_i+1}} - \xi_{ij}^{p_{i,l_i+1}}\right)$ for some $c \in \mathbb{C}^*$ and $\{\xi_{ij}\}_{i,j} \subseteq \mathbb{C}^*$ that satisfies the 'base point free condition' (12). In other words, assertion 1 of Claim 5.7.1 is true for $i' := i$.*

Proof. We prove the claim by induction on $|\mathcal{N}_i|$. Indeed, if $|\mathcal{N}_i| = 1$, then $\eta_i(\xi) = 1$, $m_{i,c_{ii}}(\vec{a}) = 0$, and $m_i(\vec{a}) = p_{i,l_i+1}a_i$. By assumption there exists $h_1, h_2 \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$ such that

(1) $\delta_i(h_1) = \delta_i(h_2) = d_i$,
(2) $\mathrm{Lc}_i(h_1) = c_1(\xi - c_{ii})^{m_i(\vec{a})}$, $c_1 \in \mathbb{C}^*$, and
(3) $\mathrm{ord}_{\xi - c_{ii}}(\mathrm{Lc}_i(h_2)) = 0$.

It follows that for a generic $\lambda \in \mathbb{C}^*$, the roots ξ_{ij} of $\mathrm{Lc}_i(h_1 + \lambda h_2)$ satisfies the base point free condition (12). Remark-Definition 5.6 then implies that Claim 5.7.3 holds for i.

Now pick i' such that Claim 5.7.3 holds for all i with $|\mathcal{N}_i| < |\mathcal{N}_{i'}|$.

Claim 5.7.4. *Let $h' \in \Gamma(X, \mathcal{O}(\sum_j d_j\Gamma_j))$ such that $\delta_{i'}(h') = d_{i'}$. Then $\mathrm{Lc}_{i'}(h') = \eta_{i'}(\xi)\mu_{h'}(\xi)$ for some $\mu_{h'} \in \mathbb{C}[\xi]$. In other words, assertion 2 of Claim 5.7.1 holds for i'.*

Proof. Pick a *generic* $h' \in \Gamma(X, \mathcal{O}(\sum_j d_j\Gamma_j))$ such that $\delta_{i'}(h') = d_{i'}$. It suffices to show that Claim 5.7.4 holds for h'. Fix $i \in \mathcal{N}_{i'} \setminus \{i'\}$. Property (8) implies that $\delta_i(h) = d_i$. Let ζ be a primitive p_{i,l_i+1}-th root of unity. The inductive hypothesis implies that h' has $a_i p_{i,l_i+1}$ degree-wise Puiseux roots of the form

$$\psi_{ijk}(x) = \phi_i(x) + \zeta^k \xi_{ij} x^{r_i} + \text{l.d.t.}$$

for $1 \le k \le p_{i,l_i+1}$, $1 \le j \le a_i$, and ξ_{ij}'s that satisfy the 'base point free condition' (12). For each such root, there are $p_i/p_{i'}$ conjugates which are of the form

$$\phi_{i'}(x) + c_{i'i} x^{r_{i'}} + \text{l.d.t.}$$

where $c_{i'i}$ is as in Definition 5.4. It follows that each $i \in \mathcal{N}_{i'} \setminus \{i'\}$ contributes a factor of $\eta_{i'i}(\xi) := (\xi^{p_{i',l_{i'}+1}} - c_{i'i}^{p_{i',l_{i'}+1}})^{a_i\tilde{p}_i/\tilde{p}_{i'}}$ to $\mathrm{Lc}_{i'}(h')$. Consequently $\mathrm{Lc}_{i'}(h')$ has a factor of the form $\prod_{i \in \mathcal{N}_{i'} \setminus \{i'\}} \eta_{i'i}(\xi) = \eta_{i'}(\xi)$. □

Proof of Claim 5.7.3 continued. Using property (11) we can pick $(\alpha, \beta, \vec{\gamma}) \in \mathcal{E}_{\vec{d}}$ such that $\delta_{i'}(g_{\alpha\beta\vec{\gamma}}) = d_{i'}$ and $\mathrm{ord}_{\xi - c_{i'i'}}\left(\mathrm{Lc}_{i'}(g_{\alpha\beta\vec{\gamma}})\right) = a_{i'} p_{i',l_{i'}+1} + m_{i',c_{i'i'}}(\vec{a})$. Since $\mathrm{ord}_{\xi - c_{i'i'}}(\eta_{i'}(\xi)) = m_{i',c_{i'i'}}(\vec{a})$, applying Claim 5.7.4 to $h := g_{\alpha\beta\vec{\gamma}}$ and using property (9), we see

$$\deg_\xi\left(\mathrm{Lc}_{i'}(g_{\alpha\beta\vec{\gamma}})\right) \ge a_{i'} p_{i',l_{i'}+1} + \deg_\xi(\eta_{i'}(\xi)) = \sum_{i \in \mathcal{N}_{i'}} \tilde{p}_i a_i / p_{i'} = m_{i'}(\vec{a}) \ge \deg_\xi\left(\mathrm{Lc}_{i'}(g_{\alpha\beta\vec{\gamma}})\right).$$

It follows that $\deg_\xi\left(\mathrm{Lc}_{i'}(g_{\alpha\beta\vec{\gamma}})\right) = m_{i'}(\vec{a})$, and $\mathrm{Lc}_{i'}(g_{\alpha\beta\vec{\gamma}}) = c(\xi - c_{i'i'})^{a_{i'}p_{i',l_{i'}+1}}\eta_{i'}(\xi)$ for some $c \in \mathbb{C}^*$. On the other hand, property (10) and Claim 5.7.4 imply that there is $(\alpha', \beta', \vec{\gamma}') \in \mathcal{E}_{\vec{d}}$ such that $\delta_{i'}(g_{\alpha'\beta'\vec{\gamma}'}) = d_{i'}$ and $\mathrm{Lc}_{i'}\left(g_{\alpha'\beta'\vec{\gamma}'}\right) = c'\eta_{i'}(\xi)\mu(\xi)$ for some $c' \in \mathbb{C}^*$ and $\mu(\xi) \in \mathbb{C}[\xi]$ with $\mathrm{ord}_{\xi - c_{i'i'}}(\mu(\xi)) = 0$. It follows that for a generic $\lambda \in \mathbb{C}^*$, the roots $\xi_{i'j}$ of $\mathrm{Lc}_{i'}(g_{\alpha\beta\vec{\gamma}} + \lambda g_{\alpha'\beta'\vec{\gamma}'})$ satisfy the base point free condition (12). Remark-Definition 5.6 then implies Claim 5.7.3 holds for i', and completes the proof of Claim 5.7.3. □

Now pick a generic $h \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$. Claim (5.7.3) implies that h has a factorization of the form

$$h = g(x,y) \prod_{i=1}^{N} \prod_{j=1}^{a'_i} \prod_{k=1}^{b_{ij}\tilde{p}_i} (y - \psi_{ijk}(x)), \tag{16}$$

where

1. $g(x,y) \in \mathbb{C}\langle\langle x \rangle\rangle[y]$,
2. $a_i = \sum_{j=1}^{a'_i} b_{ij}$, $1 \le i \le N$, and
3. ψ_{ijk}'s are conjugates of some ψ_{ij} of the form $\phi_i(x) + \xi_{ij}x^{r_i} + \text{l.o.t.}$ for $\xi_{ij} \in \mathbb{C}^*$ which satisfy the 'base point free' condition (12).

Using the 'base point freeness' property of ξ_{ij}'s, it is straightforward to compute that for each i, i', j, $1 \leq i, i' \leq N$, $1 \leq j \leq a'_i$,

$$\delta_{i'}\left(\prod_{k=1}^{b_{ij}\tilde{p}_i}(y - \psi_{ijk}(x))\right) = b_{ij}\omega_{i'i}, \text{ so that}$$

$$\delta_{i'}(h/g) = \sum_{i=1}^{N} a_i\omega_{i'i} = d_{i'}.$$

It follows that $\delta_{i'}(g) = 0$ for all i', $1 \leq i' \leq N$, and consequently, g is a constant. But then (16) and the 'base point freeness' property of ξ_{ij}'s imply that for generic $h_1, h_2 \in \Gamma(X, \mathcal{O}(\sum d_j\Gamma_j))$, the closures C_i, $1 \leq i \leq 2$, on X of the curves defined by h_i on $\mathbb{C}^2$ intersect distinct and *non-singular* points of $X \setminus \mathbb{C}^2$. This proves (a), and completes the proof of Theorem 5.7. □

6. The case of X_V, $V \in \mathcal{V}$

In this section we illustrate Theorem 4.5, Corollary 5.3 and Theorem 5.7 for the cases of X_V for $V \in \mathcal{V}_1$ and $\mathcal{V}$.

Let $V := \mathbb{C}\langle G, H^d\rangle \in \mathcal{V}_1$. Identify $\mathbb{P}^2 \setminus \{H = 0\}$ with $\mathbb{C}^2$. Then $g := G|_{\mathbb{C}^2}$ defines a curve C with one place at infinity. Denote the irreducible components of $X_V \setminus \mathbb{C}^2$ by $\Gamma_0, \ldots, \Gamma_N$, where Γ_0 is the strict transform of the line at infinity $H = 0$. Recall that the dual graph of the curves at infinity are of the form as in Figure 2. Recall that there are polynomials $g_1, \ldots, g_s, g_{s+1} = g$ (the *approximate roots* of g introduced by Abhyankar and Moh) such that each g_k is a Γ_{i_k}-curvette. Choose coordinates (x, y) on $\mathbb{C}^2$ such that the point at infinity on C is on the closure of the x-axis. Then each g_k, $1 \leq k \leq s+1$, is a monic polynomial in y (as an element of $\mathbb{C}[x][y]$). Set $g_0 := x$.

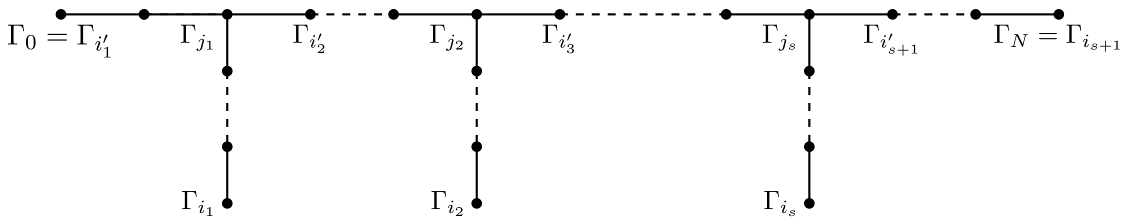

Figure 2. Dual graph of curves at infinity on X_V for $V \in \mathcal{V}_1$

Fix k, $0 \leq k \leq N$. Let l_k, $0 \leq l_k \leq s$, be such that $\Gamma_k \in [\Gamma_{i'_{l_k+1}}, \Gamma_{i_{l_k+1}}]$ (where $[\Gamma_{i'_j}, \Gamma_{i_j}]$ is defined as in the proof of Proposition 3.8). Also, let δ_k be the semidegree on $\mathbb{C}[x, y]$ associated to Γ_k.

Remark 6.1. The above definition of l_k agrees with the definition in Notation 4.0, i.e. the degree-wise Puiseux series associated to δ_k has precisely l_k+1 *formal Puiseux pairs* (Definition 7.2) $(q_{k,1}, p_{k,1}), \ldots, (q_{k,l_k+1}, p_{k,l_k+1})$. Moreover,

(1) $p_{k,j} = \frac{\gcd(\delta_k(g_0),\ldots,\delta_k(g_{j-1}))}{\gcd(\delta_k(g_0),\ldots,\delta_k(g_j))}$, $1 \leq j \leq l_k + 1$.
(2) We can set $f_{kl_k} := g_{l_k}$ (where the correspondence between f_{kl_k} and δ_k is as in Lemma 4.1)

Recall that $\{g_1, \ldots, g_{s+1}\}$ satisfies the properties of $\{f_{ij}\}_{i,j}$ of Theorem 4.5 for $X = X_V$ (Remark 4.2). For each k_1, k_2, $1 \leq k_1 \leq k_2 \leq s+1$, define

$$e_{k_1,k_2} := \frac{\deg(g_{k_2})}{\deg(g_{k_1})}. \tag{17}$$

Let $\mathcal{B} := \{i : 0 \leq i \leq N$, and Γ_i is on the (only) 'horizontal segment' (which has Γ_0 and Γ_N on the two ends) of the graph in Figure 2$\}$. The following lemma, which follows from more or less straightforward computations, computes $\mathcal{N}_i, \mathcal{N}_{i,c}, c_{ii}$ and Lc_i (from Definition 5.4 and Remark-Definition 5.6), $1 \leq i \leq N$.

Lemma 6.2. *Set $f_{kl_k} := g_{l_k}$, $0 \leq k \leq N$.*

1. *Fix i, k, $0 \leq i, k \leq N$. Then $k \in \mathcal{N}_i$ iff one of the following conditions holds:*
   1. *(a) $k = i$.*
   2. *(b) $i \in \mathcal{B}$ and the position of Γ_k on the graph of Figure 2 is below and/or to the right of Γ_i.*
   3. *(c) Γ_i and Γ_k are both on the same 'vertical segment' of the graph of Figure 2 and Γ_k is below Γ_i.*
2. *(a) If $k \notin \{j_1, \ldots, j_s\}$, then $\mathcal{N}_{k,c_{kk}} = \mathcal{N}_k \setminus \{k\}$.*
   (b) If $k = j_q$, $1 \leq q \leq s$, then $c_{kk} = 0$ and $\mathcal{N}_k = \mathcal{N}_{j_q,0} \cup \mathcal{N}_{j_q,c_q^*} \cup \{j_q\}$ for some $c_q^* \neq 0$. Moreover,
      - *(i) $\mathcal{N}_{j_q,0}$ consists of all l such that $l \neq j_q$ and the position of Γ_l on the graph in Figure 2 is on the vertical segment between Γ_{j_q} and Γ_{i_q}.*
      - *(ii) $\mathcal{N}_{j_q,c_q^*} = \mathcal{N}_{j_q} \setminus \big(\mathcal{N}_{j_q,0} \cup \{j_q\}\big)$.*
3. *(a) If $k \notin \mathcal{B}$, then $\mathrm{Lc}_k(g_1^{\alpha_1} \cdots g_{s+1}^{\alpha_{s+1}}) = c(\xi - c_{kk})^{\alpha_{l_k+1}}$ for some $c \in \mathbb{C}^*$. If in addition $k \neq i_{l_k+1}$, then $c_{kk} = 0$.*
   (b) If $k \in \mathcal{B} \setminus \{j_1, j_2, \ldots, j_s\}$, then $\mathrm{Lc}_k(g_1^{\alpha_1} \cdots g_{s+1}^{\alpha_{s+1}}) = c(\xi - c_{kk})^{\sum_{i=l_k+1}^{s+1} \alpha_i e_{l_k+1,i}}$ for some $c \in \mathbb{C}^*$.
   (c) If $k = j_q$ for some q, $1 \leq q \leq s$, then $\mathrm{Lc}_k(g_1^{\alpha_1} \cdots g_{s+1}^{\alpha_{s+1}}) = c\xi^{\alpha_q}(\xi^{e_{q,q+1}} - c_q^*)^{\sum_{i=q+1}^{s+1} \alpha_i e_{q+1,i}}$, where $c \in \mathbb{C}^*$ and c_q^* is as in assertion 2b. □

The following result follows from combining Theorem 4.5, Corollary 5.3 and Theorem 5.7 with the above observations (plus Remark 5.2). At first we recall some notations: let $\vec{e}_j$, $0 \leq j \leq N$, be the j-th unit vector in $\mathbb{Z}^{N+1}$. For $\vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}^N$, write $\Gamma_{\vec{d}}$ for the divisor $\sum_{j=0}^N d_j \Gamma_j \in \mathrm{Cl}(X_V)$ and if $f \in \Gamma\left(X_V, \mathcal{O}(\Gamma_{\vec{d}})\right)$, we denote by $(f)_{\vec{d}}$ the 'copy' of f in the $\vec{d}$'th graded component of $\mathcal{R}(X_V)$. Moreover, for $f \in \mathbb{C}(x,y)$ we denote by $\vec{\delta}(f)$ the element $(\delta_0(f), \ldots, \delta_N(f)) \in \mathbb{Z}^{N+1}$.

Corollary 6.3.

1. *$(g_j)_{\vec{\delta}(g_j)}$'s together with $(1)_{\vec{e}_j}$, $0 \leq j \leq N$, generate $\mathcal{R}(X_V)$ as a $\mathbb{C}$-algebra.*
2. *Let $\vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}^{N+1}$. Let $\mathcal{E}_{\vec{d}}$ be the collection of all $\vec{\alpha} := (\alpha_0, \ldots, \alpha_{s+1}) \in \mathbb{Z}_{\geq 0}^{s+2}$ such that $\sum_{j=0}^{s+1} \alpha_j \delta_k(g_j) \leq d_k$ for all k, $0 \leq k \leq N$. Then $\Gamma\left(X_V, \mathcal{O}(\Gamma_{\vec{d}})\right)$ is a generated as a vector space over $\mathbb{C}$ by $\{\prod_{j=0}^{s+1} g_j^{\alpha_j} : \vec{\alpha} \in \mathcal{E}_{\vec{d}}\}$.*
3. *Let $\vec{d} := (d_0, \ldots, d_N) \in \mathbb{Z}^{N+1}$. Define $\mathcal{E}'_{\vec{d}} := \{(\alpha_0, \sum_{j=1}^{s+1} \alpha_j \deg_y(g_j)) : (\alpha_0, \ldots, \alpha_{s+1}) \in \mathcal{E}_{\vec{d}}\} \subseteq \mathbb{Z}_{\geq 0}^2$. Then $\dim \Gamma\left(X_V, \mathcal{O}(\Gamma_{\vec{d}})\right) = |\mathcal{E}'_{\vec{d}}|$. For each $(a,b) \in \mathcal{E}'_{\vec{d}}$, pick any $\vec{\alpha} \in \mathcal{E}_{\vec{d}}$*

such that $a=\alpha_0$ *and* $b=\sum_{j=1}^{s+1}\alpha_j \deg_y(g_j)$, *and define* $g_{a,b} := \prod_{j=0}^{s+1} g_j^{\alpha_j}$. *Then* $\{g_{a,b} : (a,b) \in \mathcal{E}'_{\vec{d}}\}$ *is a basis of* $\Gamma\left(X_V, \mathcal{O}(\Gamma_{\vec{d}})\right)$.

(4) *For each* $\vec{d} := (d_0,\ldots,d_N) \in \mathbb{Z}^{N+1}$, *the following are equivalent:*
   (a) $\Gamma_{\vec{d}} \in P^{st}(X_V)$.
   (b) $d_j = \max\left\{\sum_{k=0}^{s+1}\alpha_k\delta_j(g_k) : (\alpha_0,\ldots,\alpha_{s+1}) \in \mathcal{E}_{\vec{d}}\right\}$ *for all* j, $0 \le j \le N$.
   (c) *For each* j, $0\le j\le N$, *there exists* $(\alpha_0,\ldots,\alpha_{s+1}) \in \mathbb{Z}^{s+2}_{\ge 0}$ *such that* $\sum_{k=0}^{s+1}\alpha_k\delta_j(g_k) = d_j$ *and* $\sum_{k=0}^{s+1}\alpha_k\delta_i(g_k) \le d_i$ *for all* i, $0\le i\le N$.

(5) *For each* k, $0\le k\le N$, *and each* $\vec{\alpha} := (\alpha_0,\ldots,\alpha_{s+1}) \in \mathbb{Z}^{s+2}_{\ge 0}$, *define*

$$\mu_k(\vec{\alpha}) := \begin{cases} \alpha_{l_k+1} & \text{if } k \notin \mathcal{B} \\ \sum_{i\ge l_k+1}\alpha_i e_{l_k+1,i} & \text{if } k\in\mathcal{B}. \end{cases} \qquad \nu_k(\vec{\alpha}) := \begin{cases} \alpha_{l_k+1} & \text{if } k \notin \mathcal{B} \\ \sum_{i\ge l_k+1}\alpha_i e_{l_k+1,i} & \text{if } k\in\mathcal{B}\setminus\{j_1,\ldots,j_s\} \\ \alpha_q & \text{if } k = j_q,\ 1\le q\le s. \end{cases}$$

Let $\vec{a} := (a_0,\ldots,a_N) \in \mathbb{Z}^{N+1}_{\ge 0}$. *Define* $d_k(\vec{a}) := \sum_{i=0}^N a_i\omega_{ki}$, $0\le k\le N$, *where*

$$\omega_{ki} := \begin{cases} p_{i,l_i+1}\delta_k(f_{il_i}) & \text{if } k\notin\mathcal{N}_i \\ \dfrac{\prod_{j=1}^{l_k+1}p_{k,j}}{\prod_{j=1}^{l_i}p_{i,j}}\delta_i(f_{il_i}) & \text{if } k\in\mathcal{N}_i. \end{cases}$$

and set $\vec{d}_{\vec{a}} := (d_0(\vec{a}),\ldots,d_N(\vec{a}))$. *For each* k, $0\le k\le N$, *define*

$$m_k(\vec{a}) := \sum_{i\in\mathcal{N}_k}\frac{\prod_{j=1}^{l_i+1}p_{i,j}}{\prod_{j=1}^{l_k}p_{k,j}}a_i, \qquad n_k(\vec{a}) := \sum_{i\in\mathcal{N}_{k,c_{kk}}}\frac{\prod_{j=1}^{l_i+1}p_{i,j}}{\prod_{j=1}^{l_k}p_{k,j}}a_i.$$

Then for each $\vec{d}\in\mathbb{Z}^{N+1}$, *the following are equivalent:*
(a) $\Gamma_{\vec{d}} \in \tilde{P}(X_V)$.
(b) $\vec{d} = \vec{d}_{\vec{a}}$ *for some* $\vec{a}\in\mathbb{Z}^{N+1}_{\ge 0}$ *such that for each* k, $0\le k\le N$,

$$d_k = \max\left\{\sum_{j=0}^{s+1}\alpha_j\delta_k(g_j) : \vec{\alpha}\in\mathcal{E}_{\vec{d}}\right\},$$

$$m_k(\vec{a}) = \max\left\{\mu_k(\vec{\alpha}) : \vec{\alpha}\in\mathcal{E}_{\vec{d}},\ \sum_{j=0}^{s+1}\alpha_j\delta_k(g_j) = d_k\right\},$$

$$n_k(\vec{a}) = \min\left\{\nu_k(\vec{\alpha}) : \vec{\alpha}\in\mathcal{E}_{\vec{d}},\ \sum_{j=0}^{s+1}\alpha_j\delta_k(g_j) = d_k\right\},$$

$$a_k p_{k,l_k+1} + n_k(\vec{a}) = \max\left\{\nu_k(\vec{\alpha}) : \vec{\alpha}\in\mathcal{E}_{\vec{d}},\ \sum_{j=0}^{s+1}\alpha_j\delta_k(g_j) = d_k\right\}. \quad \square$$

Let $V_i := \mathbb{C}\langle G_i, H^d\rangle \in \mathcal{V}$, $1\le i\le 2$. Identify $\mathbb{P}^2\setminus\{H=0\}$ with $\mathbb{C}^2$. Assume that the germs at infinity of curves defined by G_1 and G_2 are equisingular. Then there is a natural bijection between the sets of irreducible curves at infinity on X_{V_i}'s which induces an isomorphism $\rho : \mathrm{Pic}(X_{V_1}) \cong \mathrm{Pic}(X_{V_2})$.

Corollary 6.4. *Assume $V_1, V_2 \in \mathcal{V}_1$. Then*

(1) For each line bundle $\mathcal{L} \in \operatorname{Pic}(X_{V_1})$, $\dim_{\mathbb{C}}(\Gamma(X_{V_1}), \mathcal{L}) = \dim_{\mathbb{C}}(\Gamma(X_{V_2}), \mathcal{O}(\rho(\mathcal{L})))$.

(2) ρ induces isomorphisms $P^{st}(X_{V_1}) \cong P^{st}(X_{V_2})$ and $\tilde{P}(X_{V_1}) \cong \tilde{P}(X_{V_2})$.

Proof. Corollary 6.3 implies that for each $\vec{d} := (d_0, \dots, d_N) \in \mathbb{Z}^{N+1}$, the dimension of $\Gamma(X_{V_i}, \mathcal{O}(\Gamma_{\vec{d}}))$, or whether $\vec{d}$ is an element of $P^{st}(X_{V_i})$ or $\tilde{P}(X_{V_i})$, depends only the values of $\deg_y(g_{ij})$ and $\delta_{ik}(g_{ij})$'s, where g_{ij}'s are approximate roots of $g_i := G_i|_{\mathbb{C}^2}$ and δ_{ik}'s are semidegrees on $\mathbb{C}[x, y]$ associated to curves at infinity on X_{V_i}. Since $\deg_y(g_{1j}) = \deg_y(g_{2j})$ and $\delta_{1k}(g_{1j}) = \delta_{2k}(g_{2j})$ for all j, k, this proves Corollary 6.4. □

Example 6.5. Let $g_1 := (y^3 - x^2)^2 - 9x$ and $g_2 := (y^3 - x^2 - 3y)^2 - 9(x + 2y)$. Figure 3 contains the dual graph of the curves on $X_{V_i} \setminus \mathbb{C}^2$, with $\Gamma_{i,j}$, $1 \le j \le 15$, being the exceptional curve for the j-th blow up, and $\Gamma_{i,0}$ being the strict transform of the line at infinity. For each i, j, let $\delta_{i,j}$ be the semidegree corresponding to $\Gamma_{i,j}$.

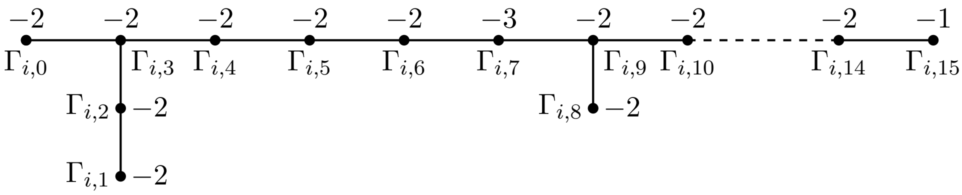

FIGURE 3. Dual graph of curves at infinity on X_{V_i} of Example 6.5

X_{V_1}		X_{V_2}	
δ	$\tilde{\phi}(x,\xi) = \phi(x) + \xi x^r$	δ	$\tilde{\phi}(x,\xi) = \phi(x) + \xi x^r$
$\delta_{1,0}$	ξx	$\delta_{2,0}$	ξx
$\delta_{1,1}$	ξ	$\delta_{2,1}$	ξ
$\delta_{1,2}$	$\xi x^{\frac{1}{2}}$	$\delta_{2,2}$	$\xi x^{\frac{1}{2}}$
$\delta_{1,3}$	$\xi x^{\frac{2}{3}}$	$\delta_{2,3}$	$\xi x^{\frac{2}{3}}$
$\delta_{1,3+k}$, $1 \le k \le 4$	$x^{\frac{2}{3}} + \xi x^{\frac{2-k}{3}}$	$\delta_{2,3+k}$, $1 \le k \le 4$	$x^{\frac{2}{3}} + \xi x^{\frac{2-k}{3}}$
$\delta_{1,8}$	$x^{\frac{2}{3}} + \xi x^{-1}$	$\delta_{2,8}$	$x^{\frac{2}{3}} + x^{-\frac{2}{3}} + \xi x^{-1}$
$\delta_{1,9}$	$x^{\frac{2}{3}} + \xi x^{-\frac{5}{6}}$	$\delta_{2,9}$	$x^{\frac{2}{3}} + x^{-\frac{2}{3}} + \xi x^{-\frac{5}{6}}$
$\delta_{1,9+k}$, $1 \le k \le 6$	$x^{\frac{2}{3}} + x^{-\frac{5}{6}} + \xi x^{-\frac{5+k}{6}}$	$\delta_{2,9+k}$, $1 \le k \le 2$	$x^{\frac{2}{3}} + x^{-\frac{2}{3}} + x^{-\frac{5}{6}} + \xi x^{-\frac{5+k}{6}}$
		$\delta_{2,9+k}$, $3 \le k \le 6$	$x^{\frac{2}{3}} + x^{-\frac{2}{3}} + x^{-\frac{5}{6}} + x^{-\frac{7}{6}} + \xi x^{-\frac{5+k}{6}}$

TABLE 1. Generic degree-wise Puiseux series of $\delta_{i,j}$ of Example 6.5

The approximate roots of g_i, $1 \le i \le 2$, are $g_{i,j}$, $0 \le j \le 3$, where

$$g_{1,0} := x,\ g_{1,1} := y,\ g_{1,2} := y^3 - x^2,\ g_{1,3} := (y^3 - x^2)^2 - 9x,$$

$$g_{2,0} := x,\ g_{2,1} := y,\ g_{2,2} := y^3 - x^2 - 3y,\ g_{2,3} := (y^3 - x^2 - 3y)^2 - 9(x + 2y).$$

Table 2 lists the values of $\delta_{i,j}(g_{i,k})$'s. The values of $p_{i,j,k}$'s can be read from the generic degree-wise Puiseux series listed in Table 1 or equivalently, from Table 2 using Remark 6.1. $\tilde{P}(X_{V_i})$ and $P^{st}(X_{V_i})$ can then be computed readily using Corollary 6.3. In particular, note that for each i,j,k, $\delta_{i,j}(g_{i,k})$ does not depend on i, and the same is true for $p_{i,j,k}$'s. It follows that the semigroups $\tilde{P}(X_{V_i})$ and $P^{st}(X_{V_i})$ are also independent of i, as Corollary 6.4 implies.

j	$\delta_{i,j}(g_{i,0})$	$\delta_{i,j}(g_{i,1})$	$\delta_{i,j}(g_{i,2})$	$\delta_{i,j}(g_{i,3})$
0	1	1	3	6
1	1	0	2	4
2	2	1	4	8
3	3	2	6	12
$3+k$ $1 \leq k \leq 4$	3	2	$6-k$	$2(6-k)$
8	3	2	1	3
9	6	4	3	6
$9+k$ $1 \leq k \leq 6$	6	4	3	$6-k$

TABLE 2. Poles of $g_{i,k}$ along curves at infinity on X_{V_i} of Example 6.5

Now consider the general case that $V_i := \mathbb{C}\langle G_i, H^d\rangle \in \mathcal{V}$, $1 \leq i \leq 2$. Write $G_i := \prod_{j=1}^{l} G_{ij}$, where each G_{ij} defines a curve C_{ij} with one place at infinity, and consider the approximate roots g_{ijk} of $g_{ij} := G_{ij}|_{\mathbb{C}^2}$. Fix i, $1 \leq i \leq 2$. Let $\Gamma_{i,t}$, $0 \leq t \leq N$, be the curves at infinity on X_{V_i}. Each $\Gamma_{i,t}$ is the exceptional divisor of the blow up of some point in an infinitesimal neighborhood of the point at infinity of some C_{ij}. Since the germs at infinity of $G_1 = 0$ and $G_2 = 0$ are equisingular, it follows, as in the case of Corollary 6.4, that for fixed t,j,k, the following values does *not* depend on i:

(1) the order of pole $\delta_{i,t}(g_{ijk})$ along $\Gamma_{i,t}$ of g_{ijk},
(2) $\deg_\xi(\mathrm{Lc}_{i,t}(g_{ijk}))$, (where $\mathrm{Lc}_{i,t}$ is defined for $\delta_{i,t}$ in the same way as Lc_i is defined for δ_i in Remark-Definition 5.6),
(3) $\mathrm{ord}_{\xi - c_{i,t,t}}(\mathrm{Lc}_{i,t}(g_{ijk}))$ (where $c_{i,t,t}$ is defined for $\delta_{i,t}$ in the same way as c_{ii} is defined for δ_i in Definition 5.4).

Combining the above observations with Remark 4.2, Theorem 4.5 and Theorem 5.7 yield

Corollary 6.6.
(1) For each line bundle $\mathcal{L} \in \mathrm{Pic}(X_{V_1})$, $\dim_{\mathbb{C}}(\Gamma(X_{V_1}), \mathcal{L}) = \dim_{\mathbb{C}}(\Gamma(X_{V_2}), \mathcal{O}(\rho(\mathcal{L})))$.
(2) ρ induces isomorphisms $P^{st}(X_{V_1}) \cong P^{st}(X_{V_2})$ and $\tilde{P}_\infty(X_{V_1}) \cong \tilde{P}_\infty(X_{V_2})$. □

7. APPENDIX - PRELIMINARIES ON DEGREE-WISE PUISEUX SERIES

Definition 7.1 (Degree-wise Puiseux series)**.** The field of *degree-wise Puiseux series* in x is

$$\mathbb{C}\langle\langle x\rangle\rangle := \bigcup_{p=1}^{\infty} \mathbb{C}((x^{-1/p})) = \left\{ \sum_{j \leq k} a_j x^{j/p} : k,p \in \mathbb{Z},\ p \geq 1 \right\},$$

where for each integer $p \geq 1$, $\mathbb{C}((x^{-1/p}))$ denotes the field of Laurent series in $x^{-1/p}$. Let ϕ be a degree-wise Puiseux series in x. The *polydromy order* (terminology taken from [CA00]) of ϕ is the smallest positive integer p such that $\phi \in \mathbb{C}((x^{-1/p}))$. For any $r \in \mathbb{Q}$, let us denote by $[\phi]_{>r}$ (resp. $[\phi]_{\geq r}$) sum of all terms of ϕ with order greater than (resp. greater than or equal to) r. Then the *Puiseux pairs* of ϕ are the unique sequence of pairs of relatively prime integers $(q_1, p_1), \ldots, (q_k, p_k)$ such that the polydromy order of ϕ is $p_1 \cdots p_k$, and for all j, $1 \leq j \leq k$,

(1) $p_j \geq 2$,

(2) $[\phi]_{> \frac{q_j}{p_1 \cdots p_j}} \in \mathbb{C}((x^{-\frac{1}{p_0 \cdots p_{j-1}}}))$ (where we set $p_0 := 1$), and

(3) $[\phi]_{\geq \frac{q_j}{p_1 \cdots p_j}} \notin \mathbb{C}((x^{-\frac{1}{p_0 \cdots p_{j-1}}}))$.

The exponents $q_j/(p_1 \cdots p_j)$, $1 \leq j \leq k$, are called the *characteristic exponents* of ϕ. Let $\phi = \sum_{q \leq q_0} a_q x^{q/p}$, where p is the polydromy order of ϕ. Then the *conjugates* of ϕ are $\phi_j := \sum_{q \leq q_0} a_q \zeta^q x^{q/p}$, $1 \leq j \leq p$, where ζ is a primitive p-th root of unity.

Definition 7.2. Let δ be a semidegree and, as in (1), let $\tilde{\phi}(x, \xi) := \phi(x) + \xi x^r$ be the *generic degree-wise Puiseux series* associated to δ. Let the Puiseux pairs of ϕ be $(q_1, p_1), \ldots, (q_l, p_l)$. Express r as $q_{l+1}/(p_1 \cdots p_l p_{l+1})$ where $p_{l+1} \geq 1$ and $\gcd(q_{l+1}, p_{l+1}) = 1$. Then the *formal Puiseux pairs* of $\tilde{\phi}$ are $(q_1, p_1), \ldots, (q_{l+1}, p_{l+1})$. Note that

(1) $\delta(x) = p_1 \cdots p_{l+1}$,

(2) it is possible that $p_{l+1} = 1$ (as opposed to other p_k's, which are always ≥ 2).

The *formal characteristic exponents* of $\tilde{\phi}$ are $q_j/(p_1 \cdots p_j)$, $1 \leq j \leq l+1$.

Example 7.3. Let (p, q) are integers such that $p > 0$ and δ be the weighted degree on $\mathbb{C}(x, y)$ corresponding to weights p for x and q for y. Then $\tilde{\phi} = \xi x^{q/p}$ (i.e. $\phi = 0$).

The rest of this section is devoted to a proof of Lemma 7.6 below, which is used in the proofs of Theorem 4.5 and Theorem 5.7.

Definition 7.4. Let $\phi = \sum_j a_j x^{q_j/p} \in \mathbb{C}\langle\langle x \rangle\rangle$ be a degree-wise Puiseux series with polydromy order p and r be a multiple of p. Then for all $c \in \mathbb{C}$ we define

$$c \star_r \phi := \sum_j a_j c^{q_j r/p} x^{q_j/p}.$$

Remark 7.5. Let ϕ, p be as in Definition 7.4. Then

(1) Every conjugate of ϕ is of the form $\zeta^k \star_p \phi$ where $0 \leq k \leq p-1$ and ζ is a primitive p-th root of unity.

(2) Let d and e be positive integers, and $c \in \mathbb{C}$. Then $c \star_{pde} \phi = c^e \star_{pd} \phi = c^{de} \star_p \phi$.

Lemma 7.6. *Let δ be a semidegree with generic degree-wise Puiseux series $\tilde{\phi}(x, \xi) = \phi(x) + \xi x^r$. Let $\psi_1, \psi_2 \in \mathbb{C}\langle\langle x \rangle\rangle$, and $g_1, g_2 \in \mathbb{C}\langle\langle x \rangle\rangle[y]$ be defined as*

$$g_i = \prod_{\substack{\psi_{ij} \text{ is a con-} \\ \text{jugate of } \psi_i}} (y - \psi_{ij}(x)).$$

Let $\epsilon_{ij} := \deg_x(\tilde{\phi}(x, \xi) - \psi_{ij}(x))$, and $\epsilon_i := \min_j\{\epsilon_{ij}\}$. If $\epsilon_1 \geq \epsilon_2$, then

$$\frac{\delta(g_1)}{\deg_y(g_1)} \geq \frac{\delta(g_2)}{\deg_y(g_2)}.$$

Proof. W.l.o.g. (replacing ψ_i's by some of their conjugates if necessary) we may assume that $\epsilon_i = \deg_x(\tilde\phi(x,\xi) - \psi_i(x))$ for each i. Then we can write:

$$\begin{aligned}
\phi(x) &= \phi_1(x) + \phi_2(x) + \phi_3(x,\xi)\\
\psi_1(x) &= \phi_1(x) + \hat\psi_1(x)\\
\psi_2(x) &= \phi_1(x) + \phi_2(x) + \hat\psi_2(x),
\end{aligned}$$

where all terms of ϕ_{j+1} and $\hat\psi_j$ have lower degree in x than each term of ϕ_j for $j = 1, 2$. Let the polydromy orders of ϕ_1, $\phi_1 + \phi_2$, ψ_1, ψ_2 be respectively p_1, p_1p_2, p_1q_1, and $p_1p_2q_2$. Let $\zeta_1, \zeta_2, \hat\zeta_1, \hat\zeta_2$ be primitive roots of unity of orders $p_1, p_1p_2, p_1q_1, p_1p_2q_2$ respectively. Then

$$\begin{aligned}
\delta(g_1) &= \delta(x) \sum_{j=0}^{p_1q_1-1} \deg_x\left(\tilde\phi(x,\xi) - \hat\zeta_1^{j} \star_{p_1q_1} \psi_1(x)\right)\\
&= \delta(x) \sum_{j_1=0}^{q_1-1}\sum_{j=0}^{p_1-1} \deg_x\left(\phi_1(x) + \phi_2(x) + \xi x^r - \hat\zeta_1^{j_1p_1+j} \star_{p_1q_1} (\phi_1(x) + \hat\psi_1(x))\right)\\
&= \delta(x) \sum_{j_1=0}^{q_1-1}\sum_{j=0}^{p_1-1} \deg_x\left(\phi_1(x) + \phi_2(x) + \xi x^r - \zeta_1^{j} \star_{p_1} \phi_1(x) + \hat\zeta_1^{j_1p_1+j} \star_{p_1q_1} \hat\psi_1(x)\right)\\
&= \delta(x) \sum_{j_1=0}^{q_1-1}\left(\deg_x\left(\phi_2(x) + \xi x^r - \hat\zeta_1^{j_1p_1} \star_{p_1q_1} \psi_1(x)\right) + \sum_{j=1}^{p_1-1}\deg_x\left(\phi_1(x) - \zeta_1^{j} \star_{p_1} \phi_1(x)\right)\right)\\
&= q_1\delta(x)\left(\epsilon_1 + \sum_{j=1}^{p_1-1}\deg_x\left(\phi_1(x) - \zeta_1^{j} \star_{p_1} \phi_1(x)\right)\right),\\
\delta(g_2) &= \delta(x) \sum_{j=0}^{p_1p_2q_2-1} \deg_x\left(\tilde\phi(x,\xi) - \hat\zeta_2^{j} \star_{p_1p_2q_2} \psi_2(x)\right)\\
&= \delta(x) \sum_{j_2=0}^{p_2q_2-1}\sum_{j_1=0}^{p_1-1} \deg_x\left(\phi_1(x) + \phi_2(x) + \xi x^r - \hat\zeta_1^{j_2p_1+j_1} \star_{p_1p_2q_2} (\phi_1(x) + \phi_2(x) + \hat\psi_2(x))\right)\\
&= \delta(x) \sum_{j_2=0}^{p_2q_2-1}\left(\deg_x\left(\phi_2(x) + \xi x^r - \hat\zeta_1^{j_2p_1} \star_{p_1p_2q_2} (\phi_2(x) + \hat\psi_2(x))\right) + \sum_{j_1=1}^{p_1-1}\deg_x\left(\phi_1(x) - \zeta_1^{j_1} \star_{p_1} \phi_1(x)\right)\right)\\
&\le p_2q_2\delta(x)\left(\epsilon_1 + \sum_{j=1}^{p_1-1}\deg_x\left(\phi_1(x) - \zeta_1^{j} \star_{p_1} \phi_1(x)\right)\right).
\end{aligned}$$

It follows that $\delta(g_2)/(p_1p_2q_2) \le \delta(g_1)/(p_1q_1)$, as required. □

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