# REAL SPECTRA OF QUANTUM GROUPS

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Abstract. The only noncommutative ring for which the real spectrum has been computed so far is the quantum affine ring $\mathbb{R}_q[x,y]$, see [19]. The aim of this paper is to describe the real spectra of quantum affine rings $k_{\mathbf{q}}[x_1,\ldots,x_n]$ where $k$ is a formally real affine $\mathbb{R}$-algebra and $\mathbf{q} \in M_n(\mathbb{R}^+)$. As a by-product we compute the real spectra of quantized enveloping algebra $U_q(\mathfrak{sl}_2(\mathbb{R}))$ and quantum special linear group $\mathcal{O}_q(SL_2(\mathbb{R}))$. Formal reality and semireality is characterized for the following classes of quantum groups: quantum affine rings, quantized enveloping algebras, quantized function algebras, quantized Weyl algebras.

## 1. Introduction

Let $R$ be a ring. A subset $P \subseteq R$ is an *ordering* if $P \cdot P \subseteq P$, $P+P \subseteq P$, $P \cup -P = R$ and $P \cap -P$ is a prime ideal of $R$. The set of all orderings of $R$ is denoted by $\mathrm{Sper}R$ and called the *real spectrum* of $R$. The rings with nonempty real spectrum are called *semireal rings.* The study of real spectra of noncommutative semireal rings is called the *noncommutative real algebraic geometry.* The pioneering work in this field has been done by Murray Marshall and his school, [15, 19, 20].

The mapping $\mathrm{supp} : \mathrm{Sper}R \to \mathrm{Spec}R$ defined by $\mathrm{supp}(P) = P \cap -P$ is called *the support.* Prime ideals in the image of supp are called *real prime ideals.* They are always completely prime. If $J$ is a real prime ideal of $R$ then the image and preimage of the canonical projection $R \to R/J$ give a one-to-one correspondence between orderings of $R$ with support $J$ and orderings of $R/J$ with support zero. If $R$ is a Noetherian ring then $R/J$ has a skew field of fractions $\mathrm{Fract}(R/J)$. In this case, we also have a one-to-one correspondence between support zero orderings of $R/J$ and orderings of $\mathrm{Fract}(R/J)$. The problem of computing the real spectrum of a Noetherian semireal ring $R$ therefore consists of two subproblems:

(1) Compute the real prime ideals of $R$.

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(2) For every real prime ideal $J$ of $R$ compute all orderings of $\mathrm{Fract}(R/J)$.

**Example.** If $R$ is the ring of all polynomial functions on a real algebraic variety $V \subseteq \mathbb{R}^n$, then its real prime ideals are in a one-to-one correspondence with subvarieties of $V$. The description of orderings of rational function fields of subvarieties of $V$ consists of the following steps:

(1) Since every rational function field $L$ is a finitely generated extension of $\mathbb{R}$, it can be written as an algebraic extension of a field $F$ which is a purely trancendental extension of $\mathbb{R}$ with a finite trancendence degree.
(2) Orderings of $F$ are computed in [14].
(3) Orderings of $\mathrm{Fract}(R/J)$ can be computed in principle by the general ramification theory of algebraic extensions.

A unital ring $R$ is *formally real* if it has a support zero ordering. Every formally real ring is a domain. A simple ring is formally real if and only if it is semireal.

## 2. Quantum affine spaces

Let $k$ be commutative unital ring, $k^\times$ its set of invertible elements $n$ a nonnegative integer and $\mathbf{q} = (q_{ij}) \in M_n(k^\times)$ multiplicatively antisymmetric (i.e. $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for every $i, j = 1, \ldots, n$). The *quantum affine space* $k_{\mathbf{q}}[x_1, \ldots, x_n]$ is the $k$-algebra on $n$ generators $x_1, \ldots, x_n$ with $n^2$ relations $x_i x_j = q_{ij} x_j x_i$. If $k$ is a domain, then $k_{\mathbf{q}}[x_1, \ldots, x_n]$ is an Ore domain. Its skew field of quotients is called the *quantum Weyl field* $k_{\mathbf{q}}(x_1, \ldots, x_n)$. We denote by $k_{\mathbf{q}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ the localization $k_{\mathbf{q}}[x_1, \ldots, x_n]_{x_1,\ldots,x_n} \subset k_{\mathbf{q}}(x_1, \ldots, x_n)$. When $n = 2$ we write $k_{q_{21}}[x_1, x_2]$ instead of $k_{\mathbf{q}}[x_1, x_2]$.

**Proposition 1.** *The quantum affine space $k_{\mathbf{q}}[x_1, \ldots, x_n]$ is semireal if and only if the ring $k$ is semireal. It is formally real if and only if $k$ has a support zero ordering such that $q_{ij} > 0$ for all $i, j = 1, \ldots, n$.*

**Proof.** A unital subring of a semireal ring is always semireal. In particular if $k_{\mathbf{q}}[x_1, \ldots, x_n]$ is semireal, then $k$ is semireal, too. A ring which has a unital homomorphism into a semireal ring is semireal. In particular, sending $x_1 \to 0, \ldots, x_n \to 0$ we get a unital ring homomorphism $\phi : k_{\mathbf{q}}[x_1, \ldots, x_n] \to k$. If $k$ is semireal, then $k_{\mathbf{q}}[x_1, \ldots, x_n]$ is semireal, too.

If $k_{\mathbf{q}}[x_1,\ldots,x_n]$ has a support zero ordering, then for every $i,j=1,\ldots,n$ the element $x_ix_j$ and $x_jx_i$ have the same sign. It follows that $q_{ij}>0$. Clearly, $k$ has a support zero ordering, too.

Assume now that $k$ has a support zero ordering such that $q_{ij}>0$ for all $i,j=1,\ldots,n$. Every nonzero element $z\in k_{\mathbf{q}}[x_1,\ldots,x_n]$ can be written uniquely as $z=\sum_{i=1}^r c_iM_i$ where $c_i\neq 0$ for $i=1,\ldots,r$ and $M_i$ are standard monomials in $x_1,\ldots,x_n$ such that $M_1<\ldots<M_n$ with respect to lexicographic ordering. Writing $z>0$ if and only if $c_r>0$ defines a support zero ordering on $k_{\mathbf{q}}[x_1,\ldots,x_n]$. $\square$

The following two propositions are variants of [6], Theorem 2.1 and [2], Theoreme I.1.

**Proposition 2.** *Let $k$ be a commutative domain with a support zero ordering $P$ and $\mathbf{q}$ an $n\times n$ matrix such that $q_{ij}\in P^\times := P\cap k^\times$ for every $i,j=1,\ldots,n$. Write $R=k_{\mathbf{q}}[x_1^{\pm 1}\ldots,x_n^{\pm 1}]$.*

*There exists a natural number $0\le r\le n$ and integers $k_{ij}$, $i,j=1,\ldots,n$ such that the elements $t_j:=x_1^{k_{j1}}\cdots x_n^{k_{jn}}$ have the following properties:*

1. *$t_1,\ldots,t_r$ belong to the center of $R$,*
2. *If $l_{r+1},\ldots,l_n\in\mathbb{Z}$ and $t_{r+1}^{l_{r+1}}\cdots t_n^{l_n}$ belongs to the center of $R$, then $l_{r+1}=\ldots=l_n=0$.*
3. *$R\cong K_{\mathbf{p}}[t_{r+1},t_{r+1}^{-1},\ldots,t_n,t_n^{-1}]$ where $K=k[t_1,t_1^{-1},\ldots,t_r,t_r^{-1}]$ and $\mathbf{p}$ is an $(n-r)\times(n-r)$ matrix with entries from $P^\times$.*

**Proof.** Let us define a mapping

$$\Phi:\mathbb{Z}^n\to(P^\times)^n.$$

$$\Phi(i_1,\ldots,i_n)=(q_{11}^{i_1}\cdots q_{1n}^{i_n},\ldots,q_{n1}^{i_1}\cdots q_{nn}^{i_n}).$$

Note that $\Phi$ is a group homomorphism from the additive abelian group $\mathbb{Z}^n$ into the multiplicative abelian group $(P^\times)^n$. Write $N(\Phi)$ for its kernel.

*Claim:* $P^\times$ has no roots of 1.

Take any $x\in P^\times$ which is a root of 1. If $x^2=1$, then either $x=1$ or $x=-1$. The second case is not possible, because $P$ is an ordering. If $x^m=1$ for some odd $m$ then either $x=1$ or $0=2(x^{m-1}+x^{m-2}+\ldots+x+1)=(x^{m-1/2})^2+(x^{m-1/2}+x^{m-3/2})^2+\ldots+(x+1)^2+1$, which is not possible because $P$ is an ordering.

*Claim:* $N(\Phi)$ has a direct complement in $\mathbb{Z}^n$.

By Corollary 28.3 in [5], it suffices to prove that $N(\phi)$ is a pure subgroup of $\mathbb{Z}^n$. If $m\cdot(i_1,\ldots,i_n)\in N(\Phi)$, then for every $j=1,\ldots,n$

we have that $(q_{j1}^{i_1}\cdots q_{jn}^{i_n})^m = 1$. By the previous paragraph it follows that $q_{j1}^{i_1}\cdots q_{jn}^{i_n} = 1$ for every $j = 1,\ldots,n$. Hence, $(i_1,\ldots,i_n) \in N(\Phi)$.

Let $\mathbf{k}_1,\ldots,\mathbf{k}_r$ be a basis of $N(\phi)$ and $\mathbf{k}_{r+1},\ldots,\mathbf{k}_n$ a basis of a direct complement. For every $j = 1,\ldots,n$ write

$$t_j := x_1^{k_{j1}}\cdots x_n^{k_{jn}} \quad \text{where } (k_{j1},\ldots,k_{jn}) = \mathbf{k}_j.$$

Assertion 3. of the proposition follows from the fact that $\mathbf{k}_1,\ldots,\mathbf{k}_n$ is a basis of $\mathbb{Z}^n$.

If $(i_1,\ldots,i_n) \in N(\Phi)$, then for every $j = 1,\ldots,n$. $x_j \cdot x_1^{i_1}\cdots x_n^{i_n} = q_{j1}^{i_1}\cdots q_{jn}^{i_n} x_1^{i_1}\cdots x_n^{i_n} \cdot x_j = x_1^{i_1}\cdots x_n^{i_n}\cdot x_j$. It follows that $x_1^{i_1}\cdots x_n^{i_n}$ is central. Assertion 1 now follows from the definition of $t_1,\ldots,t_r$.

The element $t_{r+1}^{l_{r+1}}\cdots t_n^{l_n} = (x_1^{k_{r+1,1}}\cdots x_n^{k_{r+1,n}})^{l_{r+1}}\cdots(x_1^{k_{n1}}\cdots x_n^{k_{nn}})^{l_n}$ is colinear to the element $x_1^{k_{r+1,1}l_{r+1}+\ldots+k_{n1}l_n}\cdots x_n^{k_{r+1,n}l_{r+1}+\ldots+k_{nn}l_n}$ which belongs to the center if and only if $l_{r+1}\mathbf{k}_{r+1}+\ldots+l_n\mathbf{k}_n \in N(\Phi)$. But $l_{r+1}\mathbf{k}_{r+1}+\ldots+l_n\mathbf{k}_n$ also belongs to a direct complement of $N(\Phi)$. It follows that $l_{r+1} = \ldots = l_n = 0$. This proves Assertion 2. □

An algebra is *affine* if it is commutative, finitely generated and has no zero divisors. If $K$ is formally real affine algebra and $J$ is a real prime ideal of $K$ then $K/J$ is formally real, too.

Recall the geometric description of all real prime ideals of a formally real affine $\mathbb{R}$-algebra from the introduction. Proposition 3 reduces the computation of the real spectra of quantum affine rings over a formally real affine algebras to the computation of support zero orderings of quantum affine rings over (larger) formally real affine algebras.

**Proposition 3.** *Let $k$ be a formally real affine $\mathbb{R}$-algebra. There exists an algorithm which for every formally real quantum affine space over $k$. gives all its real prime ideals.*

*For every formally real affine quantum ring $R$ over $k$ and every real prime ideal $J$ of $R$ there exists a formally real affine $\mathbb{R}$-algebra $L$ and a formally real affine quantum space $S$ over $L$ such the factor ring* $\mathrm{Fract}(R/J)$ *is isomorphic to* $\mathrm{Fract}(S)$.

**Proof.** Let $<$ be an ordering on $k$. For every $n \in \mathbb{N}$ write $A(n)$ for the set off multiplicatively antisymmetric $n\times n$ matrices over $k$ with positive entries.

If $\mathbf{q} \in A(1)$, then $k_{\mathbf{q}}[x_1] = k[x_1]$ is also a formally real affine $\mathbb{R}$-algebra. Its real prime ideals are known. Their factor rings are formally real affine $\mathbb{R}$-algebras. Suppose now that we have found all real prime ideals of all affine quantum rings corresponding to the matrices in $A(n-1)$ and that we know that their factor rings are as required. Pick any matrix $\mathbf{q} \in A(n)$ and any real prime ideal $J$ of $R = k_{\mathbf{q}}[x_1,\ldots,x_n]$.

If $x_i \in J$ for some $i$, then $J$ comes from a real prime ideal $J'$ of the factor ring $R/(x_i)$ which is isomorphic to $R' = k_{\mathbf{q}_i}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$ with $\mathbf{q}_i$ the submatrix of $\mathbf{q}$ with $i$-th row and $i$-th column deleted. The real prime ideals of $R'$ are known from the induction hypothesis. We also know that $R/J \cong R'/J'$ is as required.

If $J \cap \{x_1, \ldots, x_n\} = \emptyset$, then $J$ extends uniquely to a real prime ideal $I$ of $S = k_{\mathbf{q}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Let $t_1, \ldots, t_n$ be as in Proposition 2. We claim that $I = (I \cap K) \cdot S$, where $K = k[t_1^{\pm 1}, \ldots, t_r^{\pm 1}]$. If $z = \sum_i c_i t_{r+1}^{a_{i,r+1}} \cdots t_n^{a_{i,n}} \in I$, where $c_i \in K$, then we can group the terms of $z$ as $z = z_1 + \ldots + z_m$ where $z_i$-s $q$-commute with $t_{r+1}$ for paiwise distinct $q$-s. By conjugating $z$ with $t_{r+1}^j$ for $j = 0, \ldots, m$ we get a linear system for $z_1, \ldots, z_m$ with nonzero determinant which is homogeneous modulo $I$. It follows that $z_1, \ldots, z_m \in I$. For every $i$ group the terms of $z_i$ as $z_i = z_{i1} + \ldots + z_{ir_i}$ where $z_{ij}$-s $q$-commute with $t_{r+2}$ for pairwise distinct $q$-s. As above, it follows that $z_{ij} \in I$ for all $i, j$. We can do the same with $t_{r+3}, \ldots, t_n$. At the end we get $z = \sum z_{i_1, \ldots, i_m}$, where $z_{i_1, \ldots, i_m} \in I$ is a monomial by assertion 2 of Proposition 2.

Since $I \cap \{t_{r+1}, \ldots, t_n\} = \emptyset$, it follows that $c_i \in I$ for every $i$. The claim is proved. The method of the proof will be referred to as *the conjugation trick* in the sequel.

Let $\phi : S = K_{\mathbf{p}}[t_{r+1}^{\pm 1}, \ldots, t_n^{\pm 1}] \to (K/K \cap I)_{\mathbf{p}}[t_{r+1}^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the natural homomorphism. Clearly, $\phi$ is onto. Its kernel is the ideal $(I \cap K)S$ which is equal to $I$ by the claim above. Since $R/J \cong S/I$, it follows that $\mathrm{Fract}(R/J)$ is isomorphic to the $\mathrm{Fract}(L_{\mathbf{p}}[t_{r+1}, \ldots, t_n])$, where $L = K/K \cap I$ is a formally real affine $\mathbb{R}$-algebra. □

**Example.** Let $A = \mathbb{R}[t]_q[x_1, x_2]$ with $q \in \mathbb{R}^+$. The method from Proposition 3 gives the complete list of real prime ideals of $A$ and their factor domains.

- $J = (0)$, $A/J \cong A$.
- $J = (x_1)$, $A/J \cong \mathbb{R}[t, x_2]$.
- $J = (x_2)$, $A/J \cong \mathbb{R}[t, x_1]$.
- $J = (t - \alpha)$ $(\alpha \in \mathbb{R})$, $A/J \cong \mathbb{R}_q[x_1, x_2]$.
- $J = (x_1, g(t, x_2))$ ($g(t, x_2)$ irreducible with real zero), $A/J \cong \mathbb{R}[t, x_2]/(g(t, x_2))$. If $g$ has degree zero in $x_2$, then $g(t, x_2) = t - \alpha$, for some $\alpha \in \mathbb{R}$ and $A/J \cong \mathbb{R}[x_2]$. If $g$ has degree $\geq 1$ in $x_2$, then $A/J$ is an algebraic extension of $\mathbb{R}[t]$.
- $J = (x_2, f(t, x_1))$ ($f(t, x_1)$ irreducible with real zero), $A/J \cong \mathbb{R}[t, x_1]/(f(t, x_1))$. If $f$ has degree zero in $x_1$, then $f(t, x_1) = t - \alpha$ for some $\alpha \in \mathbb{R}$ and $A/J \cong \mathbb{R}[x_1]$. If $f$ has degree $\geq 1$ in $x_1$, then $A/J$ is an algebraic extension of $\mathbb{R}[t]$.

- $J = (t-\alpha, x_1, x_2-\eta)$ $(\alpha,\eta \in \mathbb{R})$, $A/J \cong \mathbb{R}$.
- $J = (t-\alpha, x_1-\xi, x_2)$ $(\alpha,\xi \in \mathbb{R})$, $A/J \cong \mathbb{R}$.

Let $P$ be a support zero ordering on a domain $A$. For any $a \in A$ write $|a| = a$ if $a \in P$ and $|a| = -a$ otherwise. For any $a, b \in \dot{A} := A \setminus \{0\}$ write $aLb$ if there exist $r \in \mathbb{N}$ such that $|b| \leq r|a|$. Since $L$ is transitive and reflexive it defines an equivalence relation $\sim$ by $a \sim b$ if and only if $aLb$ and $bLa$. Write $\Gamma_P$ for the factor set $\dot{A}/\sim$. Let $v_P : \dot{A} \to \Gamma_P$ be the natural projection. Since $aLb$ implies that $acLbc$ and $caLcb$ it follows that $\Gamma_P$ has the structure of an ordered semigroup. Note that $\Gamma_P$ is also cancellative. It is known that $v_P$ is a valuation on $A$. It is called *the natural valuation* of the ordering $P$.

Theorem 4 completes the classification of orderings of quantum affine rings over formally real affine real algebras and with $q_{ij} \in \mathbb{R}^+$.

**Theorem 4.** *Let $K$ be a formally real affine $\mathbb{R}$-algebra, $\mathbf{p} \in M_s(\mathbb{R}^+)$ a matrix such that $p_{ii} = 1$ and $p_{ij}p_{ji} = 1$ for every $i, j = 1, \ldots, s$ and let $R = K_{\mathbf{p}}[z_1^{\pm 1}, \ldots, z_s^{\pm 1}]$ be such that $z_1^{l_1} \cdots z_s^{l_s}$ is central if and only if $l_1 = \ldots = l_s = 0$.*

*For every support zero ordering $Q$ of $K$ there exists a natural one-to-one correspondence between the set $\mathrm{Ord}_Q(R)$ of all support zero orderings of $R$ which extend $Q$ and the set $\mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s) \times \{-1, 1\}^s$ where $\mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s)$ is the set of all total orderings of the commutative semigroup $\Gamma_Q \times \mathbb{Z}^s$ which extend the natural ordering of $\Gamma_Q$.*

**Proof.** The most difficult part of the proof is to show that there is a one-to-one correspondence between the set $V_Q$ of equivalence classses of natural valuations of orderings from $\mathrm{Ord}_Q(R)$ and the set $\mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s)$.

*Claim:* For every ordering $P \in \mathrm{Ord}_Q(R)$, any $c, d \in K$ and any $i_1, \ldots, i_s, j_1, \ldots, j_s \in \mathbb{Z}$ we have $v_P(cz_1^{i_1} \cdots z_s^{i_s}) = v_P(dz_1^{j_1} \cdots z_s^{j_s})$ if and only if $v_P(c) = v_P(d)$ and $(i_1, \ldots, i_s) = (j_1, \ldots, j_s)$.

The only if part is trivial. Write $y_1 = cz_1^{i_1} \cdots z_s^{i_s}$ and $y_2 = dz_1^{j_1} \cdots z_s^{j_s}$. Write $z := y_2 y_1^{-1} = pcd^{-1} z_1^{j_1 - i_1} \cdots z_s^{j_s - i_s}$ where $p \in \mathbb{R}^+$. If $z$ is central in $A$, then by the assumption on $R$ we have that $i_1 = j_1, \ldots, i_s = j_s$. Since $v_P(z) = 0$ and $v_P(p) = 0$, it follows that $v_P(c) = v_P(d)$. If $z$ is not central, then there exists $t \in \{z_1, \ldots, z_s\}$ such that $tz \neq zt$. We know that $tzt^{-1} = qz$ for some $q \in \mathbb{R}^+$, $q \neq 1$. Replacing $t$ by $t^{-1}$ if necessary we may assume that $q < 1$. Since $v_P(q) = 0$, it follows that $v_P(z) = v_P(tzt^{-1})$. Since $v_P(z) = v_P(1) = 0$, there exists $r \in \mathbb{Q}$ such that $|z| < r$. It follows that $|z| = q^i |t^{-i} z t^i| \leq q^i r$ for every $i \in \mathbb{N}$. Hence $|z| < \epsilon$ for every $\epsilon \in \mathbb{Q}^+$. In other words, we get $v_P(z) > 0$, a contradiction.

Every element $a \in R$ can be expressed uniquely as $a = \sum_i c_i z_1^{m_{i1}} \cdots z_s^{m_{is}}$ The claim implies that $v_P(a) = \min_i v_P(c_i z_1^{m_{i1}} \cdots z_s^{m_{is}})$. In particular $v_P(\dot{K}), v_P(z_1), \ldots, v_P(z_s)$ are $\mathbb{Z}$-linearly independent and they span $\Gamma_P$. The natural embedding of $\Gamma_Q$ into $\Gamma_P$ identifies $\Gamma_Q$ with its image $v_P(\dot{K})$. Hence there exists an isomorphism $\phi : \Gamma_P \to \Gamma_Q \times \mathbb{Z}^s$ such that $\phi(v_P(c z_1^{j_1} \cdots z_s^{j_s})) = (v_Q(c), j_1, \ldots, j_s)$. The natural ordering of $\Gamma_P$ defines via $\phi$ a total ordering $F(v_P)$ of $\Gamma_Q \times \mathbb{Z}^s$ which extends the natural ordering of $\Gamma_Q$. If $P' \in \mathrm{Ord}_Q(R)$ is such that $v_{P'}$ is equivalent to $v_P$, then $\Gamma_P = \Gamma_{P'}$ and $v_P = v_{P'}$. Hence, $v_P \to F(v_P)$ is a well defined mapping from $V_P$ to $\mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s)$.

Conversely, take any $O \in \mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s)$ and define a valuation $G(O)$ from $\dot{R}$ to the ordered group $(\Gamma_Q \times \mathbb{Z}^s, O)$ by $G(O)(\sum_{i=1}^{l} c_i z_1^{m_{i1}} \cdots z_s^{m_{is}}) = \min_O\{(v_Q(c_i), m_{i1}, \ldots, m_{is}),\ i = 1, \ldots, l\}$. Note that $G(O)$ is the natural valuation of the ordering $P_O := \{0\} \cup \{c z_1^{i_1} \cdots z_s^{i_s} + h \mid c \in \dot{Q}$ and $G(O)(c z_1^{i_1} \cdots z_s^{i_s}) < G(O)(h)\}$. Hence $O \to G(O)$ defines a mapping from $\mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s)$ to $V_P$.

Clearly, $F(G(O)) = O$ for every $O \in \mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s)$. For every $P \in \mathrm{Ord}_Q(R)$ we have $G(F(v_P)) = \phi \circ v_P$, where $\phi : \Gamma_P \to \Gamma_Q \times \mathbb{Z}^s$ is the isomorphisms from above. Hence, the valuation $G(F(v_P))$ is equivalent to $v_P$. Therefore, $F$ and $G$ give a one-to-one correspondence between $V_Q$ and $\mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s)$.

The sign of an element $z \in R$ with respect to an ordering $P \in \mathrm{Ord}_Q(R)$ is equal to the sign of the lowest (with respect to $v_P$) monomial of $z$. Therefore, $P$ is uniquely determined by $v_P$ and the signs of $z_1, \ldots, z_s$. It follows that for every $v \in V_Q$ there exists a one-to-one correspondence between orderings $P \in \mathrm{Ord}_Q(R)$ such that $v_P = v$ and the set $\{-1, 1\}^s$. The one-to-one corresponcence is given explicitly by $\mathrm{Tot}(\Gamma_Q \times \mathbb{Z}^s) \times \{-1, 1\}^s \to \mathrm{Ord}_Q(R)$, $(O, \sigma_1, \ldots, \sigma_s) \mapsto P_{O,\sigma_1,\ldots,\sigma_s} := \{c z_1^{i_1} \cdots z_s^{i_s} + h \mid c \sigma_1^{i_1} \cdots \sigma_s^{i_s} \in \dot{Q}$ and $G(O)(c z_1^{i_1} \cdots z_s^{i_s}) < G(O)(h)\}$. □

**Example.** Let $A$ be as in the previous example. We want to compute the real spectrum of $A$. Note that the classification of orderings on $\mathrm{Fract}(A/J)$ is known for all real prime ideals $J$ except for $J = (0)$. (see [14] for $\mathbb{R}(t, x)$, [19] or our Theorem 4 for $\mathbb{R}_q(x_1, x_2)$, [13] or our comments below for $\mathbb{R}(t)$. The classification of orderings on an algebraic extension of $\mathbb{R}(x)$ can be obtained in principle by the extension theory for valuations. Finally, $\mathbb{R}$ has exactly one ordering.)

It remains to describe orderings with zero support. For each $a \in \mathbb{R} \cup \{\infty\}$ we define a valuation $v_a : \mathbb{R}[t] \setminus \{0\} \to \mathbb{Z}$: $v_\infty = -\deg$ and $v_a(f(t)) = m$ if $f^{(i)}(a) = 0$ for $i = 0, 1, \ldots, m-1$ and $f^{(m)}(a) \neq 0$. The natural valuation of every support zero ordering on $\mathbb{R}[t]$ is equal to one of $v_a$. For every $v_a$, there exist exactly two orderings with $v_P = v_a$. Let $O$ be an ordering on $\mathbb{R}^3$, which extends the natural ordering on the first factor (this means that $(1,0,0) \in O$) and let $a \in \mathbb{R} \cup \infty$. The valuation $v_{a,O}$ is defined by $v_{a,O}(\sum_{(i,j)\in\Lambda} r_{ij}(t) x_1^i x_2^j) = \min_O\{(v_a(r_{ij}(t)), i, j), (i,j) \in \Lambda\}$ where $r_{ij}(t) \neq 0$ for all $(i,j) \in \Lambda$. For each $v_{a,O}$, there are exactly eight orderings with $v_P = v_{a,O}$.

## 3. Quantized enveloping algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Let $\Phi$ be the root system of $\mathfrak{g}$ and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a system of simple roots in $\Phi$. Write $d_i = (\alpha_i, \alpha_i)/2 \in \{1,2,3\}$ and $a_{ij} = (\alpha_i, \alpha_j)/d_i \in \mathbb{Z}$ for $i, j = 1, \ldots, n$.

Let $k$ be a field and $q$ an nonzero element of $k$ which is not a root of 1. Write $q_i = q^{d_i}$, $[n]_i = q_i^{n-1} + q_i^{n-3} + \ldots + q_i^{-n+1}$, $[n]_i! = [1]_i[2]_i \cdots [n]_i$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_i = \frac{[n]_i!}{[k]_i![n-k]_i!}.$$

Then $U_q(\mathfrak{g})$ is the associative unital $k$ algebra with $4n$ generators $E_i, F_i, K_i, K_i^{-1}$ subject to the following relations

$$\begin{gathered}
K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i, \\
E_j K_i = q_i^{-a_{ij}} K_i E_j, \quad K_i F_j = q_i^{-a_{ij}} F_j K_i, \\
E_i F_j - F_j E_i = \delta_{ij} \tfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\
\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i E_i^{1-a_{ij}-r} E_j E_i^r = 0, \quad (i \neq j), \\
\textstyle\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_i F_i^{1-a_{ij}-r} F_j F_i^r = 0, \quad (i \neq j).
\end{gathered}$$

Let $U^+$ be a unital subalgebra of $U_q(\mathfrak{g})$ generated by $E_i$, $i = 1, \ldots, n$, $U^-$ a unital subalgebra generated by $F_i$, $i = 1, \ldots, n$ and $U^0$ a unital subalgebra generated by $K_i$ and $K_i^{-1}$, $i = 1, \ldots, n$. Note that $U^+$ and $U^-$ are antiisomorphic.

**Proposition 5.** *Let $k$, $q$, $\mathfrak{g}$ and $U^+$ be as above. The ring $U^+$ is semireal if and only if $k$ is formally real. The ring $U^+$ is formally real if and only if $k$ has an ordering such that $q^{(\alpha_i,\alpha_j)} > 0$ for every $i, j = 1, \ldots, n$.*

**Proof.** Since $k$ is a unital subring of $U^+$ and there exists a unital homomorphism $\phi : U^+ \to k$ (defined by $E_i \mapsto 0$ for $i = 1, \ldots, n$), it follows that $U^+$ is semireal if and only if $k$ is semireal.

By example 3 in [23], $\mathrm{Fract}(U^+)$ is a quantum Weyl field with $q_{ij} = q^{-(\beta_i,\beta_j)}$ for $i, j = 1, \ldots, N$. By Proposition 1, it follows that $U^+$ is formally real if and only if $k$ is formally real and $q_{ij} > 0$. $\square$

If $k=\mathbb{R}$, then we can compute in principle all support zero orderings of $U^+$ by the results from section 1. The problem of computing all real prime ideals of $U^+$ remains open.

**Example.** Assuming that $U^+$ is formally real, we will construct two support zero orderings on $U^+$. Let us recall briefly the construction of the PBW basis of $U^+$. If $s_{i_1}\cdots s_{i_N}$ is the longest reduced expression in the Weyl group $W(\Phi)$, then the elements $\beta_m = s_{i_1}\cdots s_{i_{m-1}}(\alpha_{i_m})$ $(m=1,\ldots,N)$ are different and exhaust all positive roots of $\Phi$. For every $m=1,\ldots,N$ we define $E_{\beta_m}=T_{i_1}\cdots T_{i_{m-1}}(E_{i_m})$ where $T_i$ is the automorphism $T'_{i,-1}$ of $U^+$ as defined in [18], 37.1.3. By [17], the monomials $E^{\mathbf{a}} := E_{\beta_1}^{a_1}\cdots E_{\beta_N}^{a_N}$ $(\mathbf{a}=(a_1,\ldots a_N)\in\mathbb{N}^N)$ are a $k$-vector space basis of $U^+$ and by [16] we have the following $q$-commuting relations:

$$E_{\beta_j}E_{\beta_i}=q^{(\beta_i,\beta_j)}E_{\beta_i}E_{\beta_j}+\sum_r c_{ijr}E_{\beta_{i+1}}^{a_{r,i+1}}\ldots E_{\beta_{j-1}}^{a_{r,j-1}}$$

where $c_{ijr}\in k$ and $a_{r,s}\in\mathbb{N}$. These relations are homogeneous in the sense that all terms in each of them have the same weight.

Assume now that $k$ has an ordering such that $q^{(\alpha_i,\alpha_j)}>0$ for all $i,j=1,\ldots,n$. Let $<_{\text{lex}}$ be the lexicographic ordering on $\mathbb{N}^l$. Every element $x\in U^+$ can be written uniquely as $x=\sum_{r=1}^s c_r E^{\mathbf{m}_r}$ where $c_1,\ldots,c_s\in k$ and $\mathbf{m}_1<_{\text{lex}}\ldots<_{\text{lex}}\mathbf{m}_r$. Write $P=\{0\}\cup\{\sum_{r=1}^s c_rE^{\mathbf{m}_r}\mid c_s>0\}$. The $q$-commuting relations imply that for any $\mathbf{i},\mathbf{j}\in\mathbb{N}^l$ we have $E^{\mathbf{i}}\cdot E^{\mathbf{j}}=q^\alpha E^{\mathbf{i}+\mathbf{j}}+o$ where $\alpha$ is a $\mathbb{Z}$-linear combination of $(\alpha_1,\alpha_j)$ and $o$ is a $k$-linear combination of monomials $E^{\mathbf{k}}$ with $\mathbf{k}<_{\text{lex}}\mathbf{i}+\mathbf{j}$. It follows that $P$ is a support zero ordering on $U^+$.

For every element $\gamma=\sum_{i=1}^n m_i\alpha_i$ of the root lattice we define its *level* by $l(\gamma)=\sum_{i=1}^n m_i$. For every $\mathbf{a}=(a_1,\ldots,a_N)\in\mathbb{N}^N$ we write $l(\mathbf{a})=l(\sum_{j=1}^N a_j\beta_j)$. Define an ordering $<_l$ of $\mathbb{N}^N$ by $\mathbf{a}<\mathbf{b}$ if and only if either $l(\mathbf{a})<l(\mathbf{b})$ or $l(\mathbf{a})=l(\mathbf{b})$ and $a_1<b_1$ or …or $l(\mathbf{a})=l(\mathbf{b})$ and $a_1=b_1$ and …and $a_{N-1}=b_{N-1}$ and $a_N<b_N$. Write $P'=\{0\}\cup\{\sum_{r=1}^s c_rE^{\mathbf{m}_r}\mid \mathbf{m}_1<_l\ldots<_l\mathbf{m}_s \text{ and } c_s>0\}$. As above, we see that $P'$ is a support zero ordering on $U^+$.

**Proposition 6.** *Let $k$, $q$, $\mathfrak{g}$ and $U_q(\mathfrak{g})$ be as above. The ring $U_q(\mathfrak{g})$ is semireal if and only if $k$ is formally real. The ring $U_q(\mathfrak{g})$ is formally real if and only if $k$ has an ordering such that $q^{(\alpha_i,\alpha_j)}>0$ for every $i,j=1,\ldots,n$.*

**Proof.** $k$ is a unital subring of $U_q(\mathfrak{g})$ and there exist a unital homomorphism $\phi: U_q(\mathfrak{g})\to k$ defined by $\phi(E_i)=0$, $\phi(F_i)=0$, $\phi(K_i)=1$ for $i=1,\ldots,n$. If follows that $U_q(\mathfrak{g})$ is semireal if and only if $k$ is. If $U_q(\mathfrak{g})$ has a support zero ordering, then for every $i,j=1,\ldots,n$ the elements

$E_iK_j$ and $K_jE_i$ have the same sign with respect to this ordering. It follows that $q^{-(\alpha_i,\alpha_j)} = q_i^{-a_{ij}} > 0$ for every $i,j = 1,\ldots,n$.

Assume now that $k$ has an ordering such that $q^{(\alpha_i,\alpha_j)} > 0$ for every $i,j = 1,\ldots,n$. Let $F_{\beta_m}$ be the image of $E_{\beta_m}$ under the antiisomorphism of $U^-$ and $U^+$. Write $F^{\mathbf{b}} := F_{\beta_N}^{b_N} \cdots F_{\beta_1}^{b_1}$ for every $\mathbf{b} \in \mathbb{N}^N$ and $K^{\mathbf{m}} = K_1^{m_1} \cdots K_n^{m_n}$ for every $\mathbf{m} \in \mathbb{Z}^n$. The monomials $F^{\mathbf{b}}K^{\mathbf{m}}E^{\mathbf{a}}$ form a PBW basis of $U_q(\mathfrak{g})$. We define an ordering of PBW monomials by $F^{\mathbf{b}}K^{\mathbf{m}}E^{\mathbf{a}} < F^{\mathbf{b}'}K^{\mathbf{m}'}E^{\mathbf{a}'}$ if and only if $\mathbf{a} <_l \mathbf{a}'$ or $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} <_l \mathbf{b}'$ or $\mathbf{a} = \mathbf{a}'$ and $\mathbf{b} = \mathbf{b}'$ and $\mathbf{m} <_{\text{lex}} \mathbf{m}'$. We claim that the set

$$\{0\} \cup \{\sum_{r=1}^{s} c_r F^{\mathbf{b}_r}K^{\mathbf{m}_r}E^{\mathbf{a}_r} \mid F^{\mathbf{b}_1}K^{\mathbf{m}_1}E^{\mathbf{a}_1} < \ldots < F^{\mathbf{b}_s}K^{\mathbf{m}_s}E^{\mathbf{a}_s} \text{ and } c_s > 0\}$$

is a support zero ordering on $U_q(\mathfrak{g})$. It is enough to verify that

$$F^{\mathbf{b}}K^{\mathbf{m}}E^{\mathbf{a}} \cdot F^{\mathbf{b}'}K^{\mathbf{m}'}E^{\mathbf{a}'} = q^{\gamma}F^{\mathbf{b}+\mathbf{b}'}K^{\mathbf{m}+\mathbf{m}'}E^{\mathbf{a}+\mathbf{a}'} + o$$

where $\gamma \in \mathbb{Z}$ is a $\mathbb{Z}$-linear combination of $(\alpha_i,\alpha_j)$ and $o$ is a $k$-linear combination of smaller PBW monomials.

By [8], Lemma 1 we have that $E_{\beta_i}F_{\beta_j} = F_{\beta_j}E_{\beta_i} + \delta$ where $\delta$ is a linear combination of monomials $F^{\mathbf{b}}K^{\mathbf{m}}E^{\mathbf{a}}$ with $l(\mathbf{b}) < l(\beta_j)$ and $l(\mathbf{a}) < l(\beta_i)$. It follows that $F^{\mathbf{b}}K^{\mathbf{m}}E^{\mathbf{a}} \cdot F^{\mathbf{b}'}K^{\mathbf{m}'}E^{\mathbf{a}'} = F^{\mathbf{b}}K^{\mathbf{m}}F^{\mathbf{b}'}E^{\mathbf{a}}K^{\mathbf{m}'}E^{\mathbf{a}'} + o'$ where $o'$ is a $k$-linear combination of PBW monomials $F^{\mathbf{b}''}K^{\mathbf{m}''}E^{\mathbf{a}''}$ with $l(\mathbf{a}'') < l(\mathbf{a}) + l(\mathbf{a}')$ and $l(\mathbf{b}'') < l(\mathbf{b}) + l(\mathbf{b}')$. From the third and the fourth defining relation of $U_q(\mathfrak{g})$ it follows that $F^{\mathbf{b}}K^{\mathbf{m}}F^{\mathbf{b}'}E^{\mathbf{a}}K^{\mathbf{m}'}E^{\mathbf{a}'} = q^{\delta}F^{\mathbf{b}}F^{\mathbf{b}'}K^{\mathbf{m}+\mathbf{m}'}E^{\mathbf{a}}E^{\mathbf{a}'}$ where $\delta$ is a $\mathbb{Z}$-linear combination of $(\alpha_i,\alpha_j)$. By the example, the last expression is equal to $q^{\gamma}F^{\mathbf{b}+\mathbf{b}'}K^{\mathbf{m}+\mathbf{m}'}E^{\mathbf{a}+\mathbf{a}'} + o''$ where $\gamma$ is a $\mathbb{Z}$-linear combination of $(\alpha_i,\alpha_j)$ and $o''$ is a $k$-linear combination of smaller PBW monomials. This proves the claim. □

Not much is known about the field of fractions of $U_q(\mathfrak{g})$. The quantum Gelfand-Kirillov conjecture says that it is a quantum Weyl field. The structure theory of the prime spectrum is very developed but a complete description is known only in specal cases.

**Example.** Let $q \in \mathbb{R}\setminus\{0,1\}$ and let $A = U_q(\mathfrak{sl}_2(\mathbb{R}))$ be the $\mathbb{R}$-algebra with generators $E, F, K, K^{-1}$ and relations:

$$KK^{-1} = K^{-1}K = 1,$$
$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,$$
$$EF - FE = \tfrac{K-K^{-1}}{q-q^{-1}}.$$

Let $J$ be a real prime ideal of $A$. If $E \notin J$, then $J$ extends to a prime ideal of $A_E$. Note that $A_E \cong \mathbb{R}[C]_{q^2}[K^{\pm 1}, E^{\pm 1}]$ where $C = EF + \frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK+q^{-1}K^{-1}}{(q-q^{-1})^2}$ is the *quantum Casimir element.*

By the example after Proposition 3, it follows that either $J = (0)$ or $J = (C - \lambda), (\lambda \in \mathbb{R})$. If $E \in J$, then $K - K^{-1} = (q - q^{-1})(EF - FE) \in J$. It follows that either $K - 1 \in J$ or $K + 1 \in J$. Since $(q^2 - 1)KF = FK - KF = F(K \pm 1) - (K \pm 1)F \in J$ and $K \notin J$, it follows that $F \in J$. Hence, $J = (E, F, K + 1)$ or $J = (E, F, K - 1)$.

The description of $\mathrm{Sper}(A)$ consists of a complete list of real prime ideals and a complete list of orderings of the skew field of fractions of each factor domain:

- If $J = (E, F, K+1)$ or $J = (E, F, K-1)$ then $\mathrm{Fract}(A/J) \cong \mathbb{R}$ has exactly one ordering.
- If $J = (C - \lambda)$ where $\lambda \in \mathbb{R}$ then $\mathrm{Fract}(A/J) \cong \mathbb{R}_{q^2}(K, E)$ and we have a four-to-one correspondence between the orderings of $\mathbb{R}_{q^2}(K, E)$ and the orderings of the abelian group $\mathbb{Z} \times \mathbb{Z}$.
- If $J = (0)$ then $\mathrm{Fract}(A/J) \cong \mathbb{R}(C)_{q^2}(K, E)$ and we have an eight-to-one correspondence between the orderings of $\mathbb{R}(C)_{q^2}(K, E)$ and the cartesian product of the set $\mathbb{R} \cup \{\infty\}$ and the set of all orderings of the abelian group $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ which contain $(1, 0, 0)$.

## 4. Quantized function algebras

Let $\mathfrak{g}$ be a semisimple Lie algebra, $k$ a field and $q$ a nonzero element of $k$. For every dominant weight $\lambda$, write $L_q(\lambda)$ for the unique simple left $U_q(\mathfrak{g})$-module with highest weight $\lambda$ and let $L_q(\lambda)^*$ be its vector space dual considered as a right $U_q(\mathfrak{g})$-module. For every dominant weight $\lambda$, every $\xi \in L_q(\lambda)^*$ and every $m \in L_q(\lambda)$ we define an element $c^\lambda_{\xi,m} \in U_q(\mathfrak{g})^*$ by

$$c^\lambda_{\xi,m}(a) = \xi(am), \quad a \in U_q(\mathfrak{g}).$$

The $k$-subspace of $U_q(\mathfrak{g})^*$ spanned by all $c^\lambda_{\xi,m}$ is called the *quantized function algebra* $k_q[G]$ ($G$ is the simply connected Lie group of $\mathfrak{g}$.) Many authors write $\mathcal{O}_q(G)$ instead of $k_q[G]$.

The dual $U_q(\mathfrak{g})^*$ has an algebra structure defined by

$$cc'(a) = \sum c(a_{(1)})c'(a_{(2)}) \quad \text{if } \Delta(a) = \sum a_{(1)} \otimes a_{(2)},$$

where $\Delta : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is the comultiplication of $U_q(\mathfrak{g})$. The counit $\epsilon : U_q(\mathfrak{g}) \to k$ plays the role of 1 in $U_q(\mathfrak{g})^*$. It turns out that $k_q[G]$ is a unital subalgebra of $U_q(\mathfrak{g})^*$. The dual $U_q(\mathfrak{g})^*$ also has a $U_q(\mathfrak{g}) - U_q(\mathfrak{g})$ bimodule structure defined by

$$v \cdot c \cdot u(a) = c(uav), \quad u, v, a \in U_q(\mathfrak{g}),\ c \in U_q(\mathfrak{g})^*.$$

Since $v \cdot c^\lambda_{\xi,m} \cdot u = c^\lambda_{\xi \cdot u, v \cdot m}$, it follows that $k_q[G]$ is a subbimodule of $U_q(\mathfrak{g})^*$.

Write $\Lambda_r$ for the root lattice, $\Lambda$ for the weight lattice and $\Lambda^+$ for the set of dominant weights. If $\lambda \in \Lambda^+$ then $L_q(\lambda) = \oplus_{\mu\in\Lambda} L_q(\lambda)_\mu$ where

$$L_q(\lambda)_\mu = \{m \in L_q(\lambda) \mid K_\nu m = q^{(\mu,\nu)} m \text{ for all } \nu \in \Lambda_r\}.$$

and $L_q(\lambda)^* = \oplus_{\mu\in\Lambda} (L_q(\lambda)^*)_\mu$ where

$$(L_q(\lambda)^*)_\mu = \{f \in L_q(\lambda)^* \mid f(L_q(\lambda)_\nu) = 0 \text{ for all } \nu \neq \mu\}.$$

If $\xi \in (L_q(\lambda)^*)_\nu$ and $m \in L_q(\lambda)_\mu$, then we sometimes write $c^\lambda_{\nu,\mu}$ instead of $c^\lambda_{\xi,m}$. If $\nu, \mu \in W\lambda$, then $\dim(L_q(\lambda)^*)_\nu = \dim L_q(\lambda)_\mu = 1$, so that $c^\lambda_{\nu,\mu}$ is unique up to a scalar multiple.

Let $\alpha_1, \ldots, \alpha_n$ be a base of the root system $\Phi$, $s_1, \ldots, s_n$ the corresponding reflections ($s_i(\beta) = \beta - \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$), and $\lambda_1, \ldots, \lambda_n$ the corresponding fundamental weights ($(\lambda_i, \frac{2\alpha_j}{(\alpha_j,\alpha_j)}) = \delta_{ij}$). Let $w_0 = s_{i_1} \cdots s_{i_N}$ be the longest reduced expression in $W$. Write $y_0 = \mathrm{Id}$ and $y_k = s_{i_1} \cdots s_{i_k}$ for every $k = 1, \ldots, N$. The elements $\beta_1 = y_0(\alpha_{i_1}), \beta_2 = y_1(\alpha_{i_2}), \ldots, \beta_N = y_{N-1}(\alpha_{i_N})$ are distinct positive roots and every positive root is one of them. Write $\rho = \sum_{i=1}^n \lambda_i = \frac{1}{2}\sum_{j=1}^N \beta_j$.

In section 3.3 of [1], Caldero defines elements $c_i = c^{\lambda_i}_{w_0\lambda_i,\lambda_i}$, $i = 1, \ldots, n$, $d_i = c^\rho_{y_{i-1}\rho,-y_i\rho}$, $i = 1, \ldots, N$ and $d'_i = c^\rho_{y_{i-1}\rho,-y_{i-1}\rho}$, $i = 1, \ldots, N$ and proves that they generate a quantum affine ring whose skew field of fractions is isomorphic to $\mathrm{Fract}(k_q[G])$. One can obtain explicit $q$-commutation relations between the elements $c_i, d_i, d'_i$. By 9.1.6(**) in [10], we have $c_k c_l = c_l c_k$ for all $k, l = 1, \ldots, n$. By 9.1.4(ii) in [10], we have

$$d_k c_l = q^{-(y_k\rho,\lambda_l)-(y_{k-1}\rho,w_0\lambda_l)} c_l d_k,$$

$$d'_k c_l = q^{-(y_{k-1}\rho,\lambda_l)-(y_{k-1}\rho,w_0\lambda_l)} c_l d'_k.$$

From (1.5.1) in [1] we obtain

$$d_k d'_l = q^{(y_{k-1}\rho,y_{l-1}\rho)-(y_k\rho,y_{l-1}\rho)} d'_l d_k \text{ if } k \geq l,$$

$$d_k d'_l = q^{-(y_{k-1}\rho,y_{l-1}\rho)+(y_k\rho,y_{l-1}\rho)} d'_l d_k \text{ if } k < l,$$

$$d_k d_l = q^{(y_{k-1}\rho,y_{l-1}\rho)-(y_k\rho,y_l\rho)} d_l d_k \text{ if } k \geq l,$$

$$d'_k d'_l = d'_l d'_k.$$

All exponents of $q$ are integers. We claim that at least one of them is odd. Let $d_k c_l = q^{m(k,l)} c_l d_k$ where $m(k,l) = -(y_k\rho, \lambda_l) - (y_{k-1}\rho, w_0\lambda_l)$. For every $k = 1, \ldots, N$ we have $\sum_{l=1}^n m(k,l) = -(y_k\rho, \rho) - (y_{k-1}\rho, w_0\rho)$. Since $y_k\rho = y_{k-1}\rho - \beta_k$ and $w_0\rho = -\rho$, it follows that $\sum_{l=1}^n m(k,l) = (\beta_k, \rho)$. If $k$ is such that $\beta_k$ is a short simple root, then $\sum_{l=1}^n m(k,l) = 1$. It follows that that at least one $m(k,l)$ is odd. The following proposition is an easy consequence.

**Proposition 7.** *Let $k$ be a field, $q \in k$ and $G$ a simply connected Lie group. The ring $k_q[G]$ is semireal if and only if $k$ is a formally real field. The ring $k_q[G]$ is formally real if and only if $k$ has an ordering such that $q > 0$.*

**Example.** If $k = \mathbb{R}$, $q > 0$, $G = SL_n(\mathbb{R})$ and $R = k_q[G](= \mathcal{O}_q(SL_n(\mathbb{R})))$, then $\mathrm{Sper}(R)$ can be completely described. The ring $R$ has generators $a, b, c, d$ and relations

$$ab = qba, ac = qca, bd = qdb, cd = qdc, bc = cb,$$
$$ad - qbc = da - q^{-1}bc = 1$$

A prime ideal which contains $a$, contains also $b$ or $c$, therefore it contains 1. So there is a one-to-one correspondence between SperR and $\mathrm{SperR_a}$. But $R_a$ has generators $a, b, c$ and relations $ab = qba, ac = qca, bc = cb$, hence it is a quantum affine ring over $\mathbb{R}$. If a prime ideal $J$ contains $b$, then $R_a/J$ is a factor domain of $\mathbb{R}_q[a^{\pm 1}, c]$. If $b \notin J$, every ordering with support $J$ extends uniquely to $R_{a,b} \cong \mathbb{R}[t]_q[a^{\pm 1}, b^{\pm 1}]$ $(t = cb^{-1})$. The orderings $\mathbb{R}_q[a, c]$ are known and the orderings of $\mathbb{R}[t]_q[a, b]$ were computed in section 1.

**Example.** It has been proved in [2], that for every prime ideal $J$ of $k_q[GL(n)]$, $\mathrm{Fract}(k_q[GL(n)]/J)$ is a quantum Weyl field. If $k = \mathbb{R}$, then in principle we can describe orderings with a given support $J$. However, the classification of prime ideals of $k_q[GL(n)]$ is not known yet.

## 5. Quantized Weyl algebras

Let $k$ be a field $Q = (q_1, \ldots, q_n) \in (k^\times)^n$ and let $\Gamma = (\gamma_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix over $k$. The *multiparameter quantized Weyl algebra of degree $n$ over $k$* is the $k$-algebra $A_n^{Q,\Gamma}$ generated by elements $x_1, y_1, \ldots, x_n, y_n$ subject to the following relations

$$\begin{array}{ll} y_i y_j = \gamma_{ij} y_j y_i & (\text{all } i, j) \\ x_i x_j = q_i \gamma_{ij} x_j x_i & (i < j) \\ x_i y_j = \gamma_{ji} y_j x_i & (i < j) \\ x_i y_j = q_j \gamma_{ji} y_j x_i & (i > j) \\ x_j y_j = 1 + q_j y_j x_j + \sum_{l<j} (q_l - 1) y_l x_l & (\text{all } j) \end{array}$$

**Proposition 8.** *The $k$-algebra $R = A_n^{Q,\Gamma}$ has a support zero ordering if and only if $k$ has an ordering such that $q_i > 0$ and $\gamma_{ij} > 0$ for every $i, j = 1, \ldots, n$.*

*The $k$-algebra $R$ is semireal if and only if $k$ is semireal and for every $m \in \{2, \ldots, n\}$ such that $q_1 = q_2 = \ldots = q_{m-1} = 1$ we have $\gamma_{ij} > 0$ for all $i, j = 1, \ldots, m$.*

**Proof.** If $R$ has a support zero ordering $P$ then $\gamma_{ij} \in P \cap k^\times$ for $i, j = 1, \ldots, n$ since $y_i y_j$ and $y_j y_i$ have the same sign. Write $z_i = x_i y_i - y_i x_i$ for $i = 1, \ldots, n$ and note that $z_i y_i = q_i y_i z_i$. Since $y_i z_i$ has the same sign as $z_i y_i$, it follows that $q_i \in P \cap k^\times$ for $i = 1, \ldots, n$.

If $k$ is formally, $q_i > 0$ and $\gamma_{ij} > 0$ for all $i, j$, then

$$P := \{0\} \cup \{\textstyle\sum_{r=1}^{s} c_i y_1^{i_{r1}} \cdots y_n^{i_{rn}} x_1^{j_{r1}} \cdots x_n^{j_{rn}} \mid \\ (i_{11}, \ldots, j_{1n}) <_{\text{lex}} \ldots <_{\text{lex}} (i_{s1}, \ldots, j_{sn}) \text{ and } c_s > 0\}$$

is a support zero ordering of $R$.

Assume now that $R$ is semireal. Clearly, $k$ is semireal, too. For every $m$ such that $q_1 = \ldots = q_{m-1} = 1$, we have $x_j y_j = 1 + q_j y_j x_j$ for $j = 1, \ldots, m$. By definition, $R$ has a proper real prime ideal $J$. If $\gamma_{ij} < 0$ for some $i, j = 1, \ldots, m$ then it follows from $y_i y_j = \gamma_{ij} y_j y_i$ that either $y_i \in J$ or $y_j \in J$, a contradiction with $x_i y_i = 1 + q_i y_i x_i$ or $x_j y_j = 1 + q_j y_j x_j$. Therefore, $\gamma_{ij} > 0$ for $i, j = 1, \ldots, m$.

To prove the converse, we assume that $k$ is semireal and for every $m \in \{2, \ldots, n\}$ such that $q_1 = q_2 = \ldots = q_{m-1} = 1$ we have $\gamma_{ij} > 0$ for all $i, j = 1, \ldots, m$. If such an $m$ does not exist, then $q_1 \neq 1$. We can define a unital homomorphism $\phi : R \to k$ by $\phi(x_1) = 1, \phi(y_1) = \frac{1}{1-q_1}$ and $\phi(x_i) = \phi(y_i) = 0$ for $i > 1$. Hence, $R$ is semireal. If $q_1 = \ldots = q_n = 1$, then $\gamma_{ij} > 0$ for all $i, j = 1, \ldots, n$. In the first paragraph we proved that $R$ has a support zero ordering in this case. Again, $R$ is semireal. It remains to study the case $q_1 = \ldots = q_{m-1} = 1$, $q_m \neq 1$ where $m \in \{2, \ldots, n\}$. Let $S$ be a $k$-algebra with generators $\overline{x}_1, \ldots, \overline{x}_{m-1}, \overline{x}_m, \overline{y}_1, \ldots, \overline{y}_{m-1}$ and defining relations

$$\begin{array}{ll} \overline{y}_i \overline{y}_j = \gamma_{ij} \overline{y}_j \overline{y}_i & i, j = 1, \ldots, m-1, \\ \overline{x}_i \overline{x}_j = \gamma_{ij} \overline{x}_j \overline{x}_i & i, j = 1, \ldots, m, \\ \overline{x}_i \overline{y}_j = \gamma_{ji} \overline{y}_j \overline{x}_i & i = 1, \ldots, m, \quad j = 1, \ldots, m-1, \\ \overline{x}_j \overline{y}_j = 1 + \overline{y}_j \overline{x}_j & j = 1, \ldots, m-1. \end{array}$$

Since $\gamma_{ij} > 0$ for all $i, j = 1, \ldots, m$, we can construct a support zero ordering of $S$ as above. This ordering extends uniquely to the localization $S_{\overline{x}_m}$. We have a unital homomorphism $\phi : R \to S_{\overline{x}_m}$ defined by

$$\begin{array}{lll} \phi(x_i) = \overline{x}_i, & \phi(y_i) = \overline{y}_i, & i = 1, \ldots, m-1, \\ \phi(x_m) = \overline{x}_m, & \phi(y_m) = \frac{1}{1-q_m} \overline{x}_m^{-1}, & \\ \phi(x_j) = 0, & \phi(y_j) = 0, & j = m+1, \ldots, n. \end{array}$$

Hence, $R$ is semireal. $\square$

The classification of prime ideals of algebras $A_n^{Q,\Gamma}$ is not known. However if $q_1 \neq 1, \ldots, q_n \neq 1$, then for every prime ideal $J$ of $A_n^{Q,\Gamma}$, the skew field $\mathrm{Fract}(A_n^{Q,\Gamma}/P)$ is a quantum Weyl field over a finitely generated extension of $k$, see [2]. Therefore, in case $k = \mathbb{R}$ the computation of the real spectrum reduces to the computation of the prime spectrum.

## 6. Final comments and open problems

(1) The quantum Gelfand-Kirilllov conjecture says that the field of fractions of a quantum group is always a quantum Weyl field. If this is true then the results of this section give a classification of support zero orderings of all quantum groups over $\mathbb{R}$. See [6], Section 2.3 for a report on the present status of this conjecture.
(2) The classification of orderings in the quantum case is much easier than in the classical case. Find all orderings on the Weyl algebra $A_1(\mathbb{R}) = \mathbb{R}\langle x, y\rangle/(yx - xy - 1)$ and $U(\mathfrak{sl}_2(\mathbb{R}))$, see [20].
(3) As noted by Ringel, [25], $U_q(\mathfrak{g})$ is an iterated skew polynomial ring so further analysis is possible. What are the best results for minimal generation of basic semialgebraic sets?
(4) Quantum groups usually have nontrivial involutions. Can the results of this paper be extended to $*$-orderings? See, [22, 21, 4].
(5) The results of this paper can probably be extended to orderings of higher level. See [3] for the classification of orderings of higher level on quantum polynomials. See also [24].
(6) Quantized enveloping algebras are graded by their root lattice. Positivstellensätze for noncommutative graded rings have been developed by Igor Klep, see [11].
(7) Is there a reasonable stratification theory for real spectra of quantum groups? See [6].
(8) Let $A$ and $B$ be unital $k$-algebras with with support zero orderings which induce the same ordering on $k$. Is it always true that $A \otimes_k B$ has a support zero ordering extending the orderings on $A$ and $B$?

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## References

[1] P. Caldero, On the Gelfand-Kirillov conjecture for quantum algebras, Proc. Amer. Math. Soc., **128**, no. 4, 943–951.
[2] G. Cauchon, Quotients premiers de $O_q(\mathfrak{m}_n(k))$, J. Algebra, **180**, 530–545 (1996).
[3] J. Cimprič, Complete precones on noncommutative integral domains, Comm. Algebra, **28**, no. 1, 103–119 (2000).

[4] T. C. Craven and T. L. Smith, Ordered $*$-rings, preprint.
[5] L. Fuchs, Infinite abelian groups, Academic Press, New York and London, 1970.
[6] K. R. Goodearl, Prime spectra of quantized coordinate rings, preprint.
[7] K. R. Goodearl, E. S. Letzter, Prime factor algebras of the coordinate ring of quantum matrices, Proceedings of the AMS, **121**, no. 4, 1994, 1017–1025.
[8] W. A. De Graaf, Computing with Quantized Enveloping Algebras: PBW-Type Bases, Highest weight modules and $R$-matrices, J. Symbolic Computation, (2001) **32**, 475–490.
[9] J. C. Jantzen, "Lectures of quantum groups", Graduate Studies in Mathematics, Volume 6, American Mathematical Society, 1996.
[10] A. Joseph, *Quantum groups and their primitive ideals*, Springer-Verlag, (1995).
[11] I. Klep, Noncommutative graded Stellensätze, preprint.
[12] A. Klimyk, K. Schmüdgen, *Quantum groups and their representations*, Springer, 1997.
[13] M. Knebusch, C. Scheiderer, "Einführung in die reelle Algebra", Vieweg 1989.
[14] F. V. Kuhlmann, S. Kuhlman, M. Marshall, M. Zekavat, Embedding ordered fields in power series fields, preprint.
[15] K. H. Leung, M. Marshall, Y. Zhang, The real spectrum of a noncommutative ring, J. Algebra, **198**, 412-427 (1997).
[16] S. Z. Levendorskii and Y. S. Soibelman, Some applications of quantum Weyl group 1. The multiplicative formula for universal $R$-matrix for simple Lie algebras, J. Geom. Phys. bf 7 1991, N4.
[17] G. Lustig, Quantum groups at roots of 1, Geom. Ded. **35**, 1990, 89–114.
[18] G. Lusztig, *Introduction to quantum groups*, Birkhäuser, 1993.
[19] M. Marshall and Y. Zhang, Orderings, Real Places, and Valuations on noncommutative integral domains, J. Algebra, **212**, 190–207 (1999).
[20] M. Marshall and Y. Zhang, Ordering and valuation on twisted polynomial rings, Comm. Algebra, **28**, no. 3, 3763–3776 (2000).
[21] M. Marshall, *-orderings and *-valuations on algebras of finite Gelfand-Kirilov dimension.
[22] M. Marshall, *-orderings on a ring with involution, Comm. Algebra, **28**, 1157-1173, (2000).
[23] A. Panov, Fields of fractions of quantum solvable algebras, preprint.
[24] V. Powers, Holomorphy rings and higher level orderings on skew fields, J. Algebra, **136**, no. 1, 51–59 (1991).
[25] C. M. Ringel, PBW-bases of quantum groups, J. Reine Angew. Math (1996) **470**, 51–88.
[26] Nanhua Xi, A commutation formula for root vectors in quantized enveloping algebras, Pacific J. Math., (1999), **189**, 179–199.