On the behaviour of Brauer p-dimensions under finitely-generated field extensions*

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Abstract

The present paper shows that if $q \in \mathbb{P}$ or $q = 0$, where $\mathbb{P}$ is the set of prime numbers, then there exist characteristic q fields $E_{q,k}\colon\ k \in \mathbb{N}$, of Brauer dimension $\mathrm{Brd}(E_{q,k}) = k$ and infinite absolute Brauer p-dimensions $\mathrm{abrd}_p(E_{q,k})$, for all $p \in \mathbb{P}$ not dividing $q^2 - q$. This ensures that $\mathrm{Brd}_p(F_{q,k}) = \infty$, $p \dagger q^2 - q$, for every finitely-generated transcendental extension $F_{q,k}/E_{q,k}$. We also prove that each sequence a_p, b_p, $p \in \mathbb{P}$, satisfying the conditions $a_2 = b_2$ and $0 \leq b_p \leq a_p \leq \infty$, equals the sequence $\mathrm{abrd}_p(E), \mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, for a field E of characteristic zero.

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1 Introduction

Let E be a field, $s(E)$ the class of finite-dimensional associative central simple E-algebras, $d(E)$ the subclass of division algebras $D \in s(E)$, and for each $A \in s(E)$, let $[A]$ be the equivalence class of A in the Brauer group $\mathrm{Br}(E)$. It is known that $\mathrm{Br}(E)$ is an abelian torsion group (cf. [23], Sect. 14.4), whence it decomposes into the direct sum of its p-components $\mathrm{Br}(E)_p$, where p runs across the set $\mathbb{P}$ of prime numbers. By Wedderburn's structure theorem (see, e.g., [23], Sect. 3.5), each $A \in s(E)$ is isomorphic to the full matrix ring $M_n(D_A)$ of order n over some $D_A \in d(E)$, uniquely determined by A, up-to an E-isomorphism. This implies the dimension $[A\colon E]$ is a square of a positive integer $\deg(A)$, the degree of A. The main numerical invariants of A are $\deg(A)$, the Schur index $\mathrm{ind}(A) = \deg(D_A)$, and the exponent $\exp(A)$, i.e. the order of $[A]$ in $\mathrm{Br}(E)$.

*Throughout this paper, we write for brevity "FG-extension(s)" instead of "finitely-generated [field] extension(s)".

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The following statements describe basic divisibility relations between $\mathrm{ind}(A)$ and $\exp(A)$, and give an idea of their behaviour under the scalar extension map $\mathrm{Br}(E) \to \mathrm{Br}(R)$, in case R/E is a field extension of finite degree $[R\colon E]$ (see, e.g., [23], Sects. 13.4, 14.4 and 15.2):

(1.1) (a) $(\mathrm{ind}(A), \exp(A))$ is a Brauer pair, i.e. $\exp(A)$ divides $\mathrm{ind}(A)$ and is divisible by every $p \in \mathbb{P}$ dividing $\mathrm{ind}(A)$.

(b) $\mathrm{ind}(A \otimes_E B) = \mathrm{ind}(A)\mathrm{ind}(B)$, if $B \in s(E)$ and g.c.d.$\{\mathrm{ind}(A), \mathrm{ind}(B)\} = 1$; in this case, if $A, B \in d(E)$, then the tensor product $A \otimes_E B$ lies in $d(E)$.

(c) $\mathrm{ind}(A)$, $\mathrm{ind}(A \otimes_E R)$, $\exp(A)$ and $\exp(A \otimes_E R)$ divide $\mathrm{ind}(A \otimes_E R)[R\colon E]$, $\mathrm{ind}(A)$, $\exp(A \otimes_E R)[R\colon E]$ and $\exp(A)$, respectively.

Statements (1.1) (a), (b) imply Brauer's Primary Tensor Product Decomposition Theorem, for any $\Delta \in d(E)$ (cf. [23], Sect. 14.4). Also, (1.1) (a) fully describes general restrictions on index-exponent pairs, in the following sense:

(1.2) Given a Brauer pair $(m', m) \in \mathbb{N}^2$, there is a field F with $(\mathrm{ind}(D), \exp(D)) = (m', m)$, for some $D \in d(F)$ (Brauer, see [23], Sect. 19.6). One may take as F any rational (i.e. purely transcendental) extension in infinitely many variables over any fixed field F_0.

The Brauer p-dimensions $\mathrm{Brd}_p(E)$, $p \in \mathbb{P}$, of a field E contain essential information about pairs $\mathrm{ind}(D), \exp(D)$, $D \in d(E)$. We say that $\mathrm{Brd}_p(E)$ is finite and equal to n, for a fixed $p \in \mathbb{P}$, if n is the least integer ≥ 0, for which $\mathrm{ind}(D_p) \leq \exp(D_p)^n$ whenever $D_p \in d(E)$ and $[D_p] \in \mathrm{Br}(E)_p$. If no such n exists, we set $\mathrm{Brd}_p(E) = \infty$. The absolute Brauer p-dimension of E is defined as the supremum $\mathrm{abrd}_p(E) = \sup\{\mathrm{Brd}_p(R)\colon\ R \in \mathrm{Fe}(E)\}$, where $\mathrm{Fe}(E)$ is the set of finite extensions of E in a separable closure E_{sep}. We have $\mathrm{abrd}_p(E) = 0$, for some $p \in \mathbb{P}$, $p \neq \mathrm{char}(E)$, if and only if the absolute Galois group $\mathcal{G}_E = \mathcal{G}(E_{\mathrm{sep}}/E)$ is of cohomological p-dimension $\mathrm{cd}_p(\mathcal{G}_E) \leq 1$ (cf. [26], Ch. II, 3.1). When E is virtually perfect, i.e. $\mathrm{char}(E) = 0$ or $\mathrm{char}(E) = q > 0$ and E is a finite extension of its subfield $E^q = \{e^q\colon e \in E\}$, the following holds:

(1.3) $\mathrm{Brd}_p(E') \leq \mathrm{abrd}_p(E)$, for all $p \in \mathbb{P}$ and finite extensions E'/E.

The assertion is obvious, if $\mathrm{char}(E) = 0$. If $\mathrm{char}(E) = q > 0$, then $[E'\colon E'^q] = [E\colon E^q]$, for every finite extension E'/E (cf. [17], Ch. VII, Sect. 7). Therefore, (1.3) can be deduced from (1.1) (c) and Albert's theory of q-algebras [1], Ch. VII, Theorem 28 (see also Lemma 4.1).

It is known that $\mathrm{Brd}_p(E) = \mathrm{abrd}_p(E) = 1$, for all $p \in \mathbb{P}$, if E is a global or local field (cf. [24], (31.4) and (32.19)), or the function field of an algebraic surface defined over an algebraically closed field E_0 [14], [18]. As shown in [19], when E is the function field of an n-dimensional algebraic variety over the field E_0, we have $\mathrm{abrd}_p(E) < p^{n-1}$, $p \in \mathbb{P}$. The suprema $\mathrm{Brd}(E) = \sup\{\mathrm{Brd}_p(E)\colon\ p \in \mathbb{P}\}$ and $\mathrm{abrd}(E) = \sup\{\mathrm{Brd}(R)\colon\ R \in \mathrm{Fe}(E)\}$ are called a Brauer dimension and an absolute Brauer dimension of E, respectively. In view of (1.1), the definition of $\mathrm{Brd}(E)$ is the same as in [2], Sect. 4. It has recently been proved [12], [22] (see also Lemmas 4.3 and 4.4), that $\mathrm{abrd}(K_m) < \infty$, if (K_m, v_m) is an m-dimensional local field, in the sense of [11], with a quasifinite m-th residue field.

The present research considers the sequence $\mathrm{Brd}_p(F)$, $p \in \mathbb{P}$, for a transcendental FG-extension F of a field E, and its dependence upon $\mathrm{abrd}_p(E)$, $p \in \mathbb{P}$. It is motivated mainly by an open problem posed in Section 4 of the survey [2].

2 The main results

Fields E with $\mathrm{abrd}_p(E) < \infty$, for all $p \in \mathbb{P}$, are singled out by Galois cohomology (see Remark 4.2), and in the virtually perfect case, by the validity of the Primary Tensor Product Decomposition Theorem, for every locally finite-dimensional associative central division E-algebra of at most countable dimension (see (1.3) and [4]). The applicability of this result to basic fields of algebraic number theory and algebraic geometry raises interest in the open problem of whether FG-extensions of a global, local or algebraically closed field are of finite absolute Brauer dimensions. This draws our attention to the following open question:

(2.1) Is the class of fields E of finite absolute Brauer p-dimensions, for a fixed $p \in \mathbb{P}$, $p \neq \mathrm{char}(E)$, closed under the formation of FG-extensions?

The purpose of this paper is to answer the similar question of whether $\mathrm{Brd}(F) < \infty$, for every FG-extension F of a field E with $\mathrm{Brd}(E) < \infty$ (this is stated in [2] as Problem 4.4). Our starting point is the following result of [7]:

Proposition 2.1. *Let E be a field, $p \in \mathbb{P}$ and F/E an* FG*-extension of transcendency degree* $\mathrm{trd}(F/E) = \kappa \geq 1$. *Then:*

(a) $\mathrm{Brd}_p(F) \geq \mathrm{abrd}_p(E) + \kappa - 1$, *if* $\mathrm{abrd}_p(E) < \infty$ *and F/E is rational;*

(b) *If* $\mathrm{abrd}_p(E) = \infty$, *then* $\mathrm{Brd}_p(F) = \infty$, *and for any $n, m \in \mathbb{N}$ with $n \geq m > 0$, there is $D_{n,m} \in d(F)$, such that* $\mathrm{ind}(D_{n,m}) = p^n$ *and* $\exp(D_{n,m}) = p^m$*;*

(c) *$Brd_p(F) = \infty$, provided that* $p = \mathrm{char}(E)$ *and* $[E\colon E^p] = \infty$.

The main result of the present paper can be stated as follows:

Theorem 2.2. *For each $q \in \mathbb{P} \cup \{0\}$ and $k \in \mathbb{N}$, there exists a field $E_{q,k}$ with $char(E_{q,k}) = q$, $Brd(E_{q,k}) = k$ and $abrd_p(E_{q,k}) = \infty$, for all $p \in \mathbb{P} \setminus P_q$, where $P_0 = \{2\}$ and $P_q = \{p \in \mathbb{P}\colon\ p \mid q^2 - q)\}$, $q \in \mathbb{P}$. Moreover, if $q > 0$, then $E_{q,k}$ can be chosen so that $[E_{q,k}\colon E_{q,k}^q] = \infty$.*

When $q = 0$, the assertion of Theorem 2.2 is contained in our next result, which clarifies with Proposition 2.1 the influence of invariants $\mathrm{abrd}_p(E)$, $p \in \mathbb{P}$, on the behaviour of $\mathrm{Brd}_p(F)$, $p \in \mathbb{P}$, for any transcendental FG-extension F/E:

Theorem 2.3. *Let b_p, a_p, $p \in \mathbb{P}$, be a sequence with terms in the set $\mathbb{N}_\infty = \mathbb{N} \cup \{0, \infty\}$, such that $b_2 = a_2$ and $b_p \leq a_p \leq \infty$, $p \in \mathbb{P}$. Then there exists a Henselian field (K, v) with* $\mathrm{char}(\widehat{K}) = 0$, *$\mathcal{G}_{\widehat{K}}$ pronilpotent of cohomological dimension* $\mathrm{cd}(\mathcal{G}_{\widehat{K}}) \leq 1$, *and* $(\mathrm{abrd}_p(K), \mathrm{Brd}_p(K)) = (a_p, b_p)$, *$p \in \mathbb{P}$.*

Proposition 2.1, Theorem 2.2 and statement (1.1) (b) imply the following:

(2.2) There exist fields E_k, $k \in \mathbb{N}$, such that $\mathrm{char}(E_k) = 2$, $\mathrm{Brd}(E_k) = k$ and all Brauer pairs $(m', n') \in \mathbb{N}^2$ are index-exponent pairs over any transcendental FG-extension of E_k.

It is not known whether (2.2) holds in any characteristic $q \neq 2$. This is closely related to the following open problem:

(2.3) Find whether there exists a field E containing a primitive p-th root of unity, for a given $p \in \mathbb{P}$, such that $\mathrm{Brd}_p(E) < \mathrm{abrd}_p(E) = \infty$.

Statement (1.1) (b), Proposition 2.1 and Theorem 2.2 imply the validity of (2.2), for $q = 0$ and Brauer pairs of odd positive integers. When $q > 2$, they show that if $[E_{q,k} \colon E_{q,k}^q] = \infty$, then Brauer pairs $(m', m) \in \mathbb{N}^2$ relatively prime to $q - 1$ are index-exponent pairs over every transcendental FG-extension of $E_{q,k}$. Thus Problem 4.4 of [2] is solved in the negative. As a whole, our research shows that (2.1) can be a suitable replacement in the list of [2] for this problem.

The proofs of our main results are based on results of valuation theory like Morandi's theorem on tensor products of valued division algebras [21], Theorem 1, and the classical Ostrowski theorem. They rely on a standard method of realizing profinite groups as Galois groups [31], and on a construction of Henselian fields with prescribed properties of their value groups, residue fields and finite extensions. We also use a characterization of fields E with $\mathrm{abrd}_p(E) \leq \mu$, for a given $\mu \in \mathbb{N}$ (which generalizes Albert's theorem [1], Ch. XI, Theorem 3), as well as formulae for $\mathrm{Brd}_p(K)$ and $\mathrm{abrd}_p(K)$ concerning some Henselian fields. This approach enables one to obtain the following:

(2.4) (a) There exists a field E_1 with $\mathrm{abrd}(E_1) = \infty$, $\mathrm{abrd}_p(E_1) < \infty$, $p \in \mathbb{P}$, and $\mathrm{Brd}(L_1) < \infty$, for every finite extension L_1/E_1; hence, by [7], Corollary 5.4, $\mathrm{Brd}(F_1) = \infty$, for every transcendental FG-extension F_1/E_1;

(b) For any integer $n \geq 2$, there is a Galois extension L_n/E_n, such that $[L_n \colon E_n] = n$, $\mathrm{Brd}_p(L_n) = \infty$, for all $p \in \mathbb{P}$, $p \equiv 1 (\mathrm{mod}\ n)$, and $\mathrm{Brd}(M_n) < \infty$, provided that M_n is an extension of E in $L_{n,\mathrm{sep}}$ not including L_n.

Our basic notation and terminology are standard, as used in [5]. For any field K with a Krull valuation v, unless stated otherwise, we denote by $\widehat{K}$ and $v(K)$ the residue field and the value group of (K, v), respectively; $v(K)$ is supposed to be an additively written totally ordered abelian group. As usual, $\mathbb{Z}$ stands for the additive group of integers, $\mathbb{Z}_p$ is the additive groups of p-adic integers, for any $p \in \mathbb{P}$, and $[r]$ is the integral part of any real number $r \geq 0$. We write $I(\Lambda'/\Lambda)$ for the set of intermediate fields of a field extension Λ'/Λ, and $\mathrm{Br}(\Lambda'/\Lambda)$ for the relative Brauer group of Λ'/Λ. By a Λ-valuation of Λ', we mean a Krull valuation v with $v(\lambda) = 0$, $\lambda \in \Lambda^*$. Given a field E and $p \in \mathbb{P}$, $E(p)$ denotes the maximal p-extension of E in E_{sep}, and $r_p(E)$ the rank of the Galois group $\mathcal{G}(E(p)/E)$ as a pro-p-group ($r_p(E) = 0$, if $E(p) = E$). Brauer groups are considered to be additively written, Galois groups are viewed as profinite with respect to the Krull topology, and by a homomorphism of profinite groups, we mean a continuous one. We refer the reader to [10], [13], [17], [23] and [26], for any missing definitions concerning valuation theory, field extensions, simple algebras, Brauer groups and Galois cohomology.

Here is an overview of the rest of the paper: Section 3 includes preliminaries used in the sequel, and Galois-theoretic ingredients of the proof of Theorem 2.3. Theorems 2.2, 2.3 and statement (2.4) are proved in Section 4.

3 Preliminaries on Henselian valuations and preparation for the proof of Theorem 2.3

The results of this Section are known and will often be used without an explicit reference. Assume that (K, v) is a Henselian field, i.e. v is a Krull valuation on K, which extends uniquely, up-to an equivalence, to a valuation v_L on each algebraic extension L/K. Put $v(L) = v_L(L)$ and denote by $\widehat{L}$ the residue field of (L, v_L). It is known that $\widehat{L}/\widehat{K}$ is an algebraic extension and $v(K)$ is a subgroup of $v(L)$. When $[L\colon K]$ is finite, Ostrowski's theorem states the following (cf. [10], Theorem 17.2.1):

(3.1) $[\widehat{L}\colon \widehat{K}]e(L/K)$ divides $[L\colon K]$ and $[L\colon K][\widehat{L}\colon \widehat{K}]^{-1}e(L/K)^{-1}$ is not divisible by any $p \in \mathbb{P}$ different from $\mathrm{char}(\widehat{K})$, $e(L/K)$ being the index of $v(K)$ in $v(L)$; in particular, if $\mathrm{char}(\widehat{K}) \dagger [L\colon K]$, then $[L\colon K] = [\widehat{L}\colon \widehat{K}]e(L/K)$.

Statement (3.1) and the Henselity of v imply the following:

(3.2) The quotient groups $v(K)/pv(K)$ and $v(L)/pv(L)$ are isomorphic, if $p \in \mathbb{P}$ and L/K is a finite extension. When $\mathrm{char}(\widehat{K}) \dagger [L\colon K]$, the natural embedding of K into L induces canonically an isomorphism $v(K)/pv(K) \cong v(L)/pv(L)$.

A finite extension R/K is said to be defectless, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]e(R/K)$. It is called inertial, if $[R\colon K] = [\widehat{R}\colon \widehat{K}]$ and $\widehat{R}/\widehat{K}$ is separable. We say that R/K is totally ramified, if $[R\colon K] = e(R/K)$. The Henselity of v ensures that the compositum K_{ur} of inertial extensions of K in K_{sep} has the following properties:

(3.3) (a) $v(K_{\mathrm{ur}}) = v(K)$ and finite extensions of K in K_{ur} are inertial;

(b) K_{ur}/K is a Galois extension, $\widehat{K}_{\mathrm{ur}} \cong \widehat{K}_{\mathrm{sep}}$ over $\widehat{K}$, $\mathcal{G}(K_{\mathrm{ur}}/K) \cong \mathcal{G}_{\widehat{K}}$, and the natural mapping of $I(K_{\mathrm{ur}}/K)$ into $I(\widehat{K}_{\mathrm{sep}}/\widehat{K})$ is bijective.

When (K, v) is Henselian, each $\Delta \in d(K)$ has a unique, up-to an equivalence, valuation v_Δ extending v so that the value group $v(\Delta)$ of (Δ, v_Δ) is totally ordered and abelian (cf. [25], Ch. 2, Sect. 7). It is known that $v(K)$ is a subgroup of $v(\Delta)$ of index $e(\Delta/K) \leq [\Delta\colon K]$, and the residue division ring $\widehat{\Delta}$ of (Δ, v_Δ) is a $\widehat{K}$-algebra. By the Ostrowski-Draxl theorem [9], $[\Delta\colon K]$ is divisible by $e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}]$, and in case $\mathrm{char}(\widehat{K}) \dagger [\Delta\colon K]$, $[\Delta\colon K] = e(\Delta/K)[\widehat{\Delta}\colon \widehat{K}]$. An algebra $D \in d(K)$ is called inertial, if $[D\colon K] = [\widehat{D}\colon \widehat{K}]$ and $\widehat{D} \in d(\widehat{K})$. Inertial K-algebras and those in $d(\widehat{K})$ are related as follows (see [13], Theorem 2.8):

(3.5) (a) Each $\widetilde{D} \in d(\widehat{K})$ has an inertial lift over K, i.e. $\widetilde{D} = \widehat{D}$, for some $D \in d(K)$ inertial over K, and uniquely determined by $\widetilde{D}$, up-to a K-isomorphism.

(b) The set $\mathrm{IBr}(K) = \{[I] \in \mathrm{Br}(K)\colon\ I \in d(K) \text{ is inertial}\}$ is a subgroup of $\mathrm{Br}(K)$; the canonical mapping $\mathrm{IBr}(K) \to \mathrm{Br}(\widehat{K})$ is an isomorphism.

The study of $\mathrm{Brd}_p(K)$, for a given $p \in \mathbb{P}$, relies on constructive methods based on the following statements:

(3.6) (a) If $U_1, \dots, U_n$ are cyclic p-extensions of K in K_{ur}, and $\pi_1, \dots, \pi_n$ are elements of K^*, such that $[U_1 \dots U_n\colon K] = \prod_{j=1}^n [U_j\colon K]$ and the cosets $\bar{\pi}_j = v(\pi_j) + pv(K)$, $j = 1, \dots, n$, are linearly independent over $\mathbb{F}_p$, then $d(K)$ contains the K-algebra $B = \otimes_{j=1}^n B_j$, where $\otimes = \otimes_K$, and for each index j, B_j is the cyclic K-algebra $(U_j/K, \tau_j, \pi_j)$, τ_j being a generator of $\mathcal{G}(U_j/K)$.

(b) If $p \neq \mathrm{char}(\widehat{K})$, K contains a primitive p-th root of unity ε, and $\pi_1', \dots, \pi_{2m}'$ are elements of K^*, such that $\bar{\pi}_i' = v(\pi_i') + pv(K)$, $i = 1, \dots, 2m$, are linearly independent over $\mathbb{F}_p$, then the K-algebra $T = \otimes_{u=1}^{2m} T_u$ lies in $d(K)$, where $\otimes = \otimes_K$ and T_u is the symbol K-algebra $A_\varepsilon(\pi_{2u-1}', \pi_{2u}'; K)$, for every index u.

(c) Under the hypotheses of (a) and (b), if the system $\bar{\pi}_j, \bar{\pi}_i'$, $j = 1, \dots, n$; $i = 1, \dots 2m$, is linearly independent over $\mathbb{F}_p$, then $B \otimes_K T \in d(K)$.

Statement (3.6) (c) follows at once from [21], Theorem 1, and (3.6) (a) is a special case of [13], Example 4.3. Also, it is clear from Kummer theory and the conditions of (3.6) (b) that $T_1, \dots, T_m$ are cyclic K-algebras. Using (3.1) and the Henselity of v, one obtains further that $v(\pi'_{2u}) \neq v(\lambda_u)$, for any element λ_u of the norm group $N(K(\sqrt[p]{\pi'_{2u-1}})/K)$. Therefore, $\pi'_{2u} \notin N(K(\sqrt[p]{\pi'_{2u-1}})/K)$, so it follows from well-known general properties of cyclic K-algebras (cf. [23], Sect. 15.1, Proposition b) that $T_u \in d(K)$, $u = 1, \dots, m$. It is now easily deduced from [21], Theorem 1, that $T \in d(K)$, as claimed.

Statements (3.6) and the following lemma play a crucial role in the proof of Theorem 2.2.

Lemma 3.1. *Let K_0 be a perfect field of characteristic $q \geq 0$, and let $n(p)\colon\ p \in \mathbb{P}$, be a sequence with terms in $\mathbb{N}_\infty$. Then there exists a Henselian field (K, v) with $\mathrm{char}(K) = q$ and $\widehat{K} = K_0$, such that the group $v(K)/pv(K)$ has dimension $n(p)$ as a vector space over the field $\mathbb{F}_p$ with p elements, for each $p \in \mathbb{P}$. Moreover, if $q > 0$ and $n(q) < \infty$, then K can be chosen so that $[K\colon K^q] = q^{n(q)}$ and $r_q(K) = \infty$, and in case $n(q) > 0$, $v(K)$ possesses an isolated subgroup H satisfying the following:*

(a) *$H/pH \cong v(K)/pv(K)$ and $v(K)/H = p(v(K)/H)$, $p \in \mathbb{P} \setminus \{q\}$; $H = qH$ and $(v(K)/H)/q(v(K)/H) \cong v(K)/qv(K)$;*

(b) *The valuation v_H of K with $v_H(K) = v(K)/H$, defined by the composition $\eta_H \circ v\colon\ K^* \to v(K)/H$, where η_H is the natural homomorphism of $v(K)$ upon $v(K)/H$, has a perfect residue field $K_H \in I(K/K_0)$ with $r_q(K_H) = \infty$.*

Proof. Let K_∞ be an extension of K_0 obtained as the union $K_\infty = \cup_{n\in\mathbb{N}} K_n$ of iterated formal (Laurent) power series fields, defined inductively by the rule $K_n = K_{n-1}((X_n))$, $n \in \mathbb{N}$. Denote by ω_n the standard K_0-valuation of K_n with $\omega_n(K_n) = \mathbb{Z}^n$, for each $n \in \mathbb{N}$ ($\mathbb{Z}^n$ is viewed as an ordered group with respect to the inverse lexicographic ordering). Let ω be the natural valuation of K_∞ extending ω_n, for every n. Clearly, K_0 is the residue field of (K_∞, ω) and $\omega(K_\infty)$ equals the union $\mathbb{Z}^\infty = \cup_{n\in\mathbb{N}} Z^n$, considered with its unique ordering inducing the noted orderings on $\mathbb{Z}^n$, for all $n \in \mathbb{N}$. It is known (cf. [10], Sects. 4.2 and 18.4) that the valuations ω_n, $n \in \mathbb{N}$, are Henselian, which implies ω is of the same kind. Note further that if $q > 0$, then the set $\rho(K_\infty) = \{u^q - u\colon u \in K_\infty\}$ is a vector subspace of K_∞ over its prime subfield $\mathbb{F}$, and $\omega(u^q - u) \in q\omega(K_\infty)$ whenever $\omega(u) < 0$. This implies that, for any $\pi \in K_\infty$ with $\omega(\pi) < 0$ and $\omega(\pi) \notin q\omega(K_\infty)$, the cosets $\pi^{1+qm} + \rho(K_\infty)$, $m \in \mathbb{N}$, are linearly independent over $\mathbb{F}$, so it follows from the Artin-Schreier theorem (cf. [17], Ch. VIII, Sect. 6) that $r_q(K_\infty) = \infty$. These observations show that if $n(p) = \infty$, for all $p \in \mathbb{P}$, then it suffices for the proof of Lemma 3.1 to put $(K, v) = (K_\infty, \omega)$.

Henceforth, we assume that the set $P = \{p \in \mathbb{P} \setminus \{q\}\colon\ n(p) < \infty\}$ is nonempty, $\overline{K}_\infty$ is an algebraic closure of K_∞, $\bar{\omega}(K_\infty)$ is a divisible hull of $\omega(K_\infty)$, and for each $R \in I(\overline{K}_\infty/K_\infty)$, ω_R is the valuation of R extending ω so that $\omega(R)\colon = \omega_R(R)$ be an ordered subgroup of $\bar{\omega}(K_\infty)$. For any $p \in P$ and each index $n > n(p)$, let $\Sigma_{p,n} = \{Y_{p,n,m}\colon\ m \in \mathbb{N}\}$ be a subset of $\overline{K}_\infty$, such that $Y_{p,n,1}^p = X_n$ and $Y_{p,n,m}^p = Y_{p,n,(m-1)}$, $m \geq 2$. Put $\Sigma = \cup_{p\in P} \Sigma_p$, where $\Sigma_p = \cup_{n=n(p)+1}^{\infty} \Sigma_{p,n}$, for each $p \in P$, and denote by $\widetilde{K}$ the extension of K_∞ generated by Σ. It is easily verified that finite extensions of K_∞ in $\widetilde{K}$

are totally ramified, and for each $p \in \mathbb{P} \setminus \{q\}$, $n(p)$ equals the dimension of $\omega(\widetilde{K})/p\omega(\widetilde{K})$ as an $\mathbb{F}_p$-vector space. As $r_q(K_\infty) = \infty$ in case $q > 0$, one also sees that then $r_q(\widetilde{K}) = \infty$ as well. These observations show that $(\widetilde{K}, \omega_{\widetilde{K}})$ has the property required by Lemma 3.1 in the case where $q = 0$ or $q > 0$ and $n(q) = \infty$. Suppose now that $q > 0$ and $n(q) < \infty$, and let Θ_0 be the perfect closure of $\widetilde{K}$ in $\overline{K}_\infty$. As K_0 is perfect, the basic theory of algebraic extensions (cf. [17], Ch. VII, Proposition 12) implies that $\omega(\Theta_0) = q\omega(\Theta_0)$, and for each $p \in \mathbb{P} \setminus \{q\}$, $\omega(\Theta_0)/p\omega(\Theta_0)$ has dimension $n(p)$ over $\mathbb{F}_p$. Thus Lemma 3.1 is proved in the case where $n(q) = 0$.

It remains to consider the case of $0 < n(q) < \infty$. Let $n(q) = n$, Θ_n be an iterated formal power series field in n variables over Θ_0, κ the standard $\mathbb{Z}^n$-valued Θ_0-valuation of Θ_n, and w the valuation of Θ_n extending ω so that $\omega(\Theta_0)$ be an isolated subgroup of $w(\Theta_n)$, $w(\Theta_n)$ the direct sum $\omega(\Theta_0) \oplus \kappa(\Theta_n)$, and κ be induced canonically by w and $\omega(\Theta_0)$ (cf. [10], Sect. 4.2). Then [10], Theorem 18.1.2, and [30], Theorem 32.15, imply w inherits the Henselity of ω and κ. Applying (3.1), [10], Theorem 18.4.1, and [30], Theorem 31.21, one concludes that finite extensions of Θ_n are defectless relative to κ, and n equals the $\mathbb{F}_p$-dimension of $\kappa(\Theta_n)/p\kappa(\Theta_n)$, for $p \in \mathbb{P}$. Let now K be a maximal extension of Θ_n in $\Theta_{n,\mathrm{sep}}$ with respect to the property that finite extensions of Θ_n in K have degrees not divisible by q and are totally ramified over Θ_n relative to κ. Then $[K\colon K^q] = q^n$, $\kappa(K) = p\kappa(K)$, $p \in \mathbb{P} \setminus \{q\}$, and it follows from (3.2), [5], (1.2), and the preceding observation that the natural embedding of Θ_n into K induces an isomorphism $\kappa(\Theta_n)/q\kappa(\Theta_n) \cong \kappa(K)/q\kappa(K)$. These results and the obtained properties of (Θ_0, ω) indicate that $\kappa(K) \cong v(K)/\omega(\Theta)$ and $v(K)$ has the properties required by Lemma 3.1, where $v = w_K$. □

Remark 3.2. *Under the hypotheses of Lemma 3.1, suppose that $q > 0$ and $0 < n(q) = n < \infty$, Θ_0 and Θ_n, κ, w and ω are defined as in the proof of the lemma, $\Theta_j = \Theta_{j-1}((Z_j))$, $j = 1, \dots, n$, and θ is the Henselian discrete Θ_{n-1}-valuation of Θ_n. Denote by κ', w' and θ' the valuations of K_{sep} extending κ, w and θ, respectively, put $\Lambda_0 = \Theta_{n-1}(Z_n)$, and for any $\Lambda \in I(K_{\mathrm{sep}}/\Lambda_0)$, let w_Λ, κ_Λ and θ_Λ the valuations of Λ induced by w', κ' and θ', respectively. Analyzing the proof of the latter part of Lemma 3.1, one obtains the existence of a subset $\Gamma \subset K$, such that $\Theta_n(\Gamma) = K$, the field $\Phi_0 = \Lambda_0(\Gamma)$ is separable over Λ_0, $\Phi_0' \cap \Theta_n = \Lambda_0$, where $\Phi_0' \in I(K_{\mathrm{sep}}/\Phi_0)$ is the Galois closure of Φ_0 over Λ_0, and for each $\Lambda \in I(\Theta_n/\Lambda_0)$, the field $\Phi = \Lambda(\Gamma)$ satisfies the condition $\kappa_\Phi(\Phi) = \kappa(K)$, $\theta_\Phi(\Phi) = \theta(K)$ and $w_\Phi(\Phi) = w(K)$. Also, $\Phi_0'\Lambda$ is the root field of Φ in K_{sep} over Λ, $\mathcal{G}(\Phi_0'\Lambda/\Lambda) \cong \mathcal{G}(\Phi_0'/\Lambda_0)$ and finite extensions of Λ in Φ are totally ramified of degrees not divisible by q. In addition, the residue fields of w_Φ, κ_Φ and θ_Φ are isomorphic to K_0, Θ_0 and Θ_{n-1}, respectively.*

We conclude this Section with two lemmas which contain the main Galois-theoretic ingredients of our proofs of (2.4) (a) and Theorem 2.3. The former lemma makes it easy to prove Theorem 2.3 steering clear of (2.3).

Lemma 3.3. *There exists a field E_0 with $\mathrm{char}(E_0) = 0$, such that $\mathcal{G}_{E_0}$ is isomorphic to the additive group $\mathbb{Z}_2$ of 2-adic integers, and for each $p \in \mathbb{P}$, $[E_0(\varepsilon_p)\colon E_0] = 2^{y(p)}$, where ε_p is a primitive p-th root of unity in $E_{0,\mathrm{sep}}$, and $y(p)$ is the greatest integer for which $2^{y(p)} \mid p - 1$.*

Proof. Let ε_p be a primitive p-th root of unity in $\mathbb{Q}_{\rm sep}$, and let R_0 be the extension of $\mathbb{Q}$ in $\mathbb{Q}_{\rm sep}$ generated by the set $\Sigma = \{\sqrt{\alpha_p}\colon p \in \mathbb{P}\}$, where $\alpha_2 = -2$ and $\alpha_p = (-1)^{(p+1)/2}$, for each $p > 2$. Then it follows from Kummer theory that $\sqrt{-1} \notin R_0$, i.e. the set $\Sigma' = \{R \in I(\mathbb{Q}_{\rm sep}/R_0)\colon \sqrt{-1} \notin R\}$ is nonempty. Clearly, Σ' satisfies the conditions of Zorn's lemma, whence it contains a maximal element E_0 with respect to the partial ordering by inclusion. In view of Galois theory, this ensures that ${\rm Fe}(E_0)$ consists of cyclic 2-extensions. Observing also that E_0 is a nonreal field (since $\sqrt{\alpha_2} \in E_0$), one obtains from [32], Theorem 2, that $\mathcal{G}_{E_0} \cong \mathbb{Z}_2$, as claimed. It remains to be seen that $[E_0(\varepsilon_p)\colon E_0] = 2^{y(p)}$, for an arbitrary fixed $p \in \mathbb{P} \setminus \{2\}$. It is well-known that $\mathbb{Q}(\varepsilon_p)/\mathbb{Q}$ is a cyclic extension and $\mathbb{Q}(\sqrt{\beta_p})$ is the unique quadratic extension of $\mathbb{Q}$ in $\mathbb{Q}(\varepsilon_p)$, where $\beta_p = (-1)^{(p-1)/2}p$. It is therefore clear from Galois theory that the equality $[E_0(\varepsilon_p)\colon E_0] = 2^{y(p)}$ will follow, if we show that $\sqrt{\beta_p} \notin E_0$. This, however, is obvious, since $\beta_p = -\alpha_p$, $\sqrt{\alpha_p} \in E_0$ and $\sqrt{-1} \notin E_0$, so Lemma 3.3 is proved. □

Lemma 3.4. *Assume that E_0 is a field, such that ${\rm cd}(\mathcal{G}_{E_0}) \le 1$, and let G be a profinite group with ${\rm cd}(G) \le 1$ and ${\rm cd}_p(G) = 0$ whenever $p \in \mathbb{P}$ and ${\rm cd}_p(\mathcal{G}_{E_0}) \neq 0$. Then there exists a field extension E/E_0, such that E_0 is algebraically closed in E and $\mathcal{G}_E$ is isomorphic to the topological group product $\mathcal{G}_{E_0} \times G$.*

Proof. It is known (cf. [31]) that E_0 has extensions R and R', such that R'/E_0 is rational, $R \in I(R'/E_0)$ and R'/R is Galois with $\mathcal{G}(R'/R) \cong G$. Identifying $E_{0,{\rm sep}}$ with its E_0-isomorphic copy in $R'_{\rm sep}$, and observing that E_0 is algebraically closed in R', one obtains that $E_{0,{\rm sep}}R'/R$ is Galois with $\mathcal{G}(E_{0,{\rm sep}}R'/R) \cong \mathcal{G}_{E_0} \times G$. In view of the assumptions on $\mathcal{G}_{E_0}$ and G, this yields ${\rm cd}(\mathcal{G}(E_{0,{\rm sep}}R'/R)) = 1$, which means that $\mathcal{G}(E_{0,{\rm sep}}R'/R)$ is a projective profinite group (cf. [26], Ch. I, 5.9). Hence, by Galois theory, there is a field $E \in I(R'_{\rm sep}/R)$, such that $E_{0,{\rm sep}}R'E = R'_{\rm sep}$ and $(E_{0,{\rm sep}}R') \cap E = R$. This shows that E_0 is algebraically closed in E and $\mathcal{G}_E \cong \mathcal{G}(E_{0,{\rm sep}}R'/R) \cong \mathcal{G}_{E_0} \times G$, which proves Lemma 3.4. □

4 Proofs of Theorems 2.2 and 2.3

First we characterize the condition ${\rm abrd}_p(E) \le \mu$, for a field E and a given $\mu \in \mathbb{N}$. When E is virtually perfect, by (1.3), this result in fact is equivalent to [22], Lemma 1.1, and in case $\mu = 1$, it restates Theorem 3 of [1], Ch. XI.

Lemma 4.1. *Let E be a field, $p \in \mathbb{P}$ and $\mu \in \mathbb{N}$. Then ${\rm abrd}_p(E) \le \mu$ if and only if, for each $E' \in {\rm Fe}(E)$, ${\rm ind}(\Delta) \le p^\mu$ whenever $\Delta \in d(E')$ and $\exp(\Delta) = p$.*

Proof. The left-to-right implication is obvious, so we prove only the converse one. Fix a pair $E' \in {\rm Fe}(E)$, $\Delta' \in d(E')$ with $\exp(\Delta') = p^n$, for some $n \in \mathbb{N}$. We show that ${\rm ind}(\Delta') \mid p^{n\mu}$. This is obvious, if $n = 1$, so we assume that $n \ge 2$. Take $\Delta \in d(E')$ so that $[\Delta] = p^{n-1}[\Delta']$, and let Y be a maximal subfield of Δ. It is well-known that $[Y\colon E'] = {\rm ind}(\Delta)$ and Y can be chosen so as to be separable over E' (see [23], Sect. 13.5). Therefore, our assumptions show that $[Y\colon E'] \mid p^\mu$.

Note that, by the choice of Δ, $\Delta' \otimes_{E'} Y \in s(Y)$ and $\exp(\Delta' \otimes_{E'} Y) = p^{n-1}$. These remarks and a standard inductive argument lead to the conclusion that it suffices to prove the divisibility $\mathrm{ind}(\Delta') \mid p^{n\mu}$, provided $\mathrm{ind}(\Delta' \otimes_{E'} Y) \mid p^{(n-1)\mu}$. Fix $\Delta'_Y \in d(Y)$ so that $[\Delta'_Y] = [\Delta' \otimes_{E'} Y]$, and take a maximal subfield Y' of Δ'_Y. Then $[Y' \colon E'] = \mathrm{ind}(\Delta' \otimes_{E'} Y).[Y \colon E']$, which implies $[Y' \colon E'] \mid p^{n\mu}$. Observing finally that $[\Delta'] \in \mathrm{Br}(Y'/E')$ (cf. [23], Sects. 9.4 and 13.1), one obtains that $\mathrm{ind}(\Delta') \mid [Y' \colon E'] \mid p^{n\mu}$, so Lemma 4.1 is proved. □

Remark 4.2. *Note that a field E satisfies $abrd_p(E) < \infty$, for some $p \in \mathbb{P}$, if and only if there exists $c_p(E) \in \mathbb{N}$, such that each $A_R \in s(R)$ with $exp(A_R) = p$ is Brauer equivalent to a tensor product of $c_p(E)$ algebras from $s(R)$ of degree p, where R ranges over $Fe(E_p)$ and E_p is the fixed field of a Sylow pro-p-subgroup G_p of $\mathcal{G}_E$. Since E_p contains a primitive p-th root of unity unless $p = \mathrm{char}(E)$, this can be deduced from Lemma 4.1 and "quantative" versions of [20], (16.1), and [1], Ch. VII, Theorem 28 (see [28], page 506, and [27], respectively). When $abrd_p(E) < \infty$ and $p \neq \mathrm{char}(E)$, $c_p(E)$ is in fact a cohomological invariant of G_p (cf. [20], (11.5)). As noted in [15], the Bloch-Kato Conjecture, proved in [29], implies that if $abrd_p(E) < \infty$, then $cd_p(\mathcal{G}_E) < \infty$ unless E is formally real and $p = 2$ (see also [17], Ch. XI, Sect. 2, and [26], Ch. I, 3.3).*

Lemma 3.1 and the following two lemmas form the valuation-theoretic basis for the proof of the main results of this paper.

Lemma 4.3. *Let (K, v) be a valued field with $\mathrm{char}(K) = q > 0$, and let K_v be a Henselization of K in K_{sep} relative to v. Then:*

(a) $\mathrm{Brd}_q(K) \leq n$, *provided that* $[K \colon K^q] = q^n < \infty$;

(b) $\mathrm{Brd}_q(K) \geq n$, *if $v(K)/qv(K)$ has order q^n and $r_q(\widehat{K}) \geq n$; in this case, (q^n, q) is an index-exponent pair over K.*

Proof. Lemma 4.3 (a) follows from (1.3), Lemma 4.1, and [1], Ch. VII, Theorem 28, so it remains for us to prove Lemma 4.3 (b). It is clear from the Artin-Schreier theorem that K possesses degree q extensions $U_1, \dots, U_n$ in $K(q)$, such that $U'_j = U_j K_v$ is inertial over K_v with $[U'_j \colon K_v] = q$, $j = 1, \dots, n$, and $[U' \colon K_v] = q^n$, where $U' = U'_1 \dots U'_n$. As $v(K)/qv(K)$ is of order q^n, this enables one to deduce Lemma 4.3 (b) from (3.6) (a). □

Lemma 4.4. *Let (K, v) be a Henselian field with $\mathcal{G}_{\widehat{K}}$ pronilpotent and $\mathrm{cd}_p(\mathcal{G}_{\widehat{K}}) \leq 1$, for some $p \in \mathbb{P}$, $p \neq \mathrm{char}(\widehat{K})$. Let also $\tau(p)$ be the $\mathbb{F}_p$-dimension of $v(K)/pv(K)$, $\varepsilon_p \in \widehat{K}_{\mathrm{sep}}$ a primitive p-th root of unity, and $m_p = \min\{\tau(p), r_p(\widehat{K})\}$. Then:*

(a) $\mathrm{Brd}_p(K) = \infty$ *if and only if* $m_p = \infty$ *or* $\tau(p) = \infty$ *and* $\varepsilon_p \in \widehat{K}$; $\mathrm{abrd}_p(K) = \infty$, *if and only if* $\tau(p) = \infty$;

(b) $\mathrm{Brd}_p(K) = \mathrm{abrd}_p(K) = [(m_p + \tau(p))/2]$, *in case* $\varepsilon_p \in \widehat{K}$, $r_p(\widehat{K}) \leq 1$ *and* $\tau(p) < \infty$; $\mathrm{Brd}_p(K) = m_p$, *if* $\varepsilon_p \notin \widehat{K}$ *and* $m_p < \infty$.

(c) $\mathrm{abrd}_p(K) = \tau(p)$, *if* $r_p(\widehat{K}) \geq 2$ *and* $\tau(p) < \infty$.

Proof. It is clear from (3.3) and (3.6) (a) that $\mathrm{Brd}_p(K) = \infty$, provided that $m_p = \infty$. Henceforth, we assume that $m_p < \infty$. Suppose first that $\varepsilon_p \notin \widehat{K}$. Since $p \neq \mathrm{char}(K)$ and $\mathrm{cd}_p(\mathcal{G}_{\widehat{K}}) \leq 1$ (whence, $\mathrm{abrd}_p(\widehat{K}) = 0$), the Henselity of v ensures that every $D \in d(K)$ with $[D] \in \mathrm{Br}(K)_p$ is a nicely semi-ramified algebra over K, in the sense of [13] (see Lemmas 5.14 and 6.2 therein). Hence, by [13], Theorem 4.4, D is defined, for some $n \in \mathbb{N}$, by cyclic p-extensions $U_1, \dots, U_n$ of K in K_{ur}, and by elements $\pi_1, \dots, \pi_n \in K^*$, in accordance with (3.6) (a). This indicates that $U = U_1 \dots U_n$ is a Galois extension of K, $\mathcal{G}(U/K)$ has a system of at most n generators, $n \leq m_p$, and $\mathrm{ind}(D)$ and $\exp(D)$ are equal to the order and to the period of $\mathcal{G}(U/K)$, respectively. These observations prove that $\mathrm{Brd}_p(K) = m_p$. Since $[\widehat{K}(\varepsilon_p) \colon \widehat{K}] \mid p - 1$, they imply in conjunction with (1.1) (c), (3.2) and (3.3) that one may assume, for the rest of the proof of Lemma 4.4, that $\varepsilon_p \in \widehat{K}$. Then (3.6) (b) yields $\mathrm{Brd}_p(K) = \infty$, if $\tau(p) = \infty$, so it remains for us to consider the case of $\tau(p) < \infty$. As $p \neq \mathrm{char}(\widehat{K})$ and $\mathrm{cd}_p(\mathcal{G}_{\widehat{K}}) \leq 1$, it is clear from (3.5) (b) and [13], Lemmas 5.14 and 6.2, that $\mathrm{abrd}_p(K) = 0$, provided that $\tau(p) = 0$. This agrees with the conclusions of the lemma, so we assume further that $\tau(p) > 0$. Our proof relies on the following observations:

(4.1) (a) For each $D \in d(K)$ with $\exp(D) = p$, the group $v(D)/v(K)$ has period p, $\widehat{D}$ is a field and $\widehat{D}/\widehat{K}$ is a Galois extension, such that $\mathcal{G}(\widehat{D}/\widehat{K})$ is a homomorphic image of $v(D)/v(K)$; hence, $\mathrm{ind}(D)^2 = [\widehat{D} \colon \widehat{K}]e(D/K) \mid p^{m_p + \tau(p)}$.

(b) If $r_p(\widehat{K}) \geq 2$, then there exists a finite extension U of K in $K_{\mathrm{ur}} \cap K(p)$, such that $r_p(U) > \tau(p)$.

The inequality $\mathrm{cd}_p(\mathcal{G}_{\widehat{K}}) \leq 1$ ensures that $\mathcal{G}(\widehat{K}(p)/\widehat{K})$ is a free pro-p-group, so (4.1) (b) can be deduced from (3.3) and Nielsen-Schreier's formula for open subgroups of free pro-p-groups (cf. [26], Ch. I, 4.2, and Ch. II, 2.1). Statement (4.1) (a) is contained in [13], (1.6) and Corollary 6.10). It follows from (4.1) (a) and Lemma 4.1 that $\mathrm{abrd}_p(K) \leq \tau(p)$, whereas (3.6) (a) and (4.1) (b) imply $\mathrm{Brd}_p(U) \geq \tau(p)$. These results prove Lemma 4.4 (a) and (c).

We turn to the proof of Lemma 4.4 (b), so we assume that $r_p(\widehat{K}) \leq 1$. Then it follows from (3.2), (3.3), [32], Theorem 2, and the conditions on $\mathcal{G}_{\widehat{K}}$ that $r_p(\widehat{K'}) = r_p(\widehat{K})$ and $v(K')/pv(K') \cong v(K)/pv(K)$, for every $K' \in \mathrm{Fe}(K)$. Hence, by (4.1) (a) and Lemma 4.1, $\mathrm{Brd}_p(K') \leq [(m_p + \tau(p))/2]$, proving that $\mathrm{abrd}_p(K) \leq [(m_p + \tau(p))/2]$. On the other hand, it is clear from (3.6) (b) that $\mathrm{Brd}_p(K) \geq [\tau(p)/2]$. These observations prove Lemma 4.4 (b) in case $r_p(\widehat{K}) = 0$. Suppose finally that $r_p(\widehat{K}) = 1$. Then it turns out that $d(K)$ contains an algebra $B \otimes_K T$, defined in accordance with (3.6) (c), for $n = 1$, $[U_1 \colon K] = p$ and $m = [(\tau(p) - 1)/2]$. In particular, $\mathrm{ind}(B \otimes_K T) = p^{m+1}$ and $\exp(B \otimes_K T) = p$, which implies $\mathrm{Brd}_p(K) \geq 1 + m = [(1 + \tau(p))/2$ and so completes the proof of Lemma 4.4 (b). □

We are now in a position to prove Theorem 2.3. Let G be a pronilpotent group with $\mathrm{cd}(G) = 1$, G_p the Sylow pro-p-subgroup of G and r_p the rank of G_p, for each $p \in \mathbb{P}$. Suppose that $G_2 \cong \mathbb{Z}_2$ and put $\mathbb{P}' = \mathbb{P} \setminus \{2\}$. Then, it follows from Burnside-Wielandt's theorem (cf. [16], Ch. 6, Theorem 17.1.4) that G is isomorphic to the topological group product $\prod_{p \in \mathbb{P}} G_p$. As $\mathrm{cd}(G) = 1$, Lemma 3.3 and Lemma 3.4, applied to $P = \{2\}$, G_2 and $\prod_{p \in \mathbb{P}'} G_p$, imply that $G \cong \mathcal{G}_{K_0}$, for some characteristic zero field K_0 not containing a primitive p-th root of unity, for any $p \in \mathbb{P}'$. This ensures that $r_p(K_0) = r_p$, $p \in \mathbb{P}$. Note finally that G can be chosen so that $r_p = b_p$, for all $p > 2$, and by Lemma 3.1, there exists a Henselian field (K, v) with $\widehat{K} \cong K_0$. Moreover, it follows from Lemmas 3.1 and 4.4 that (K, v) (specifically, the invariants $\tau(p)$ of $v(K)/pv(K)$, $p \in \mathbb{P}$) can be chosen so that $(\mathrm{Brd}_p(K), \mathrm{abrd}_p(K)) = (b_p, a_p)$, $p \in \mathbb{P}$, as claimed.

Our objective now is to prove Theorem 2.2 in the case of $q > 0$. The former part of this theorem is proved by applying our next result to the field $K_0 = \mathbb{F}_q$ and a system c_p, $p \in \mathbb{P}$, with $c_p = \infty$, for all $p \dagger q^2 - q$.

Lemma 4.5. *Let K_0 be a finite field with q^m elements, where $q = \mathrm{char}(K_0)$. Put $P_{q,m} = \{p \in \mathbb{P} \colon p \mid q(q^m - 1)\}$, and fix a system $c_p \in \mathbb{N}_\infty \colon\ p \in \mathbb{P}$. Then:*

(a) *There exists a Henselian field (K, v) with $\mathrm{char}(K) = q$, $\widehat{K} = K_0$, $\mathrm{Brd}_q(K) = c_q$ and $\mathrm{abrd}_p(K) = c_p$, for each $p \in \mathbb{P}$; this ensures that $\mathrm{Brd}_p(K) \leq 1$ when $p \in \mathbb{P} \setminus P_{q,m}$, $\mathrm{Brd}_p(K) = c_p$, $p \in \mathbb{P}_{q,m}$, and $\mathrm{Brd}_p(K) \neq 0$ in case $c_p \neq 0$;*

(b) *If $0 < c_q \neq \infty$, then (K, v) can be chosen so that $[K \colon K^q] = q^{c_q}$;*

(c) *When $c_q = 0$, (K, v) can be chosen so that $r_q(K) = \infty$ and K be perfect.*

Proof. Let $\bar{n} = n(p) \in \mathbb{N}_\infty \colon\ p \in \mathbb{P}$, be a sequence, such that $n(p) = \infty$, provided $c(p) = \infty$, and $2c_p - 1 \leq n(p) \leq 2c_p$ in case $c(p) < \infty$ and $p \neq q$. Let also (K, v) be a Henselian field with $\mathrm{char}(K) = q$ and $\widehat{K} = K_0$, attached to $\bar{n}$ as in Lemma 3.1 (and subject to its additional restrictions in case $c(q) < \infty$). Then it follows from Lemmas 4.3, 4.4 and the equalities $r_p(K_0) = 1$, $p \in \mathbb{P}$, that (K, v) has the properties required by Lemma 4.5. □

The extension Θ_n/Λ_0 considered in Remark 3.2 satisfies the condition $\mathrm{trd}(\Theta_n/\Lambda_0) = \infty$ (see [3], and further references there). Hence, Λ_0 has a rational extension Λ_∞ in Θ_n with $\mathrm{trd}(\Lambda_\infty/\Lambda_0) = \infty$. This implies $[\Lambda \colon \Lambda^q] = [\Lambda_\infty \colon \Lambda_\infty^q] = \infty$, where Λ is the separable closure of Λ_∞ in Θ_n. Therefore, the latter assertion of Theorem 2.2 can be deduced from Lemma 4.5 and the following lemma.

Lemma 4.6. *Let K_0 be a finite field, and in the setting of Remark 3.2, put $\Theta = \Theta_n$, and suppose that $\Lambda \in I(\Theta/\Lambda_0)$ is separably closed in Θ. Then:*

(a) *The valuations w_Λ, κ_Λ and θ_Λ of Λ are Henselian;*

(b) *For each finite separable extension R of Λ in K_{sep}, $R\Theta$ is a completion of R relative to the topology induced by w_R, and $w_{R\Theta}$ is the continuous prolongation of w_R on $R\Theta$; in addition, $D_R \otimes_R R\Theta \in d(R\Theta)$, for every $D_R \in d(R)$;*

(c) *The field $\Phi = \Lambda(\Gamma)$ satisfies the equalities $\mathrm{Brd}_p(\Phi) = \mathrm{Brd}_p(K)$ and $abrd_p(\Phi) = \mathrm{abrd}_p(K)$, $p \in \mathbb{P}$, $\mathrm{Brd}_q(\Phi) = \mathrm{abrd}_q(\Phi) = n$, and $[\Phi \colon \Phi^q] = [\Lambda \colon \Lambda^q]$.*

Proof. Lemma 4.6 (a) follows from [10], Theorem 15.3.5, and the Henselity of the valuations w, κ and θ of Θ. The former claim of Lemma 4.6 (b) is obvious, and it enables one to deduce the latter part of Lemma 4.6 (b) from [8], Theorem 2. As $v = w_K$, $w_\Phi(\Phi) = w_K(K)$ and K_0 is the residue fields of (K, v) and (Φ, w_Φ), Lemma 4.4 implies $\mathrm{Brd}_p(\Phi) = \mathrm{Brd}_p(K)$ and $\mathrm{abrd}_p(\Phi) = \mathrm{abrd}_p(K)$, for each $p \neq q$. Observing that $[\Theta \colon \Theta^q] = q^n$, one obtains from Lemma 4.6 (b) and [1], Ch. VII, Theorem 28, that $\mathrm{Brd}_q(R) \leq \mathrm{Brd}_q(R\Theta) \leq n$, for every finite separable extension R of Λ in K_{sep}. This proves that $\mathrm{abrd}_q(\Lambda) \leq \mathrm{abrd}_q(\Theta) = n$, which leads to the conclusion that $\mathrm{Brd}_q(\Phi) \leq \mathrm{abrd}_q(\Phi) \leq \mathrm{abrd}_q(\Lambda)$ (see also [5], (1.2)). On the other hand, by Remark 3.2, $\kappa_\Phi(\Phi) = \kappa(K)$ and the residue field of (Φ, κ_Φ) is isomorphic to Θ_0. Since, by the proof of Lemma 3.1, $r_q(\Theta_0) = \infty$ and $\kappa(K)/q\kappa(K)$ is of order q^n, this allows us to obtain from Lemma 4.3 that $\mathrm{Brd}_q(\Phi) \geq n$. Note finally that Φ/Λ is a separable extension, so we have $[\Phi\colon \Phi^q] = [\Lambda\colon \Lambda^q]$, which completes our proof. □

Remark 4.7. *The proof of Theorem 2.2 is technically simpler in characteristic 2. Lemma 4.4 shows that if $K_0 = \mathbb{F}_2$ and Θ_0 is a perfect closure of the extension K_∞ of K_0 defined in the proof of Lemma 3.1, then $abrd_2(\Theta_0) = 0$, $Brd_p(\Theta_0) = 1$ and $abrd_p(\Theta_0) = \infty$, for all $p > 2$. When $n \in \mathbb{N}$, Θ_n and Λ_0 are defined as in Remark 3.2, Λ_∞ is a rational extension of Λ_0 in Θ_n with $trd(\Lambda_\infty/\Lambda_0) = \infty$, and Λ is the separable closure of Λ in Θ_n, then $[\Lambda\colon \Lambda^2] = \infty$, $Brd_2(\Lambda) = \mathrm{abrd}_2(\Lambda) = n$, and for each $p > 2$, $Brd_p(\Lambda) = 1$ and $abrd_p(\Lambda) = \infty$. Note also, omitting the details, that Θ_0 can be used for finding an alternative proof of Theorem 2.2 in zero characteristic (see [7], Example 6.2).*

When $c_p \in \mathbb{N}$, $p \in \mathbb{P}$, is an unbounded sequence, the fields E singled out by Lemma 4.5 have the properties required by (2.4) (a). As to (2.4) (b), it is implied by Lemma 3.1 and our next result.

Corollary 4.8. *In the setting of Lemma 4.4, let $\widehat{K}$ be a quasifinite field with $\mathrm{char}(\widehat{K}) = 0$ and $\varepsilon_p \notin \widehat{K}$, for any $p \in \mathbb{P}\backslash\{2\}$, and let U_n be the degree n extension of K in K_{ur}, for a fixed integer $n \geq 2$. Suppose that $P_n = \{p_n \in \mathbb{P}\colon\ n \mid p_n - 1\}$, $[\widehat{K}(\varepsilon_{p_n})\colon \widehat{K}] = n$, for all $p_n \in \mathbb{P}_n$, and the sequence $\tau(p)\colon\ p \in \mathbb{P}$, satisfies the condition $\tau(p) = \infty$ if and only if $p \in \mathbb{P}_n$. Then a field $L \in \mathrm{Fe}(K)$ satisfies $\mathrm{Brd}_p(L) < \infty$, $p \in \mathbb{P}$, if and only if $U_n \notin I(L/K)$. When $U_n \notin I(L/K)$ and the system $\tau(p)$, $p \in \mathbb{P} \setminus P_n$, is bounded,* $\mathrm{Brd}(L) < \infty$.

Proof. Lemma 4.4 and our assumptions show that if $p \notin P_n$, then $\mathrm{Brd}_p(L) \leq \mathrm{abrd}_p(K) < \infty$. When $p \in P_n$ and $L \in \mathrm{Fe}(K)$, they prove that $\mathrm{Brd}_p(L) = \infty$ if and only if $\varepsilon_p \in \widehat{L}$, and this occurs if and only if $U_n \subseteq L$. The concluding assertion of Corollary 4.8 follows from Lemma 4.4. □

Lemmas 3.3 and 3.4 indicate that there exists a quasifinite field E of zero characteristic, such that $[E(\varepsilon)\colon E] = 2^{y(p)}$, $p \in \mathbb{P}$, where ε_p is a primitive p-th root of unity in E_{sep} and $y(p)$ is defined as in Lemma 3.3, for each p. Also, Lemma 3.1 and Corollary 4.8 imply the existence of Henselian fields (E_n, v_n) with $\widehat{E}_n = E$, which possess the properties required by (2.4) (b), for $n = 2^t$, $t \in \mathbb{N}$. Using [6], Lemma 3.2, instead of Lemma 3.3, and arguing in the same way, one proves (2.4) (b) in general.

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