THE RELATIVE APPROXIMATION DEGREE IN VALUED FUNCTION FIELDS

FRANZ-VIKTOR KUHLMANN AND IZABELA VLAHU

Abstract. We continue the work of Kaplansky on immediate valued field extensions and determine special properties of elements in such extensions. In particular, we are interested in the question when an immediate valued function field of transcendence degree 1 is henselian rational (i.e., generated, modulo henselization, by one element). If so, then ramification can be eliminated in this valued function field. The results presented in this paper are crucial for the first author's proof of henselian rationality over tame fields, which in turn is used in his work on local uniformization.

1. Introduction

This paper continues the work of Kaplansky [3] in which, based on earlier work of Ostrowski [12], he laid the foundations for an understanding of immediate extensions of valued fields. Such an understanding has turned out to be essential for many questions about the structure of valued fields, which vary from their model theory and applications in real algebra to the very difficult task of elimination of ramification in valued function fields (which we will define below). The latter plays an essential role in the quest for local uniformization, which in turn is a local form of resolution of singularities. These problems being still wide open in positive characteristic, any refined valuation theoretical tools that can bring new insight are very important.

The theory developed by Kaplansky and Ostrowski is very useful for valuations with residue fields of characteristic 0, but its real strength (as well as its limitations) become visible when the residue characteristic is positive.

While Kaplansky was mainly concerned with embeddings in power series fields and the question when maximal immediate extensions are unique up to isomorphism, the above mentioned problems have added new questions to the spectrum. In the present paper we develop Kaplansky's tools further in order to answer various questions about the structure of immediate function fields. Several results of this paper are indispensable for the paper [10] on henselian rationality, which is central

Date: 12. 5. 2013.

2010 *Mathematics Subject Classification.* Primary 12J10; Secondary 12J20.

The authors wish to thank the referee for his thorough reading of the paper and several very helpful suggestions. The first author wishes to thank Peter Roquette for his invaluable help and support during the preparation of his doctoral thesis, in which several of the results presented here appeared first. He also wishes to thank F. Delon, B. Green, A. Prestel and F. Pop for inspiring discussions and suggestions. Thanks also to Salih Durhan for his careful reading of the manuscript.

During the work on this paper, the first author was partially supported by a Canadian NSERC grant.

in the first author's work on elimination of ramification and local uniformization (see [4]), as well as the model theory of valued fields (see [9]).

By $(L|K,v)$ we denote an extension of valued fields, i.e., $L|K$ is a field extension, v is a valuation on L, and K is endowed with the restriction of v (which we again denote by v.) An extension $(L|K,v)$ is said to be **immediate** if the canonical embeddings $vK \hookrightarrow vL$ of the value groups and $Kv \hookrightarrow Lv$ of the residue fields are onto. An important example for an immediate algebraic extension of a valued field (K,v) is its **henselization**, denoted by $(K,v)^h$ or just K^h, which is a minimal extension in which Hensel's Lemma holds. A valued field is **henselian** if it is equal to its henselization, or equivalently, if it admits a unique extension of its valuation to every algebraic extension field.

An immediate function field $(F|K,v)$ of transcendence degree 1 will be called **henselian rational** if there exists an element $x \in F^h$ such that $F^h = K(x)^h$, that is, F^h is the henselization of the rational function field $K(x)$, and $F \subset K(x)^h$. We then call x a **henselian generator** of F^h.

The Henselian Rationality Theorem of [5, 10] states that every immediate function field $(F|K,v)$ of transcendence degree 1 over a tame field (K,v) is henselian rational and the henselian generator x can already be found in F. The field (K,v) is called **tame** if it is henselian and the ramification field K^r of the normal extension $K^{\mathrm{sep}}|K$ is algebraically closed, where K^{sep} denotes the separable algebraic closure of K. In particular, every tame field is perfect.

In the above definition, the henselian field K has a unique extension of its valuation to K^{sep}. For an arbitrary valued field (K,v), K^r is defined with respect to a previously fixed extension of the valuation, and similarly, K^i is then defined to be the inertia field of the extension $(K^{\mathrm{sep}}|K,v)$. Note that $K^h \subseteq K^i \subseteq K^r$.

For an arbitrary valued function field $(F|K,v)$, **elimination of ramification** is the task of finding a transcendence basis T of $F|K$ such that F lies in $K(T)^i$. Since $K(x)^h \subseteq K(x)^i$, the Henselian Rationality Theorem eliminates ramification from immediate function fields of transcendence degree 1 over tame fields.

For the proof of the Henselian Rationality Theorem, one first reduces the problem to the case of valued fields of rank 1 (i.e., having archimedean ordered value groups), and then starts with an arbitrary element $x \in F$ transcendental over K; it can be chosen such that $F|K(x)$ is separable. If x is not a henselian generator, then $(F^h|K(x)^h,v)$ is a proper finite immediate extension. Let us describe the further steps of the proof in the important special case where $\mathrm{char}K = p > 0$. If one replaces $(F|K,v)$ by the valued function field $(F.K^r|K^r,v)$, which again is immediate, then the extension $((F.K^r)^h|K^r(x)^h,v)$ becomes a tower of Artin-Schreier extensions. The lowest of them is shown to be generated by a root y of a polynomial $X^p - X - f(x)$ where p is the residue characteristic and $f(x) \in K[x]$. We observe that $f(x) = y^p - y \in K(y)$, hence if $K(x)^h = K(f(x))^h$, then $K(x)^h \subsetneqq K(y)^h$. Replacing x by y, we have then reduced the degree of $F^h|K(x)^h$ by a factor of p. This shows that it is crucial to determine the degree $[K(x)^h : K(f(x))^h]$ for a given $f(x) \in K[x]$ and to choose $f(x)$ in such a way that the degree becomes 1.

In order to gain insight on the degree $[K(x)^h : K(f(x))^h]$, we study the elements $f(x) \in K[x]$ in (not necessarily transcendental) immediate extensions $(K(x)|K,v)$, through extending Kaplansky's technical lemmas. After introducing approximation types and their basic properties in Sections 3 and 4, this study is carried out in Sections 5 to 8. In Section 7, we define the "relative approximation degree of $f(x)$

in x" to be the integer h that appears in Kaplansky's Lemma 8. We then show in Theorem 9.1 that under suitable assumptions about the extension $(K(x)|K,v)$ and the element $f(x)$, the degree $[K(x)^h : K(f(x))^h]$ is smaller than or equal to the relative approximation degree of $f(x)$ in x.

Having proved (in [10]) that the immediate function field $(F.K^r|K^r(x),v)$ is henselian rational, one has to pull this property down to $(F|K,v)$. Observe that if $(F.K^r|K^r(x),v)$ is henselian rational, then the same already holds for $(F.L|L(x),v)$ for a suitable finite subextension $L|K$ of $K^r|K$. Moreover, $L|K$ can be chosen to be Galois since also $K^r|K$ is Galois (we allow Galois extensions to be infinite). An extension of a henselian field (K,v) is called **tame** if it lies in K^r. Consequently, a Galois extension is tame if and only if its ramification group is trivial. So what we need is a pull down principle for henselian rationality through tame extensions of the base field. This is presented in Theorem 14.5. More precisely, we show in Section 14 that if x is a henselian generator for $(F.L|L,v)$, where $(L|K,v)$ is a finite tame Galois extension, then for a suitable element $d \in L$, the trace $\mathrm{Tr}(d \cdot x)$ is a henselian generator for $(F|K,v)$. We use a valuation theoretical characterization of the Galois groups of tame Galois extensions that is developed in Section 13.

Once a henselian generator $x \in F^h$ is found, the question arises whether x can already be chosen in F. We show in Theorem 11.1 that this can be done. In fact, there is some $\gamma \in vK$ such that $K(x)^h = K(y)^h$ for every $y \in F$ with $v(x-y) \geq \gamma$. This result is crucial for the proof given in [4] that local uniformization can always be achieved after a finite Galois extension of the function field. In order to prove Theorem 11.1, we generalize the relative approximation degree to other elements $y \in K(x)^h$ in place of $f(x)$ in Section 10. We then prove the corresponding generalization of Theorem 9.1: Theorem 10.7 states that under suitable assumptions, we again have that the degree $[K(x)^h : K(y)^h]$ is smaller than or equal to the relative approximation degree of y in x.

Theorem 11.1 can be seen as a special case of a "dehenselization" procedure (analogous to the "decompletion" used by M. Temkin in [15]). If for a given valued function field $(F|K,v)$ there is a finite extension F' of F within its henselization such that $(F'|K,v)$ admits local uniformization, one would like to deduce that also $(F|K,v)$ admits local uniformization. This can be done if Theorem 11.1 can be generalized in a suitable way to the case of non-immediate valued function fields. This problem will be investigated in a subsequent paper.

Our investigation of the properties of elements in immediate extensions is facilitated by the introduction of the notion of "approximation type", which we use in place of Kaplansky's "pseudo-convergent sequences" (also called "pseudo-Cauchy sequences" or "Ostrowski nets" in the literature). This new notion makes computations and the formulation of results easier. For instance, to every element x in an immediate extension $(L|K,v)$, we associate the unique approximation type of x over K, while there are many pseudo-convergent sequences in K that have x as a pseudo-limit, and in addition one needs to require maximality of such sequences (for $x \notin K$ one asks that they do not have a pseudo limit in K). Furthermore, the definition of approximation types is not restricted to immediate extensions only. In fact, approximation types can be further enhanced to a tool for describing properties of elements in non-immediate extensions. In Section 6, we take the occasion to show how Kaplansky's fundamental Theorems 2 and 3 can be proved by using approximation types in place of pseudo-convergent sequences.

This paper is based on results that appeared in the first author's doctoral thesis (cf. [5]) and presents updated, improved and extended versions of them, with simplified proofs.

2. Some preliminaries

For basic facts from valuation theory, see [1], [2], [14], [16], [17].

Take a valued field (K,v). We denote its value group by vK, its residue field by Kv, and its valuation ring by $\mathcal{O}_K$. For $a\in K$, we write va for its value and av for its residue.

By $\tilde{K}$ we will denote the algebraic closure of K. For each extension of v to $\tilde{K}$, we have that $\tilde{K}v=\widetilde{Kv}$, and $v\tilde{K}$ is the divisible hull of vK, which we denote by $\widetilde{vK}$.

Note that the extension $(L|K,v)$ is immediate if and only if for all $b\in L$ there is $c\in K$ such that $v(b-c)>vb$ (as is implicitly shown in the proof of Lemma 4.1 below). This property can be used to define immediate extensions of other valued structures, such as valued abelian groups and valued vector spaces.

An algebraic extension $(L|K,v)$ of henselian fields is called **defectless** if every finite subextension $E|K$ satisfies the fundamental equality $[E:K]=\mathrm{e}\cdot\mathrm{f}$, where $\mathrm{e}=(vE:vK)$ is the ramification index and $\mathrm{f}=[Ev:Kv]$ is the inertia degree. In this case, $(E|K,v)$ admits a **standard valuation basis**, which we construct as follows: we take $a_1,\ldots,a_{\mathrm{e}}\in E$ such that $va_1+vK,\ldots,va_{\mathrm{e}}+vK$ are the cosets of vK in vE, and $b_1,\ldots,b_{\mathrm{f}}\in E$ such that $b_1v,\ldots,b_{\mathrm{f}}v$ are a basis of $Ev|Kv$. Then a_ib_j , $1\le i\le\mathrm{e}$, $1\le j\le\mathrm{f}$, is a basis of $E|K$, and it has the following property: for all choices of $c_{ij}\in K$,

$$v\sum_{i,j}c_{ij}a_ib_j \;=\; \min_{i,j} vc_{ij}a_ib_j \;=\; \min_{i,j} vc_{ij}a_i\;.$$

Note that we can always choose $a_1=b_1=1$ so that $a_1b_1=1$.

All tame extensions of henselian fields are defectless, see [9]. The following facts are well known and easy to prove:

Lemma 2.1. *Take a defectless extension $(L|K,v)$ of henselian fields and $a\in L$. Then the set $\{v(a-c)\mid c\in K\}$ has a maximum. More precisely, if we choose a standard valuation basis for $E=K(a)$ as above with $a_1=b_1=1$ and write*

$$a\;=\;\sum_{i,j}c_{ij}a_ib_j\;,$$

then $v(a-c_{1\,1})$ is the maximum of $\{v(a-c)\mid c\in K\}$.

Proof. For every $c\in K$,

$$\begin{aligned}
v(a-c_{1\,1}) &= v\sum_{(i,j)\neq(1,1)}c_{ij}a_ib_j \;=\; \min_{(i,j)\neq(1,1)} vc_{ij}a_ib_j\\
&\geq \min\{v(c_{1\,1}-c)\,,\, vc_{ij}a_ib_j\mid (i,j)\neq(1,1)\}\\
&= v\left(c_{1\,1}-c+\sum_{(i,j)\neq(1,1)}c_{ij}a_ib_j\right) \;=\; v(a-c)\;.
\end{aligned}$$

□

We will also need the following tool (cf. [7, Lemma 2.5]):

Lemma 2.2. *Take a henselian field (K,v), a valued field extension $(K'|K,v)$, an immediate subextension $(F|K,v)$, and a defectless algebraic subextension $(L|K,v)$. Then $F|K$ and $L|K$ are linearly disjoint, $(F.L|F,v)$ is defectless, and $(F.L|L,v)$ is immediate.*

3. Approximation types and distances

We will now introduce approximation types, which constitute a suitable structure for dealing with immediate extensions of valued fields.

We define $B_\alpha(c,K)=\{a\in K\mid v(a-c)\geq\alpha\}$ to be the "closed" ultrametric ball in (K,v) of radius $\alpha\in vK\infty:=vK\cup\{\infty\}$ centered at $c\in K$. An **approximation type over** (K,v) is a full nest of closed balls in (K,v), that is, a collection

$$\mathbf{A}\;=\;\{B_\alpha(c_\alpha,K)\mid\alpha\in S\}$$

with S an initial segment of $vK\infty$, $c_\alpha\in K$, and the balls $B_\alpha(c_\alpha,K)$ linearly ordered by inclusion. We write $\mathbf{A}_\alpha=B_\alpha(c_\alpha,K)$ for $\alpha\in S$, and $\mathbf{A}_\alpha=\emptyset$ otherwise. We call S the **support** of $\mathbf{A}$ and denote it by $\mathrm{supp}\mathbf{A}$.

Note that if $\beta<\alpha\in\mathrm{supp}\mathbf{A}$, then $\mathbf{A}_\beta=B_\beta(c_\beta,K)=B_\beta(c_\alpha,K)$, i.e., $\mathbf{A}_\beta$ is uniquely determined by $\mathbf{A}_\alpha$ and β. Hence, $\mathbf{A}$ is uniquely determined by the balls $\mathbf{A}_\alpha$ where α runs through an arbitrary cofinal sequence in $\mathrm{supp}\mathbf{A}$.

Take any extension $(L|K,v)$ and $x\in L$. For all $\alpha\in vK\infty$, we set

$$\mathrm{appr}(x,K)_\alpha\;:=\;\{c\in K\mid v(x-c)\geq\alpha\}\;=\;B_\alpha(x,L)\cap K\;. \tag{3.1}$$

It is easy to check that $\mathrm{appr}(x,K)_\alpha$ is empty or a closed ball of radius α. If $\mathrm{appr}(x,K)_\alpha\neq\emptyset$ and $\beta<\alpha$, then also $\mathrm{appr}(x,K)_\beta\neq\emptyset$. This shows that the set

$$\{\alpha\in vK\infty\mid\mathrm{appr}(x,K)_\alpha\neq\emptyset\}$$

is an initial segment of $vK\infty$ and therefore,

$$\mathrm{appr}(x,K)\;:=\;\{\mathrm{appr}(x,K)_\alpha\mid\alpha\in vK\infty\text{ and }\mathrm{appr}(x,K)_\alpha\neq\emptyset\} \tag{3.2}$$

is an approximation type over (K,v). We call $\mathrm{appr}(x,K)$ the **approximation type of x over** (K,v).

As the support S of $\mathrm{appr}(x,K)$ is an initial segment of $vK\infty$, $S\cap vK=S\setminus\{\infty\}$ is an initial segment of vK and thus induces a cut in vK with lower cut set $S\setminus\{\infty\}$. Now this cut induces a cut in the divisible hull $\widetilde{vK}$ of vK, where the lower cut set is the smallest initial segment of $\widetilde{vK}$ containing $S\setminus\{\infty\}$. We call this cut the **distance of x from** (K,v) and denote it by

$$\mathrm{dist}(x,K)\;.$$

We write $\mathrm{dist}(x,K)=\infty$ if the lower cut set is $\widetilde{vK}$, and $\mathrm{dist}(x,K)<\infty$ otherwise. Note that $\mathrm{dist}(x,K)=\infty$ if and only if S contains vK, which holds if and only if x lies in the completion of (K,v).

For a subset $A\subset K$ we define $\mathrm{dist}_K(x,A)$, the distance of x from A over K, to be the cut in $\widetilde{vK}$ having as lower cut set the smallest initial segment in $\widetilde{vK}$ containing the set $\{v(x-c)\mid c\in A\}\cap vK$.

Note that if $(L|K,v)$ is an algebraic extension of valued fields, then the divisible hull of vK coincides with the divisible hull of vL and so for an element x in an extension of K, we have that $\mathrm{dist}(x,K)$ and $\mathrm{dist}(x,L)$ are both cuts in the same group. This allows us to compare these distances by set inclusion of the lower cut sets. Another reason to take the distance in the divisible hull is that the classification of Artin–Schreier defect extensions through distances presented in [7] does not work if they are taken in ordered abelian groups with archimedean components which are not dense; this situation does not appear in divisible groups.

If n is a natural number and the lower cut set of $\mathrm{dist}(x,K)$ is D, then

$$n\cdot \mathrm{dist}(x,K)$$

will denote the cut with lower cut set $nD := \{n\gamma \mid \gamma\in D\}$; note that nD is again an initial segment of $\widetilde{vK}$ because of divisibility.

If C and C' are two cuts in a linearly ordered set T defined by their lower cut sets D and D', respectively, then $C = C'$ if $D = D'$, and we write $C < C'$ if $D \subsetneqq D'$. For an element $\alpha\in T$ we write $\alpha > C$ if $\alpha > \beta$ for all $\beta\in D$, and $\alpha\geq C$ if $\alpha\geq\beta$ for all $\beta\in D$; note that if D has no last element, then $\alpha > C \Leftrightarrow \alpha\geq C$. We write $\alpha\leq C$ if $\alpha\in D$, and $\alpha < C$ if $\alpha\in D$ but is not the last element of D.

Lemma 3.1. *Take an extension $(L|K,v)$ of valued fields, and $x,x'\in L$.*
a) For every α in the support of $\mathrm{appr}(x,K)$, $\mathrm{appr}(x,K)_\alpha = \mathrm{appr}(x',K)_\alpha$ holds if and only if $v(x-x')\geq\alpha$.
b) Further,

$$\mathrm{appr}(x,K) = \mathrm{appr}(x',K) \implies v(x-x')\geq \mathrm{dist}(x,K) = \mathrm{dist}(x',K)\,, \tag{3.3}$$

$$v(x-x')\geq \max\{\mathrm{dist}(x,K),\mathrm{dist}(x',K)\} \implies \mathrm{appr}(x,K) = \mathrm{appr}(x',K)\,. \tag{3.4}$$

Proof. a): Take $\alpha\in vK\infty$. If $v(x-x')\geq\alpha$, then $B_\alpha(x,L) = B_\alpha(x',L)$, which yields that $\mathrm{appr}(x,K)_\alpha = B_\alpha(x,L)\cap K = B_\alpha(x',L)\cap K = \mathrm{appr}(x',K)_\alpha$. If $v(x-x') < \alpha$, then $B_\alpha(x,L)\cap B_\alpha(x',L) = \emptyset$, whence $\mathrm{appr}(x,K)_\alpha\cap\mathrm{appr}(x',K)_\alpha = \emptyset$; for $\mathrm{appr}(x,K)_\alpha\neq\emptyset$, this yields that $\mathrm{appr}(x,K)_\alpha\neq\mathrm{appr}(x',K)_\alpha$.
b): If $\mathrm{dist}(x,K)\neq\mathrm{dist}(x',K)$, then $\mathrm{appr}(x,K)\neq\mathrm{appr}(x',K)$. If $v(x-x')\geq \mathrm{dist}(x,K)$ does not hold, then there is some α in the support of $\mathrm{appr}(x,K)$ such that $\alpha > v(x-x')$. By part a), it follows that $\mathrm{appr}(x,K)_\alpha\neq\mathrm{appr}(x',K)_\alpha$. This proves (3.3).

If $v(x-x')\geq\mathrm{dist}(x,K)$ holds, then $v(x-x')\geq\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}(x,K)$. Again by part a), it follows that $\mathrm{appr}(x,K)_\alpha = \mathrm{appr}(x',K)_\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}(x,K)$. Similarly, $v(x-x')\geq\mathrm{dist}(x',K)$ implies that $\mathrm{appr}(x,K)_\alpha = \mathrm{appr}(x',K)_\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}(x',K)$. If none of the supports contains ∞, then we obtain that $\mathrm{appr}(x,K) = \mathrm{appr}(x',K)$. If on the other hand, at least one support contains ∞, then the corresponding distance is ∞, whence $v(x-x') = \infty$, i.e., $x = x'$ and again, $\mathrm{appr}(x,K) = \mathrm{appr}(x',K)$. We have proved (3.4). □

If $\mathbf{A}$ is an approximation type over (K,v) and there exists an element x in some valued extension field L such that $\mathbf{A} = \mathrm{appr}(x,K)$, then we say that x **realizes** $\mathbf{A}$ (in (L,v)). If $\mathbf{A}$ is realized by some $c\in K$, then $\mathbf{A}$ will be called **trivial**. This holds if and only if $\mathbf{A}_\infty\neq\emptyset$, in which case $\mathbf{A}_\infty = \{c\}$. As $\mathbf{A}_\infty$ can contain at most one element, a trivial approximation type can be realized by only one element.

We leave the easy proof of the following lemma to the reader.

Lemma 3.2. *Take an approximation type $\mathbf{A}$ over (K,v) and an extension $(L|K,v)$ of valued fields. The element $x\in L$ realizes $\mathbf{A}$ if and only if the following conditions hold:*

1) *if $\alpha\in\mathrm{supp}\mathbf{A}$, then $v(x-c)\geq\alpha$ for some $c\in\mathbf{A}_\alpha$,*
2) *if $\beta\notin\mathrm{supp}\mathbf{A}$, then $v(x-c) < \beta$ for all $c\in K$.*

For our work with approximation types, we introduce the following notation which is particularly useful in the immediate case. We introduce it in connection with valued fields, but its application to ultrametric spaces and other valued structures is similar. So take an arbitrary valued field (K,v) and an approximation type $\mathbf{A}$ over (K,v). Further, take a formula φ with one free variable. Then the sentence

$$\varphi(c) \text{ for } c \nearrow \mathbf{A}$$

will denote the assertion

$$\text{there is } \alpha \in vK \text{ such that } \mathbf{A}_\alpha \neq \emptyset \text{ and } \varphi(c) \text{ holds for all } c \in \mathbf{A}_\alpha\ .$$

Note that if $\varphi_1(c)$ for $c \nearrow \mathbf{A}$ and $\varphi_2(c)$ for $c \nearrow \mathbf{A}$, then also $\varphi_1(c) \wedge \varphi_2(c)$ for $c \nearrow \mathbf{A}$.

In the case of $\mathbf{A} = \operatorname{appr}(x,K)$, we will also write "$c \nearrow x$" in place of "$c \nearrow \mathbf{A}$".

If $\gamma = \gamma(c) \in vK$ is a value that depends on $c \in K$ (e.g., the value $vf(c)$ for a polynomial $f \in K[X]$), then we will say that γ **increases for** $c \nearrow x$ if there exists some $\alpha \neq \infty$ in the support of $\operatorname{appr}(x,K)$ such that for every choice of $c' \in \operatorname{appr}(x,K)_\alpha$ with $x \neq c'$,

$$\gamma(c) > \gamma(c') \quad \text{for} \quad c \nearrow x\ .$$

Note that the condition $x \neq c'$ is automatically satisfied if $\operatorname{appr}(x,K)$ is nontrivial.

4. Immediate approximation types

An approximation type $\mathbf{A}$ with support S will be called **immediate** if its intersection

$$\bigcap \mathbf{A} \;=\; \bigcap_{\alpha\in S} \mathbf{A}_\alpha$$

is empty. If $\mathbf{A}$ is trivial, then $\bigcap \mathbf{A} = \mathbf{A}_\infty \neq \emptyset$; therefore, an immediate approximation type is never trivial. However, a nontrivial approximation type is not necessarily immediate; as we will see in the following lemma, in an extension $(L|K,v)$ that is not immediate there is always some $x \in L \setminus K$ for which $\operatorname{appr}(x,K)$ is not immediate. As a trivial approximation type is relized by only one element, which lies in K, we see that $\operatorname{appr}(x,K)$ is nontrivial.

Lemma 4.1. *Let $(L|K,v)$ be an extension of valued fields.*
a) If $x \in L$, then $\operatorname{appr}(x,K)$ is immediate if and only if for every $c \in K$ there is some $c' \in K$ such that $v(x-c') > v(x-c)$, that is, the set

$$v(x-K) \;:=\; \{v(x-c) \mid c \in K\}$$

has no maximal element.
b) The extension $(L|K,v)$ is immediate if and only if for every $x \in L \setminus K$, its approximation type $\operatorname{appr}(x,K)$ over (K,v) is immediate.
c) If $\operatorname{appr}(x,K)$ is immediate, then its support is equal to $v(x-K)$.

Proof. a): Suppose that $\operatorname{appr}(x,K)$ is immediate and that c is an arbitrary element of K. Then by definition there is some α such that $c \notin \operatorname{appr}(x,K)_\alpha \neq \emptyset$, so $v(x-c) < \alpha$. Choosing some $c' \in \operatorname{appr}(x,K)_\alpha$, we obtain that $v(x-c) < \alpha \leq v(x-c')$.

Now take $x \in L \setminus K$ and suppose that for every $c \in K$ there is $c' \in K$ such that $v(x-c') > v(x-c)$. Then there is also some $c'' \in K$ such that $v(x-c'') > v(x-c')$. By the ultrametric triangle law we obtain that $v(c'-c) = v(x-c) < v(x-c') =$

$v(c''-c')$. Hence $v(c'-c)\in v(x-K)$ and $c\notin \mathrm{appr}(x,K)_{v(c''-c')}\neq\emptyset$. As $c\in K$ was arbitrary, this shows that $\mathrm{appr}(x,K)$ is immediate.

b): Assume that $(L|K,v)$ is immediate. Take $x\in L\setminus K$ and an arbitrary $c\in K$. Then $v(x-c)\in vL=vK$, i.e., there is $d\in K$ such that $v(x-c)=vd$ so that $vd^{-1}(x-c)=0$. Then $d^{-1}(x-c)v\in Lv=Kv$, i.e., there is $d'\in K$ such that $d^{-1}(x-c)v=d'v$, which means that $v(d^{-1}(x-c)-d')>0$. This implies that $v(x-c-dd')>vd=v(x-c)$. Setting $c'=c+dd'$, we obtain $v(x-c')>v(x-c)$. By part a) it now follows that $\mathrm{appr}(x,K)$ is immediate.

For the converse, assume that for every $x\in L\setminus K$, $\mathrm{appr}(x,K)$ is immediate. By the proof of a), for every $c\in K$ we have that $v(x-c)\in vK$, so in particular, $v(x-0)\in vK$; this shows that $vL|vK$ is trivial. It remains to show that $Lv|Kv$ is trivial. Take any $x\in L\setminus K$ with $vx=0$. Since $\mathrm{appr}(x,K)$ is immediate, there is $c'\in K$ such that $v(x-c')>v(x-0)=vx$. From this we obtain that $xv=c'v\in Kv$. Hence $Lv|Kv$ is trivial.

c): If $\alpha\in vK$ is an element of the support of $\mathrm{appr}(x,K)$, then $\mathrm{appr}(x,K)_\alpha\neq\emptyset$, and so by (3.1), there is $c\in K$ such that $v(x-c)\geq\alpha$. In the case of $v(x-c)=\alpha$, we immediately see that $\alpha\in v(x-K)$. In the case of $v(x-c)>\alpha$, choose some $d\in K$ with $vd=\alpha$; then $v(x-(c+d))=vd=\alpha$, which again shows that $\alpha\in v(x-K)$.

For the converse inclusion, take $c\in K$. By the proof of part a), there is $c'\in K$ such that $v(x-c)=v(c'-c)$, which shows that $v(x-c)\in vK$. It follows from (3.1) that $c\in\mathrm{appr}(x,K)_{v(x-c)}$, so $v(x-c)$ is in the support of $\mathrm{appr}(x,K)$. $\square$

For immediate approximation types, we can improve part b) of Lemma 3.1, and Lemma 3.2.

Lemma 4.2. *Take an extension $(L|K,v)$ of valued fields, and $x,x'\in L$. If $\mathrm{appr}(x,K)$ is immediate, then*

$$\mathrm{appr}(x,K)=\mathrm{appr}(x',K)\iff v(x-x')\geq\mathrm{dist}(x,K)\,. \tag{4.1}$$

Proof. We only have to prove the implication “$\Leftarrow$”. As in the proof of (3.4), we deduce from $v(x-x')\geq\mathrm{dist}(x,K)$ that $v(x-x')\geq\alpha$ and $\mathrm{appr}(x,K)_\alpha=\mathrm{appr}(x',K)_\alpha$ for all $\alpha\neq\infty$ in the support of $\mathrm{appr}(x,K)$. Since $\mathrm{appr}(x,K)$ is immediate, we also know that ∞ is not in its support. It remains to show that $\mathrm{appr}(x',K)_\alpha=\emptyset$ for every α not in the support of $\mathrm{appr}(x,K)$. If this were not true, there would be $c\in K$ such that $v(x'-c)>\mathrm{supp}\,\mathrm{appr}(x,K)$. Since also $v(x-x')>\mathrm{supp}\,\mathrm{appr}(x,K)$, we would obtain that $v(x-c)>\mathrm{supp}\,\mathrm{appr}(x,K)$. But then $c\in\bigcap\mathrm{appr}(x,K)$, contradicting the assumption that $\mathrm{appr}(x,K)$ is immediate. $\square$

Lemma 4.3. *Take an immediate approximation type $\mathbf{A}$ over (K,v) and an extension $(L|K,v)$. The element $x\in L$ realizes $\mathbf{A}$ if and only if for every $\alpha\in\mathrm{supp}\mathbf{A}$, $v(x-c)\geq\alpha$ for some $c\in\mathbf{A}_\alpha$.*

Proof. We have to show that for every immediate approximation type $\mathbf{A}$, condition 2) of Lemma 3.2 holds if condition 1) holds. Assume that $\beta\notin\mathrm{supp}\mathbf{A}$. Since the support is an initial segment of $vK\infty$, this means that $\beta>\mathrm{supp}\mathbf{A}$. Take any $c\in K$. Since $\mathbf{A}$ is immediate, there is some $\alpha\in\mathrm{supp}\mathbf{A}$ such that $c\notin\mathbf{A}_\alpha$. By condition 1), there is some $c'\in\mathbf{A}_\alpha$ such that $v(x-c')\geq\alpha$. Now $v(x-c)\geq\alpha$ would imply that $v(c-c')\geq\min\{v(x-c),v(x-c')\}\geq\alpha$, whence $c\in\mathbf{A}_\alpha$, a contradiction. It follows that $v(x-c)<\alpha<\beta$. Hence condition 2) holds. $\square$

Corollary 4.4. *Take an immediate approximation type* $\mathbf{A}$ *over* (K,v)*, an extension* $(L|K,v)$ *of valued fields, and* $x \in L$*. If* $v(x-c)$ *is not fixed for* $c \nearrow \mathbf{A}$*, then* $\mathbf{A} = \mathrm{appr}(x,K)$*.*

Proof. Our assumption means that for all $\alpha \in \mathrm{supp}\mathbf{A}$ there are $c, c' \in \mathbf{A}_\alpha$ such that $v(x-c') > v(x-c)$. This implies that $v(x-c') > \min\{v(x-c), v(c-c')\}$, whence $v(x-c) = v(c-c') \geq \alpha$. Now our assertion follows from the previous lemma. □

In the remainder of this section, we wish to explore how immediate approximation types behave under valued field extensions $(L|K,v)$. Take x in some extension of L such that $x \notin L$ and $\mathrm{appr}(x,K)$ is immediate. Obviously,

$$\mathrm{dist}(x,L) \;\geq\; \mathrm{dist}(x,K)$$

and

$$\mathrm{appr}(x,K)_\alpha = B_\alpha(c_\alpha,K) \implies \mathrm{appr}(x,L)_\alpha = B_\alpha(c_\alpha,L)\ . \tag{4.2}$$

If $\mathrm{dist}(x,L) = \mathrm{dist}(x,K)$, then by (4.2), $\mathrm{appr}(x,K)$ fully determines $\mathrm{appr}(x,L)$. But if $\mathrm{dist}(x,L) > \mathrm{dist}(x,K)$, then $\mathrm{appr}(x,K)$ does not provide enough information for those $\mathrm{appr}(x,L)_\beta$ with $\beta > \mathrm{dist}(x,L)$.

Lemma 4.5. *If in the above situation* $(L|K,v)$ *is a defectless extension, then* $\mathrm{dist}(x,L) = \mathrm{dist}(x,K)$ *and by (4.2),* $\mathrm{appr}(x,K)$ *fully determines* $\mathrm{appr}(x,L)$*.*

Proof. Suppose that $\mathrm{dist}(x,L) > \mathrm{dist}(x,K)$. Then there is some $a \in L$ such that $v(x-a) > \mathrm{dist}(x,K)$, which by (3.4) implies that $\mathrm{appr}(a,K) = \mathrm{appr}(x,K)$, which is immediate. But by Lemma 2.1, $\{v(a-c) \mid c \in K\}$ has a maximum. This contradicts part a) of Lemma 4.1. □

5. Polynomials and immediate approximation types

Take an arbitrary polynomial $f \in K[X]$ and an approximation type $\mathbf{A}$ over (K,v). We will say that $\mathbf{A}$ **fixes the value of** f if there is some $\alpha \in vK$ such that $vf(c) = \alpha$ for $c \nearrow \mathbf{A}$. We will call an immediate approximation type $\mathbf{A}$ a **transcendental approximation type** if $\mathbf{A}$ fixes the value of every polynomial $f(X) \in K[X]$. Otherwise, $\mathbf{A}$ is called an **algebraic approximation type**. If there exists any polynomial $f \in K[X]$ whose value is not fixed by $\mathbf{A}$, then there exists also a monic polynomial of the same degree having the same property (since this property is not lost by multiplication with nonzero constants from K). If $f(X)$ is a monic polynomial of minimal degree $\mathbf{d}$ such that $\mathbf{A}$ does not fix the value of f, then it will be called an **associated minimal polynomial** for $\mathbf{A}$, and $\mathbf{A}$ is said to be of **degree d**. We define the degree of a transcendental approximation type to be $\mathbf{d} = \infty$. According to this terminology, an approximation type over K of degree $\mathbf{d}$ fixes the value of every polynomial $f \in K[X]$ with $\deg f < \mathbf{d}$. Note that an associated minimal polynomial f for $\mathbf{A}$ is always irreducible over K. Indeed, if the degree of $g, h \in K[X]$ is smaller than $\deg f$, then $\mathbf{A}$ fixes the value of g and h and thus also of $g \cdot h$. Since every polynomial $g \in K[X]$ of degree $\mathbf{d}$ whose value is not fixed by $\mathbf{A}$ is just a multiple cf of an associated minimal polynomial f for $\mathbf{A}$ (with $c \in K^\times$), the irreducibility holds for every such polynomial as well.

If the support of $\mathbf{A}$ is bounded from above in vK, then associated minimal polynomials are not uniquely determined. Indeed, if f is an associated minimal polynomial and g is a polynomial of lower degree with coefficients of large enough value, then $f+g$ is again an associated minimal polynomial.

We note that an immediate approximation type $\mathbf{A}$ fixes the value of every linear polynomial in $K[X]$. Indeed, for every $c \in K$ there is $\alpha \in \mathrm{supp}\mathbf{A}$ such that $c \notin \mathbf{A}_\alpha$. Hence for all $c', c'' \in \mathbf{A}_\alpha$, $v(c'-c'') > v(c-c')$ and thus $v(c'-c) = v(c''-c)$. This shows that $\mathbf{A}$ fixes the value of $X-c$. We conclude that the degree of an algebraic approximation type is not less than 2.

We will now study the behaviour of polynomials with respect to immediate approximation types $\mathrm{appr}(x,K)$. We need the following lemma for ordered abelian groups, which is a reformulation of Lemma 4 of Kaplansky [3]. For archimedean ordered groups, it was proved by Ostrowski [12].

Lemma 5.1. *Take elements $\alpha_1, \ldots, \alpha_m$ of an ordered abelian group Γ and a subset $\Upsilon \subset \Gamma$ without maximal element. Let $t_1, \ldots, t_m$ be distinct integers. Then there exists an element $\beta \in \Upsilon$ and a permutation σ of the indices $1, \ldots, m$ such that for all $\gamma \in \Upsilon$, $\gamma \geq \beta$,*

$$\alpha_{\sigma(1)} + t_{\sigma(1)}\gamma > \alpha_{\sigma(2)} + t_{\sigma(2)}\gamma > \ldots > \alpha_{\sigma(m)} + t_{\sigma(m)}\gamma\ .$$

For an arbitrary polynomial $f(X) = c_n X^n + c_{n-1}X^{n-1} + \ldots + c_0$, we call

$$f_i(X) \ :=\ \sum_{j=i}^{n} \binom{j}{i} c_j X^{j-i} \ =\ \sum_{j=0}^{n-i} \binom{j+i}{i} c_{j+i} X^j \tag{5.1}$$

the **i-th formal derivative of f** and

$$f(X) \ =\ \sum_{i=0}^{n} f_i(c)(X-c)^i \tag{5.2}$$

$$f_i(X) \ =\ \sum_{j=i}^{n} \binom{j}{i} f_j(c)(X-c)^{j-i} \tag{5.3}$$

the **Taylor expansions** of f and f_i at c.

If the immediate approximation type $\mathbf{A}$ is of degree $\mathbf{d}$ and $f \in K[X]$ is of degree at most $\mathbf{d}$, then $\mathbf{A}$ fixes the value of every formal derivative f_i of f ($1 \leq i \leq \deg f$), since every such derivative has degree less than $\mathbf{d}$. So we can define β_i to be the fixed value $vf_i(c)$ for $c \nearrow x$. In certain cases, a derivative may be identically 0. In this case, we have $\beta_i = \infty$. However, the Taylor expansion of f shows that not all derivatives vanish identically, and the vanishing ones will not play a role in our computations.

By use of Lemma 5.1, we can now prove:

Lemma 5.2. *Take an immediate approximation type $\mathbf{A} = \mathrm{appr}(x,K)$ of degree $\mathbf{d}$ over (K,v) and $f \in K[X]$ a polynomial of degree at most $\mathbf{d}$. Further, let β_i denote the fixed value $vf_i(c)$ for $c \nearrow x$. Then there is a positive integer $\mathbf{h} \leq \deg f$ such that*

$$\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) < \beta_i + i \cdot v(x-c) \tag{5.4}$$

whenever $i \neq \mathbf{h}$, $1 \leq i \leq \deg f$ and $c \nearrow x$. Hence,

$$v(f(x) - f(c)) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) \quad \textit{for } c \nearrow x\ . \tag{5.5}$$

Consequently, if $\mathbf{A}$ fixes the value of f, then

$$v(f(x) - f(c)) > vf(x) = vf(c) \quad \textit{for } c \nearrow x\ ,$$

and if $\mathbf{A}$ *does not fix the value of* f*, then*

$$vf(x) > vf(c) = \beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) \quad \textit{for } c \nearrow x\ .$$

Proof. Set $n = \deg f$. We consider the Taylor expansion

$$f(x) - f(c) = f_1(c)(x-c) + \ldots + f_n(c)(x-c)^n \tag{5.6}$$

with $c \in K$. We have that $vf_i(c)(x-c)^i = \beta_i + i \cdot v(x-c)$ for $c \nearrow x$. So we apply the foregoing lemma with $\alpha_i = \beta_i$ and $t_i = i$, and with Υ equal to the support of $\mathbf{A}$ (which has no maximal element since $\mathbf{A}$ is an immediate approximation type). We find that there is an integer $\mathbf{h} \leq \deg f$ such that $\beta_{\mathbf{h}} + \mathbf{h}v(x-c) < \beta_i + iv(x-c)$ for $c \nearrow x$ and $i \neq \mathbf{h}$. This is equation (5.4), which in turn implies equation (5.5).

If $\mathbf{A}$ fixes the value of f, then $vf(x) \neq vf(c)$ is impossible for $c \nearrow x$ since otherwise, the left hand side of (5.5) would be equal to $\min\{vf(x), vf(c)\}$ and thus fixed while the right hand side of (5.5) increases for $c \nearrow x$. This proves that $vf(x) = vf(c)$ and thus also $v(f(x) - f(c)) \geq vf(x)$ for $c \nearrow x$. But since the right hand side increases, we find that $v(f(x) - f(c)) > vf(x)$ for $c \nearrow x$.

If $\mathbf{A}$ does not fix the value of f, then $vf(x) \neq vf(c)$ and thus $v(f(x) - f(c)) = \min\{vf(x), vf(c)\}$ for $c \nearrow x$. Since $v(f(x) - f(c))$ increases for $c \nearrow x$ and $vf(x)$ is a constant, the minimum must be $vf(c)$, and $vf(x) = vf(c)$ is impossible. □

If $g \in K[X]$ has a degree smaller than the degree of $\mathbf{A}$, then by the foregoing lemma, the value of $g(x)$ in $(K(x), v)$ is given by $vg(x) = vg(c)$ for $c \nearrow x$. Since $g(c) \in K$, that means that the value of $g(x)$ is uniquely determined by $\mathbf{A}$ and the restriction of v to K. If g is a nonzero polynomial, then $g(c) \neq 0$ for $c \nearrow x$ (since there is a nonempty $\mathbf{A}_\alpha$ which does not contain the finitely many zeros of g, as $\mathbf{A}$ is immediate). Consequently, $g(x) \neq 0$, which shows that the elements $1, x, \ldots, x^{\mathbf{d}-1}$ are K-linearly independent.

We even know that $v(g(x) - g(c)) > vg(x)$ for $c \nearrow x$. This means that $(K, v) \subset (K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$ is an immediate extension of valued vector spaces. If $\mathbf{d} = [K(x) : K] < \infty$, then $K(x) = K[x] = K + Kx + \ldots + Kx^{\mathbf{d}-1}$, and so the valued field extension $(K(x)|K, v)$ is immediate. If $\mathbf{d} = \infty$, then $(K, v) \subset (K[x], v)$ is immediate. But then again it follows that the valued field extension $(K(x)|K, v)$ is immediate. Indeed, if $v(g(x) - g(c)) > vg(x)$ and $v(h(x) - h(c)) > vh(x)$, then $vg(x) = vg(c)$, $vh(x) = vh(c)$ and

$$\begin{aligned}
v\left(\frac{g(x)}{h(x)} - \frac{g(c)}{h(c)}\right) &= v\left[g(x)h(c) - g(c)h(x)\right] - vh(x)h(c) \\
&= v\left[g(x)h(c) - g(c)h(c) + g(c)h(c) - g(c)h(x)\right] - vh(x)h(c) \\
&= v\left[(g(x) - g(c))h(c) + g(c)(h(c) - h(x))\right] - vh(x)h(c) \\
&> vg(x)h(x) - vh(x)h(x) \;=\; v\,\frac{g(x)}{h(x)}\ .
\end{aligned}$$

We have proved:

Lemma 5.3. *Take an immediate approximation type* $\mathbf{A} = \mathrm{appr}(x, K)$ *of degree* $\mathbf{d}$ *over* (K, v)*. Then the valuation on the valued* (K, v)*-vector subspace* $(K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$ *of* $(K(x), v)$ *is uniquely determined by* $\mathbf{A}$ *because*

$$vg(x) = vg(c) \quad \textit{for } c \nearrow x$$

for every $g(x) \in K + Kx + \ldots + Kx^{\mathbf{d}-1}$*. The elements* $1, x, \ldots, x^{\mathbf{d}-1}$ *are* K*-linearly independent. In particular,* x *is transcendental over* K *if* $\mathbf{d} = \infty$*.*

Moreover, the extension $(K,v) \subset (K + Kx + \ldots + Kx^{\mathbf{d}-1}, v)$ of valued vector spaces is immediate. In particular, if $\mathbf{d} = \infty$ or if $\mathbf{d} = [K(x) : K] < \infty$, then $(K[x]|K,v)$ is immediate and the same is consequently true for the valued field extension $(K(x)|K,v)$.

So far we have only considered polynomials of degree at most $\mathbf{d}$; the next lemma will cover the remaining case.

Lemma 5.4. *Take an immediate algebraic approximation type $\mathbf{A} = \mathrm{appr}(x,K)$ over (K,v) and an associated minimal polynomial $f \in K[X]$ for $\mathbf{A}$. Further, take an arbitrary polynomial $g \in K[X]$ and write*

$$g(X) = c_k(X)f(X)^k + \ldots + c_1(X)f(X) + c_0(X) \tag{5.7}$$

with polynomials $c_i \in K[X]$ of degree less than $\deg f$. Then there is some integer m, $1 \leq m \leq k$, and a value $\beta \in vK$ such that with $\mathbf{h}$ as in Lemma 5.2,

$$v(g(c) - c_0(c)) = vc_m(c) + m \cdot vf(c) = \beta + m \cdot \mathbf{h} \cdot v(x-c) \quad \textit{for } c \nearrow x\ . \tag{5.8}$$

Consequently, if $\mathbf{A}$ fixes the value of g, then

$$vg(x) = vg(c) = vc_0(c) = vc_0(x) < v(g(c) - c_0(c)) \quad \textit{for } c \nearrow x\ ,$$

and if $\mathbf{A}$ does not fix the value of g, then

$$vg(x) > vg(c) = \beta + m \cdot \mathbf{h} \cdot v(x-c) \quad \textit{for } c \nearrow x\ .$$

Proof. Since $\deg c_i(X) < \deg f(X) = \deg \mathbf{A}$, we have that $\mathbf{A}$ fixes the value of $c_i(X)$, for $0 \leq i \leq k$. We denote by γ_i the fixed value $vc_i(c)$ for $c \nearrow x$. Since f is an associated minimal polynomial for $\mathbf{A}$, we know that $\mathbf{A}$ does not fix the value of f. From Lemma 5.2 we infer that the value of $c_i(c)f(c)^i$ is equal to $\gamma_i + i\beta_{\mathbf{h}} + i\mathbf{h}v(x-c)$. We apply Lemma 5.1 with $\alpha_i = \gamma_i + i\beta_{\mathbf{h}}$, $t_i = i\mathbf{h}$ and $\Upsilon = \mathrm{supp}\mathbf{A}$ to deduce that there is an integer m such that $1 \leq m \leq k$ and $vc_m(c)f(c)^m < vc_i(c)f(c)^i$ for $c \nearrow x$ and $1 \leq i \neq m$. Consequently,

$$v(g(c) - c_0(c)) = vc_m(c)f(c)^m = \gamma_m + m \cdot \beta_{\mathbf{h}} + m \cdot \mathbf{h} \cdot v(x-c)\ . \tag{5.9}$$

We set $\beta := \gamma_m + m\beta_{\mathbf{h}}$.

The value of the right hand side of (5.9) is not fixed for $c \nearrow x$. Consequently, if $\mathbf{A}$ fixes the value of g, then from our representation (5.7) of g we see that the value $vc_m(c)f(c)^m$ must be greater than the fixed value of $c_0(c)$ for $c \nearrow x$, which yields that $vg(c) = vc_0(c)$. From Lemma 5.2, we know that $vc_0(x) = vc_0(c)$ and $vf(x) > vf(c)$ for $c \nearrow x$. Therefore,

$$vc_i(x)f(x)^i > vc_i(c)f(c)^i > vc_0(c) = vc_0(x) \tag{5.10}$$

for $1 \leq i \leq k$ and $c \nearrow x$, whence $vg(x) = vc_0(x) = vc_0(c) = vg(c)$.

If $\mathbf{A}$ does not fix the value of g, then $vc_m(c)f(c)^m < vc_0(c)$ and

$$vg(c) = vc_m(c)f(c)^m = \beta + m \cdot \mathbf{h} \cdot v(x-c)$$

for $c \nearrow x$. The inequality $vg(x) > vg(c)$ for $c \nearrow x$, is seen as follows. Using the first inequality of (5.10) together with $vc_m(c)f(c)^m < vc_0(c)$, we obtain:

$$\begin{array}{rcl} vg(x) & \geq & \min\{v(c_k(x)f(x)^k)\,, \ldots,\, v(c_1(x)f(x))\,,\, vc_0(x)\} \\ & > & \min\{v(c_k(c)f(c)^k)\,, \ldots,\, v(c_1(c)f(c))\,,\, vc_0(c)\} = vg(c)\ . \end{array}$$

This completes the proof of our lemma. $\square$

Corollary 5.5. *Take an immediate approximation type* $\mathrm{appr}(x,K)$ *over* (K,v). *If* x *is algebraic over* K *with minimal polynomial* $g\in K[X]$, *then* $\mathrm{appr}(x,K)$ *does not fix the value of* g *and is thus of degree* $\mathbf{d}\le[K(x):K]$.

Proof. Since $\mathrm{appr}(x,K)$ is immediate, it is nontrivial, so $x\notin K$ and $g(c)\neq 0$ for all $c\in K$. But by hypothesis, $g(x)=0$. Hence $vg(x)>vg(c)$ for all $c\in K$. Now the assertion follows by an application of Lemma 5.4. □

Unfortunately, $\mathbf{d}$ may be smaller than $[K(x):K]$, as the following example will show:

Example 5.6. We choose (K,v) to be $(\mathbb{F}_p(t),v_t)$ or $(\mathbb{F}_p((t)),v_t)$ or any henselian intermediate field (where $\mathbb{F}_p$ is the field with p elements). We take L to be the perfect hull $K(t^{1/p^i}\mid i\in\mathbb{N})$ of K.

If ϑ is a root of the polynomial

$$X^p-X-\frac{1}{t}$$

then the Artin–Schreier extension $L(\vartheta)|L$ is immediate with $v(\vartheta-L)=\{\alpha\in vL\mid \alpha<0\}$ (see [8, Example 3.12]). It follows from Proposition 6.5 below and the fact that (L,v) is henselian (being an algebraic extension of the henselian field (K,v)) that $\deg\mathrm{appr}(\vartheta,L)=p=[L(\vartheta):L]$. But an element $x=\vartheta+y$ in some extension of (L,v) has the same approximation type as ϑ over L if $vy\ge 0$ (cf. Lemma 3.1). We may take y of arbitrarily high degree over L. Indeed, we may even take y to be transcendental over L to obtain that $\vartheta+y$ is transcendental over L. This shows that a transcendental element may have an algebraic approximation type. Moreover, we may choose y such that $vy\notin vL$ or $yv\notin Lv$ to obtain an extension which is not immediate, although its generating element has an immediate approximation type.

6. Realization of immediate approximation types

In this section we will present the two basic theorems due to Kaplansky ([3]) which show that each immediate approximation type can be realized in a simple immediate extension. Kaplansky proved these theorems to derive a characterization of maximal fields, which we will also present here.

Theorem 6.1. (Theorem 2 of [3], approximation type version)
For every immediate transcendental approximation type $\mathbf{A}$ *over* (K,v) *there exists a simple immediate transcendental extension* $(K(x),v)$ *such that* $\mathrm{appr}(x,K)=\mathbf{A}$.

If $(K(y),v)$ *is another valued extension field of* (K,v) *such that* $\mathrm{appr}(y,K)=\mathbf{A}$, *then* y *is also transcendental over* K *and the isomorphism between* $K(x)$ *and* $K(y)$ *over* K *sending* x *to* y *is valuation preserving.*

Proof. We take $K(x)|K$ to be a transcendental extension and define the valuation on $K(x)$ as follows. In view of the rule $v(g/h)=vg-vh$, it suffices to define v on $K[x]$. Take $g\in K[X]$. By assumption, $\mathbf{A}$ fixes the value of g, that is, there is $\beta\in vK$ such that $vg(c)=\beta$ for $c\nearrow\mathbf{A}$. We set $vg(x)=\beta$. If g is a constant in K, we just obtain the value given by the valuation v on K. Our definition implies that $vg\neq\infty$ for every nonzero $g\in K[x]$.

Take $g,h\in K[X]$. Again by our definition, $vg(x)=vg(c)$, $vh(x)=vh(c)$, and $vg(x)h(x)=v(g\cdot h)(x)=v(g\cdot h)(c)=vg(c)h(c)$ for $c\nearrow\mathbf{A}$. Thus, $vg(x)h(x)=vg(c)h(c)=vg(c)+vh(c)=vg(x)+vh(x)$ and $v(g(x)+h(x))=v((g+h)(x))=$

$v((g+h)(c)) = v(g(c)+h(c)) \geq \min\{vg(c), vh(c)\} = \min\{vg(x), vh(x)\}$ for $c \nearrow \mathbf{A}$. So indeed, our definition yields a valuation v on $K(x)$ which extends the valuation v of K. Under this valuation, we have that $\mathbf{A} = \mathrm{appr}(x,K)$. This is seen as follows. In view of Lemma 4.3, it suffices to prove that for every $\alpha \in \mathrm{supp}\mathbf{A}$, we have that $v(x-c_\alpha) \geq \alpha$ for each $c_\alpha \in \mathbf{A}_\alpha$. But this follows directly from our definition of $v(x-c_\alpha)$ because for $c \nearrow \mathbf{A}$, $c \in \mathbf{A}_\alpha$ and thus $v(x-c_\alpha) = v(c-c_\alpha) \geq \alpha$.

From Lemma 5.3, we now infer that $(K(x)|K,v)$ is an immediate extension. Given another element y in some valued field extension of (K,v) such that $\mathbf{A} = \mathrm{appr}(y,K)$, we want to show that the epimorphism from $K[x]$ onto $K[y]$ induced by $x \mapsto y$ is valuation preserving. For this, we only have to show that $vg(x) = vg(y)$ for every $g \in K[X]$. By hypothesis, the degree of $\mathbf{A}$ is ∞. From Lemma 5.3 we can thus infer that $vg(x) = vg(c) = vg(y)$ holds for $c \nearrow \mathbf{A}$; this proves the desired equality. Again from Lemma 5.3, we deduce that y is transcendental over K. Hence, the assignment $x \mapsto y$ induces an isomorphism from $K(x)$ onto $K(y)$. Since the valuations of $K(x)$ and $K(y)$ are uniquely determined by its restriction to $K[x]$ and $K[y]$ respectively, it follows from what we have already proved that this isomorphism is valuation preserving. □

Corollary 6.2. *Take an extension $(L|K,v)$ of valued fields and $y \in L$. If $\mathrm{appr}(y,K)$ is an immediate transcendental approximation type, then y is transcendental over K and $(K(y)|K,v)$ is immediate.*

Proof. By the foregoing theorem, there is an immediate extension $(K(x)|K,v)$ such that $\mathrm{appr}(x,K) = \mathrm{appr}(y,K)$, with x transcendental over K. By the same theorem, there is a valuation preserving isomorphism of $K(x)$ and $K(y)$ over K. This proves our assertions. □

The next lemma will show that every immediate algebraic approximation type is of the form $\mathrm{appr}(y,K)$.

Lemma 6.3. *Take an immediate algebraic approximation type $\mathbf{A}$ over (K,v), a polynomial $f \in K[X]$ whose value is not fixed by $\mathbf{A}$, and a root y of f. Then there is an extension of v from K to $K(y)$ such that $\mathbf{A} = \mathrm{appr}(y,K)$.*

Proof. We choose some extension w of v from K to $K(y)$. We write $f(X) = d\prod_{i=1}^{\deg f}(X-a_i)$ with $d \in K$ and $a_i \in \tilde{K}$. If for all i, the values $w(c-a_i)$ would be fixed for $c \nearrow \mathbf{A}$, then $\mathbf{A}$ would fix the value of f, contrary to our assumption. Hence there is a root a of f such that $w(a-c)$ is not fixed for $c \nearrow \mathbf{A}$. Take some automorphism σ of $\tilde{K}|K$ such that $\sigma y = a$ and set $v := w \circ \sigma$. Then v extends the valuation of K, and $v(y-c) = w \circ \sigma(y-c) = w(\sigma y - c) = w(a-c)$ is not fixed for $c \nearrow \mathbf{A}$. By Corollary 4.4, $\mathbf{A} = \mathrm{appr}(y,K)$. □

The following is the analogue of Theorem 6.1 for immediate algebraic approximation types.

Theorem 6.4. (Theorem 3 of [3], approximation type version)
For every immediate algebraic approximation type $\mathbf{A}$ over (K,v) of degree $\mathbf{d}$ with associated minimal polynomial $f(X) \in K[X]$ and y a root of f, there exists an extension of v from K to $K(y)$ such that $(K(y)|K,v)$ is an immediate extension and $\mathrm{appr}(y,K) = \mathbf{A}$.

If $(K(z),v)$ is another valued extension field of (K,v) such that $\mathrm{appr}(z,K) = \mathbf{A}$, then any field isomorphism between $K(y)$ and $K(z)$ over K sending y to z will *preserve the valuation. (Note that there exists such an isomorphism if and only if z is also a root of f.)*

Proof. We take the valuation v of $K(y)$ given by Lemma 6.3. Then $\mathrm{appr}(y,K) = \mathbf{A}$. The fact that $(K(y)|K,v)$ is immediate follows from Lemma 5.3.

The last assertion of our theorem is shown in the same way as the corresponding assertion of Theorem 6.1: if $\mathrm{appr}(y,K) = \mathrm{appr}(z,K)$ and $g \in K[X]$ with $\deg g < \mathbf{d}$ then, again by Lemma 5.3, $vg(y) = vg(c) = vg(z)$ for $c \nearrow x$. Hence an isomorphism over K sending y to z will preserve the valuation. □

From this theorem, we can derive important information about the degree of immediate algebraic approximation types.

Proposition 6.5. *The degree of an immediate algebraic approximation type over a henselian field (K,v) is a power of the characteristic of the residue field Kv.*

Proof. Take an immediate algebraic approximation type $\mathbf{A}$ over a henselian field (K,v) of degree $\mathbf{d}$. Then by Theorem 6.4 there is an immediate extension $(L|K,v)$ of degree $\mathbf{d}$. As (K,v) is henselian, the extension of v from K to L is unique. Hence by the Lemma of Ostrowski (cf. [1], [14]),

$$\mathbf{d} \;=\; [L:K] \;=\; p^{\nu}\cdot(vL:vK)\cdot[Lv:Kv] \;=\; p^{\nu}\;,$$

where $\nu \in \mathbb{N}\cup\{0\}$ and $p = \mathrm{char} Kv$. Note that $\nu > 0$ because the degree of $\mathbf{A}$ is not less than 2. □

Theorem 6.1 and Theorem 6.4 together imply:

Proposition 6.6. *Every immediate approximation type is realized in some immediate simple valued field extension.*

We say that a valued field (K,v) is **maximal** if it admits no proper immediate extensions. In this case, by the two theorems, it admits no immediate approximation types. On the other hand, if (K,v) admits no immediate approximation types, then by part b) of Lemma 4.1, it admits no proper immediate extensions. This proves:

Theorem 6.7. (Theorem 4 of [3], approximation type version)
A valued field (K,v) is maximal if and only if it does not admit immediate approximation types.

Similarly, we say that a valued field (K,v) is **algebraically maximal** if it does not admit proper immediate algebraic extensions. In this case, Theorem 6.4 shows that it does not admit immediate algebraic approximation types. On the other hand, if (K,v) admits a proper immediate algebraic extension $(L|K,v)$, and $x \in L\backslash K$, then by part b) of Lemma 4.1, $\mathrm{appr}(x,K)$ is an immediate approximation type, and by Corollary 5.5, it is algebraic. This proves:

Theorem 6.8. *A valued field (K,v) is algebraically maximal if and only if it does not admit immediate algebraic approximation types.*

7. The relative approximation degree of polynomials

In view of Proposition 6.6, we can from now on assume that every immediate approximation type $\mathbf{A}$ is of the form $\mathbf{A} = \mathrm{appr}(x,K)$. For the integer $\mathbf{h}$ that appears in Lemma 5.2, where $\deg f \leq \deg \mathbf{A}$, we will write $\mathbf{h}_K(x:f)$ or just $\mathbf{h}(x:f)$. We call $\mathbf{h}(x : f)$ the **relative approximation degree of** $f(x)$ **in** x (**over** K). From Lemma 5.2 we know that

$$1 \leq \mathbf{h}_K(x : f) \leq \deg f \ .$$

One can extend the definition of the relative approximation degree to polynomials of arbitrary degree as follows. Take any polynomial $g \in K[X]$. Suppose that there exist $\beta \in vK$ and a positive integer k such that

$$v(g(x) - g(c)) = \beta + k \cdot v(x - c)$$

for $c \nearrow x$. Note that β and k are uniquely determined because as $\mathrm{appr}(x, K)$ is immediate, there are infinitely many values $v(x - c)$ for $c \nearrow x$. We will call k the **relative approximation degree of** $g(x)$ **in** x, denoted by $\mathbf{h}_K(x : g)$ as before. Further, we will call β the **relative approximation constant of** $g(x)$ **in** x, denoted by

$$\beta_K(x : g) \ .$$

By virtue of equation (5.5) of Lemma 5.2, our new definition of the relative approximation degree coincides with the definition as given for polynomials of degree at most $\mathbf{d}$. On the other hand, our new definition assigns a relative approximation degree to every polynomial of arbitrary degree whose value is not fixed, as Lemma 5.4 shows because in this case, $v(g(x) - g(c)) = vg(c)$ for $c \nearrow x$. However, for polynomials of degree bigger than $\mathbf{d}$, the relative approximation degree may not be a power of p. Unfortunately, Lemma 5.4 does not give information about the value $v(g(x) - g(c))$ if $\mathbf{A}$ fixes the value of g; this is an open problem.

From Lemma 5.4 we derive:

Corollary 7.1. *The value of g is fixed by $\mathbf{A}$ if and only if $vg(x) = vg(c)$ for $c \nearrow x$. On the other hand, $\mathbf{A}$ does not fix the value of g if and only if $vg(x) > vg(c)$ for $c \nearrow x$, and this holds if and only if*

$$v(g(x) - g(c)) = vg(c) = \beta_{\mathbf{h}}(x : g) + \mathbf{h}_K(x : g) \cdot v(x - c) \tag{7.1}$$

for $c \nearrow x$.

For the distances associated with $g(x)$, the following inequalities will hold in all cases where $\beta_K(x : g)$ and $\mathbf{h}_K(x : g)$ are defined:

$$\mathrm{dist}(g(x), K) \geq \mathrm{dist}_K(g(x), g(K)) \geq \beta_K(x : g) + \mathbf{h}_K(x : g) \cdot \mathrm{dist}(x, K) \tag{7.2}$$

(the first inequality is trivial and the second follows directly from the definition of relative approximation degree and relative approximation constant). In the next section, we will consider various cases where equalities hold.

We will now investigate the relative approximation degree more closely for the case of $\deg f \leq \deg \mathbf{A}$. We will first consider the relation between $\mathbf{h}_K(x : f)$ and the approximation type $\mathrm{appr}(f(x), K)$. Then we show that $\mathbf{h}_K(x : f)$ is a power of the characteristic exponent of the residue field, where the **characteristic exponent** of a field is defined to be its characteristic if this is positive, and 1 otherwise. Finally we will give some hints for the computation of $\mathbf{h}_K(x : f)$.

Throughout this and the next two sections, we will assume the following situation:

$$(7.3)\qquad \begin{cases} \mathbf{A} = \mathrm{appr}(x,K) & \text{an immediate approximation type over } (K,v) \\ p & \text{the characteristic exponent of } Kv, \\ \mathbf{d} & \text{the degree of } \mathrm{appr}(x,K), \\ f \in K[X] & \text{a nonconstant polynomial of degree } n \leq \mathbf{d}\,, \\ \mathbf{h} & = \mathbf{h}_K(x:f) \\ \beta_i & \text{the fixed value } vf_i(c) \text{ for } c \nearrow x. \end{cases}$$

Lemma 7.2. *Take another polynomial $g \in K[X]$ of degree at most $\mathbf{d}$ such that $\mathrm{appr}(x,K)$ fixes the value of $f-g$. If $\mathrm{appr}(f(x),K) = \mathrm{appr}(g(x),K)$, then $\mathbf{h}_K(x:f) = \mathbf{h}_K(x:g)$ and $\beta_K(x:f) = \beta_K(x:g)$.*

Proof. By part b) of Lemma 3.1, $\mathrm{appr}(f(x),K) = \mathrm{appr}(g(x),K)$ implies that

$$v(f(x)-g(x)) \geq \mathrm{dist}(f(x),K)\;.$$

By hypothesis, $\mathrm{appr}(x,K)$ fixes the value of $f-g$, hence by Lemma 5.2,

$$v(f(c)-g(c)) = v(f(x)-g(x)) \geq \mathrm{dist}(f(x),K) \geq v(f(x)-f(c)) \quad \text{for} \quad c \nearrow x\;.$$

As (5.5) shows that the values $v(f(x)-f(c))$ are increasing for $c \nearrow x$, the last inequality can be replaced by a strict inequality. So we obtain that

$$\begin{aligned} v(g(x)-g(c)) &= \min\{v(g(x)-f(x))\,,\, v(f(x)-f(c))\,,\, v(f(c)-g(c))\} \\ &= v(f(x)-f(c)) \;=\; \beta_K(x:f) + \mathbf{h}_K(x:f)\cdot v(x-c) \end{aligned}$$

for $c \nearrow x$. This implies our assertion. □

To achieve our second goal, we need the following lemma:

Lemma 7.3. *If p is prime and r is a positive integer prime to p, $r>1$, then*

$$\binom{p^t r}{p^t}$$

is prime to p, for every integer $t \geq 0$.

Proof. Consider

$$\binom{p^t r}{p^t} = \frac{p^t r(p^t r-1)\cdot\ldots\cdot(p^t r - p^t + 1)}{p^t(p^t-1)\cdot\ldots\cdot 1}\;.$$

In the numerator of this fraction, the first factor $p^t r$ is divisible by precisely p^t, while the remaining factors $p^t r - m$, $1 \leq m \leq p^t - 1$, are not divisible by p^t. Hence, for every such factor occurring in the numerator, the corresponding factor $p^t - m = p^t r - m - p^t(r-1)$ which occurs in the denominator will be divisible by p to precisely the same power. This gives the desired result. □

Now we are able to prove:

Proposition 7.4. *If $i = p^t$, $j = p^t r \leq n$ with $r > 1$, $(r,p) = 1$, and if $\beta_i \neq \infty$, then for $c \nearrow x$,*

$$\beta_i + i\cdot v(x-c) < \beta_j + j\cdot v(x-c)\;.$$

Consequently, $\mathbf{h}_K(x:f)$ is a power of p (including the case of $\mathbf{h}_K(x:f) = 1 = p^0$).

Proof. We consider the Taylor expansion (5.3) for $f_i(x)$:

$$f_i(x) - f_i(c) = \\ (i+1)f_{i+1}(c)(x-c) + \ldots + \tbinom{j}{i} f_j(c)(x-c)^{j-i} + \ldots + \tbinom{n}{i} f_n(c)(x-c)^{n-i} \,.$$

For $c \nearrow x$, the values $vf_{i+1}(c), \ldots, vf_n(c)$ will be equal to $\beta_{i+1}, \ldots, \beta_n$ as defined in (7.3). We apply Lemma 5.1 with $m = n-i$, $t_k = k$ for $1 \le k \le m$, and

$$\alpha_1 = v(i+1) + \beta_{i+1}\,, \ldots,\; \alpha_{j-i} = v\binom{j}{i} + \beta_j\,, \ldots,\; \alpha_m = v\binom{n}{i} + \beta_n \,.$$

We find that among the terms on the right hand side of the Taylor expansion, there will be precisely one which has least value for $c \nearrow x$. The value of this term must then equal the value of the left hand side of the Taylor expansion, which yields that the latter increases for $c \nearrow x$. But both values $vf_i(x)$ and $vf_i(c)$ are fixed for $c \nearrow x$. Hence, $v(f_i(x) - f_i(c)) > vf_i(x) = vf_i(c) = \beta_i$ for $c \nearrow x$. It follows that in particular, the term

$$\binom{j}{i} f_j(c)(x-c)^{j-i}$$

on the right hand side of the Taylor expansion will also have value $> \beta_i$ for $c \nearrow x$. But $v\tbinom{j}{i} = 0$: if $p > 0$, this is shown in Lemma 7.3, and if $p = 1$, then $\mathrm{char}Kv = 0$ which means that $\mathrm{char}K = 0$ and v is trivial on the subfield $\mathbb{Q}$ of K. Therefore,

$$\beta_i < \beta_j + (j-i) \cdot v(x-c)$$

for $c \nearrow x$. This yields our assertion. □

The following lemma will give more detailed information on the computation of $\mathbf{h}_K(x:f)$.

Lemma 7.5. *Assume that $v(x-c) \ge 0$ for $c \nearrow x$. If i is an integer such that β_i is minimal among all β_j, $j > 0$, then $\mathbf{h}_K(x:f) \le i$.*

Proof. By assumption, we have that $\beta_j - \beta_i \ge 0$ for all $j > 0$. Further,

$$\beta_{\mathbf{h}} + \mathbf{h} \cdot v(x-c) < \beta_j + j \cdot v(x-c)$$

for $j > 0$, $j \ne \mathbf{h}$, and $c \nearrow x$. Thus,

$$0 \le \beta_{\mathbf{h}} - \beta_i \le (i - \mathbf{h}) \cdot v(x-c)$$

for $c \nearrow x$, which in view of $v(x-c) \ge 0$ for $c \nearrow x$ yields that $i - \mathbf{h} \ge 0$, which is the assertion. □

Lemma 7.6. *Assume that $p \ge 2$, and write $f(X) = c_n X^n + \ldots + c_0$. Suppose that there exists $i > 0$ such that $vc_i < vc_k$ for all $k > 0$, $k \ne i$, and write $i = p^t r$ with r prime to p. Then $vf_j(c) \ge vc_i$ holds for all $j > 0$ and every c with $vc = 0$. And if $vx = 0$, then*

$$\mathbf{h}_K(x:f) \le p^t \,.$$

Proof. For $vc = 0$ and $j > 0$, by the definition (5.1) of the j-th formal derivative,

$$vf_j(c) \;=\; v\sum_{k=j}^{n} \binom{k}{j} c_k c^{k-j} \;\ge\; \min_{j \le k \le n} v\binom{k}{j} c_k c^{k-j} \;\ge\; vc_i \,.$$

By Lemma 7.3, the binomial coefficient $\binom{p^t r}{p^t}$ is not divisible by p. This shows that $v\binom{p^t r}{p^t} = 0$ and thus,

$$v f_{p^t}(c) = v c_i \ .$$

Now assume in addition that $vx = 0$. Then $vc = 0$ for $c \nearrow x$. This yields that

$$\beta_{p^t} = v c_i \leq \beta_j$$

for all $j > 0$. The foregoing lemma now gives our assertion. □

Corollary 7.7. *Assume that $vx = 0$, and take an integer $e \geq 1$. Suppose that all nonzero coefficients c_i of f, $i > 0$, have different values and that for all i with $p^e|i$, the coefficient c_i is equal to zero. Then $\mathbf{h}_K(x : f) < p^e$.*

8. Approximation types and distances of polynomials

Recall that throughout this section, we assume the situation of (7.3).

Lemma 8.1. *The following holds:*

$$c \in \mathrm{appr}(x,K)_\gamma \Longleftrightarrow f(c) \in \mathrm{appr}(f(x),K)_{\beta_{\mathbf{h}}+\mathbf{h}\cdot\gamma} \text{ for } c \nearrow x \ . \tag{8.1}$$

In particular,

$$\mathrm{dist}(f(x),K) \geq \mathrm{dist}_K(f(x),f(K)) = \beta_{\mathbf{h}} + \mathbf{h}\cdot \mathrm{dist}(x,K) \ . \tag{8.2}$$

Proof. Equation (5.5) of Lemma 5.2 yields (8.1), while the inequality $\mathrm{dist}(f(x),K) \geq \mathrm{dist}_K(f(x),f(K))$ was already stated in (7.2). It remains to prove that

$$\mathrm{dist}_K(f(x),f(K)) = \beta_{\mathbf{h}} + \mathbf{h}\cdot \mathrm{dist}(x,K) \ .$$

If $\mathrm{dist}(x,K) = \infty$, this equality follows immediately from (7.2). So let us assume from now on that $\mathrm{dist}(x,K) < \infty$. In order to deduce a contradiction, assume that there exists an element $c_0 \in K$ such that

$$v(f(x) - f(c_0)) > \beta_{\mathbf{h}} + \mathbf{h}\cdot \mathrm{dist}(x,K) \ ,$$

or equivalently,

$$v(f(x) - f(c_0)) > v(f(x) - f(c))$$

for $c \nearrow x$. Hence

$$\begin{aligned} v(f(c_0) - f(c)) &= \min\{v(f(x) - f(c)), v(f(x) - f(c_0))\} \\ &= v(f(x) - f(c)) \ = \ \beta_{\mathbf{h}} + \mathbf{h}\cdot v(x-c) \end{aligned}$$

for $c \nearrow x$. Replacing x by c_0 in the Taylor expansion (5.6), we find

$$\begin{aligned} v(f_1(c_0)\cdot(c-c_0) + \ldots + f_n(c_0)\cdot(c-c_0)^n) &= v(f(c_0) - f(c)) \\ &= \beta_{\mathbf{h}} + \mathbf{h}\cdot v(x-c) \end{aligned}$$

for $c \nearrow x$. As noted already at the beginning of Section 4, an immediate approximation type fixes the value of every linear polynomial. Hence, $v(c - c_0)$ will be fixed for $c \nearrow x$. On the other hand, the value $\beta_{\mathbf{h}} + \mathbf{h}\cdot v(x-c)$ is not fixed for $c \nearrow x$, so we conclude that the value

$$v(\, f_1(c_0) + f_2(c_0)\cdot(c-c_0) + \ldots + f_n(c_0)\cdot(c-c_0)^{n-1}\,)$$

is not fixed for $c \nearrow x$. This proves the existence of a polynomial of degree $n-1$ whose value is not fixed by $\mathrm{appr}(x,K)$. But $n-1 = \deg f - 1 < \mathbf{d}$, a contradiction. This proves the desired equality. □

Lemma 8.2. *Assume that* $\deg f < \mathbf{d}$*. Then* $\mathrm{appr}(f(x),K)$ *is an immediate approximation type over* K *with*

$$\mathrm{dist}(f(x),K) = \mathrm{dist}_K(f(x),f(K)) = \beta_{\mathbf{h}} + \mathbf{h}\cdot \mathrm{dist}(x,K), \tag{8.3}$$

and $\mathrm{appr}(f(x),K)$ *is determined by (8.1).*

Proof. In view of (8.2), to prove the first equality in (8.3) we have to show that for every $b\in K$ there exists an element $c\in K$ such that $v(f(x)-f(c))\geq v(f(x)-b)$. Since $\deg(f-b)=\deg f<\mathbf{d}$, it follows that $\mathrm{appr}(x,K)$ fixes the value of $f-b$. Applying Lemma 5.2 to $f-b$ in place of f, we deduce that $v(f(x)-b)=v(f(c)-b)$ for $c\nearrow x$. Consequently, for such an element $c\in K$ we get that

$$v(f(x)-f(c))\geq \min\{v(f(x)-b),v(f(c)-b)\}=v(f(x)-b)\;,$$

as desired.

By the second equality of (8.3), which has already been proved in Lemma 8.1, we know that there exists $c'\in K$ such that $v(f(x)-f(c'))>v(f(x)-f(c))\geq v(f(x)-b)$. We have proved that for every $b\in K$ there is $b'=f(c')\in K$ such that $v(f(x)-b')>v(f(x)-b)$. Part a) of Lemma 4.1 now shows that $\mathrm{appr}(f(x),K)$ is immediate.

By (8.3), the values $\beta_{\mathbf{h}}+\mathbf{h}\cdot v(x-c)$ are cofinal in $\mathrm{supp}\,\mathrm{appr}(f(x),K)$ for $c\nearrow x$. Therefore, $\mathrm{appr}(f(x),K)$ is determined by the balls $\mathrm{appr}(f(x),K)_{\beta_{\mathbf{h}}+\mathbf{h}\cdot v(x-c)}$ for those c, which in turn are determined by (8.1). □

Corollary 8.3. *Assume that* $\deg f<\mathbf{d}$*, and let* $d'\geq 1$ *be a natural number such that* $d'\cdot\deg f\leq \mathbf{d}$*. Then*

$$\deg \mathrm{appr}(f(x),K)\geq d'\;.$$

In particular, if $\mathrm{appr}(x,K)$ *is transcendental, then so is* $\mathrm{appr}(f(x),K)$*.*

Proof. Take a polynomial g of degree smaller than $d'\leq\mathbf{d}$. Suppose that $\mathrm{appr}(f(x),K)$ does not fix the value of g. Then by Lemma 5.2,

$$vg(f(x))\;>\;vg(a)$$

for $a\nearrow f(x)$. Since $\deg f<\mathbf{d}$, Lemma 8.2 shows that $\mathrm{dist}_K(f(x),f(K))=\mathrm{dist}(f(x),K)$, so

$$vg(f(x))\;>\;vg(f(c))$$

for $c\nearrow x$. But then by Lemma 5.2, $\mathrm{appr}(x,K)$ does not fix the value of the polynomial $g(f(X))$. This contradicts the fact that its degree is smaller than $\mathbf{d}$. □

Lemma 8.4. *Assume that* $\mathrm{appr}(x,K)$ *does not fix the value of* f *(hence* $\deg f=\mathbf{d}$*). Then*

$$vf(x)>\beta_{\mathbf{h}}+\mathbf{h}\cdot v(x-c)\quad \textit{for } c\nearrow x.$$

Proof. We rewrite (5.6) as follows:

$$-f(c)=f_1(c)\cdot(x-c)+\ldots+f_n(c)\cdot(x-c)^n-f(x)\;.$$

Suppose that $vf(x)<\beta_{\mathbf{h}}+\mathbf{h}\cdot v(x-c)$ for $c\nearrow x$. This in turn implies that the value of the right hand side is equal to $vf(x)$ and hence the value $vf(c)$ is fixed for for $c\nearrow x$, which contradicts our assumption. This proves that $vf(x)\geq\beta_{\mathbf{h}}+\mathbf{h}\cdot v(x-c)$, and since $v(x-K)$ has no maximal element, also $vf(x)>\beta_{\mathbf{h}}+\mathbf{h}\cdot v(x-c)$ for $c\nearrow x$. □

Note that in the case of $\deg f = \mathbf{d}$ we can only say that "$\mathrm{appr}(f(x), K)$ is determined by (8.1) up to $\mathrm{dist}_K(f(x), f(K))$". But it may happen that

$$\mathrm{dist}(f(x), K) > \mathrm{dist}_K(f(x), f(K)) \ .$$

This will usually be the case when f is the minimal polynomial of x, which yields that $f(x) = 0$ and hence $\mathrm{dist}(f(x), K) = \mathrm{dist}(0, K) = \infty$.

Example 8.5. Take (L, v) and $f(X) = X^p - X - t^{-1}$ with root ϑ as in Example 5.6. As noted there, $v(\vartheta - L) = \{\alpha \in vL \mid \alpha < 0\}$, so $\mathrm{dist}(\vartheta, L)$ is the cut in $\widetilde{vL}$ whose lower cut set consists of all negative elements. This implies that $\mathrm{dist}(\vartheta, L) = p \cdot \mathrm{dist}(\vartheta, L)$.

We have that $f(X) - f(c) = X^p - X - (c^p - c) = (X - c)^p - (X - c)$. Since $v(\vartheta - c) < 0$, it follows that $v(\vartheta - c)^p = p \cdot v(\vartheta - c) < v(\vartheta - c)$ and therefore, $v(f(\vartheta) - f(c)) = v((\vartheta - c)^p - (\vartheta - c)) = \min\{v(\vartheta - c)^p, v(\vartheta - c)\} = p \cdot v(\vartheta - c)$. This shows that $\mathbf{h}_L(x : f) = p$ and $\beta_L(x : f) = 0$. We obtain that

$$\mathrm{dist}(f(\vartheta), L) \;=\; \infty \;>\; \mathrm{dist}(\vartheta, L) \;=\; p \cdot \mathrm{dist}(\vartheta, L) \;=\; \mathrm{dist}_L(f(\vartheta), f(L)) \ ,$$

where the last equality holds by Lemma 8.1.

9. The degree $[K(x)^h : K(f(x))^h]$

In the situation of (7.3), we ask for the degree

$$[K(x)^h : K(f(x))^h] \ .$$

This can indeed be calculated by means of $\mathbf{h}_K(x : f)$. Inequality (9.1) below will explain the origin of the notation "$\mathbf{h}_K(x : f)$". Note that $[K(x) : K(f(x))] = \deg f$, while in general, we may have that $[K(x)^h : K(f(x))^h] < \deg f$.

Theorem 9.1. *Assume (7.3). Then*

$$[K(x)^h : K(f(x))^h] \leq \mathbf{h}_K(x : f) \ . \tag{9.1}$$

Proof. We consider the Taylor expansion (5.2) of f for an arbitrary $c \in K$. From Lemma 5.2, we know that (5.4) holds for $1 \leq i \leq \deg f$, $i \neq \mathbf{h} = \mathbf{h}_K(x : f)$ and $c \nearrow x$. We choose such an element $c \in K$ and also an element $d \in K$ with $vd = -v(x - c)$. We set $x_0 = d \cdot (x - c)$; hence $vx_0 = 0$ and $K(x) = K(x_0)$. Now (5.4) takes the form

$$v(f_i(c)d^{-i}) > v(f_{\mathbf{h}}(c)d^{-\mathbf{h}}) \ \text{ for } i \neq \mathbf{h},\ 1 \leq i \leq \deg f, \tag{9.2}$$

and (5.5) reads as

$$v(f(x) - f(c)) = vf_{\mathbf{h}}(c)d^{-\mathbf{h}} \ . \tag{9.3}$$

Further, from (5.2), (9.2) and (9.3) we obtain:

$$\begin{aligned} \left(\frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot (f(c) - f(x))\right) v \;&=\; \left(-\frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot \sum_{i=1}^{\deg f} f_i(c)(x - c)^i\right) v \\ &=\; \left(-\sum_{i=1}^{\deg f} \frac{f_i(c)d^{-i}}{f_{\mathbf{h}}(c)d^{-\mathbf{h}}}\, x_0^i\right) v \;=\; -(x_0 v)^{\mathbf{h}} \ . \end{aligned} \tag{9.4}$$

Now we set

$$\tilde{f}(Z) = \sum_{i=0}^{\deg f} f_i(c)d^{-i}Z^i \ ;$$

hence $\tilde{f}(x_0) = f(x)$. Let us consider the polynomial

$$F(Z) = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot (\tilde{f}(Z) - \tilde{f}(x_0))$$

whose coefficients lie in $K(\tilde{f}(x_0)) = K(f(x))$ and for which x_0 is a zero. Using (9.2) and (9.3), we compute

$$F(Z) = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)} \cdot (f(c) - f(x)) + \sum_{i=1}^{\deg f} \frac{f_i(c)d^{-i}}{f_{\mathbf{h}}(c)d^{-\mathbf{h}}} Z^i \in \mathcal{O}_{K(f(x))}[Z]$$

and, using also (9.4),

$$F(Z)v = Z^{\mathbf{h}} - (x_0v)^{\mathbf{h}} = (Z - x_0v)^{\mathbf{h}}$$

(where the latter equation holds because by Proposition 7.4, $\mathbf{h}$ is a power of p). Using the strong Hensel's Lemma, that is, property 3) of Theorem 4.1.3 in [2], we deduce that there is a factorization

$$F(Z) = G(Z)H(Z)$$

over $K(f(x))^h$ with

$$G(Z)v = Z^{\mathbf{h}} - (x_0v)^{\mathbf{h}}$$

and

$$\deg G(Z) = \deg G(Z)v = \mathbf{h}\ .$$

A zero of $F(Z)$ which has residue x_0v cannot be a zero of $H(Z)$ since $H(Z)v = 1$, hence it must appear as a zero of $G(Z)$. In particular, $G(x_0) = 0$. Since $G(Z) \in K(f(x))^h[Z]$ and $\deg G(Z) = \mathbf{h}$, and since $K(x_0)^h = K(f(x))^h(x_0)$, this shows that

$$[K(x)^h : K(f(x))^h] = [K(x_0)^h : K(f(x))^h] \le \mathbf{h} = \mathbf{h}_K(x : f)\ .$$

□

Corollary 9.2. *In addition to (7.3), assume that (K, v) is henselian and x is algebraic over K. If $\mathbf{d} = [K(x) : K]$ and f is the minimal polynomial of x over K, then $p \ge 2$ and*

$$[K(x) : K] = \mathbf{h}_K(x : f) = p^t$$

for some integer $t \ge 1$.

Proof. By hypothesis, we have $\mathbf{d} = [K(x) : K] = \deg f$. Since K is henselian and x is algebraic over K, we have that $K(x)$ is henselian as well. In view of $f(x) = 0$, an application of the foregoing theorem shows that

$$\deg f = [K(x) : K] \le \mathbf{h}_K(x : f) \le \deg f\ .$$

Consequently, equality holds everywhere.

Since $\mathrm{appr}(x, K)$ is immediate by assumption, it is nontrivial, hence $x \notin K$ and $\mathbf{h}_K(x : f) = [K(x) : K] > 1$. Proposition 7.4 yields that $p \ge 2$ and $\mathbf{h}_K(x : f) = p^t$ with $t \ge 1$. □

10. The degree $[K(x)^h : K(y)^h]$

Throughout this section, we will work with the following situation. By K^c we will denote the completion of (K, v).

$$\text{(10.1)} \qquad \begin{cases} (K,v) \text{ a valued field of rank 1} \\ (K(x)|K,v) \text{ an immediate extension such that } x \notin K^c \\ \text{and } \mathrm{appr}(x,K) \text{ is transcendental} \\ y \in K(x)^h \text{ transcendental over } K. \end{cases}$$

Note that by Corollary 6.1, the assumption that $\mathrm{appr}(x, K)$ is transcendental implies that x is transcendental over K. Furthermore, if (K, v) is algebraically maximal, then $\mathrm{appr}(x, K)$ is always transcendental, provided that $(K(x)|K, v)$ is immediate and nontrivial.

We ask for the degree

$$[K(x)^h : K(y)^h] \, .$$

To treat this question and in particular to define the relative approximation degree of x over y, we look for a polynomial $f \in K[X]$ such that

$$v(y - f(x)) \geq \mathrm{dist}(y, K) \, . \tag{10.2}$$

We need some preparation.

Lemma 10.1. *If K is of rank 1 and $K(x)|K$ is immediate, then $K[x]$ is dense in $K(x)^h$.*

Proof. Since any valued field of rank 1 is dense in its henselization, it suffices to show that $K[x]$ is dense in $K(x)$. For this we only have to show that for every $f(x) \in K[x]$ and every $\alpha \in vK$ there exists an element $g(x) \in K[x]$ such that $v(g(x) - 1/f(x)) > \alpha$. Since $K(x)|K$ is immediate there is an element $c \in K$ satisfying $v(c - f(x)) > vf(x) = vc$, which yields that $v(1 - f(x)/c) > 0$. By our hypothesis on the rank which means that the value group vK is archimedian, there exists $j \in \mathbb{N}$ such that $j \cdot v(1 - f(x)/c) > \alpha + vc$. Now we put $h(x) = 1 - f(x)/c \in K[x]$ and compute

$$\begin{aligned} v\left(\frac{1}{f(x)} - c^{-1}\sum_{i=0}^{j-1} h(x)^i\right) &= v\left(\frac{1}{c(1-h(x))} - c^{-1}\sum_{i=0}^{j-1} h(x)^i\right) \\ &= vc^{-1}h(x)^j = j \cdot v(1 - f(x)/c) - vc > \alpha \, . \end{aligned}$$

As the sum is an element of $K[x]$, this proves our lemma. □

By $K[x]^c$ we denote the completion of $K[x]$ within $K(x)^c$.

Lemma 10.2. *Assume (10.1). Then $y \in K[x]^c \setminus K^c$ and there exists a polynomial $f \in K[X]$ such that (10.2) holds.*

Proof. From Lemma 10.1, we infer that $y \in K[x]^c$. Suppose that $y \in K^c$. Then K is dense in $K(y)$ and also in $K(y)^h$ since $K(y)$ is dense in its henselization, being of rank 1 like K. Let $g(X) \in K(y)^h[X]$ be the minimal polynomial of x over $K(y)^h$. We can choose polynomials $\tilde{g}(X) \in K[X]$ with coefficients arbitrarily close to the corresponding coefficients of g. By the continuity of roots (cf. Theorem 4.5 of [PZ]) and our assumption that $x \notin K^c$, i.e., $\mathrm{dist}(x, K) < \infty$, we can find a suitable polynomial $\tilde{g}$ with a suitable root $\tilde{x} \in \tilde{K}$ such that

$$v(x - \tilde{x}) \geq \mathrm{dist}(x, K) \, .$$

By Lemma 3.1 b), this implies that

$$\mathrm{appr}(x,K) = \mathrm{appr}(\tilde{x},K)\ .$$

Since $\tilde{x}$ is algebraic over K, it follows by Corollary 5.5 that $\mathrm{appr}(\tilde{x},K)$ and hence $\mathrm{appr}(x,K)$ is an algebraic approximation type over K, a contradiction to hypothesis (10.1). This shows that $y \notin K^c$, i.e., $\mathrm{dist}(y,K) < \infty$. As $y \in K[x]^c$, this shows the existence of a polynomial $f \in K[X]$ such that $v(y - f(x)) \geq \mathrm{dist}(y,K)$. □

With f as in this lemma, we define

$$\mathbf{h}_K(x:y) \ := \ \mathbf{h}_K(x:f) \ \text{ and } \ \beta_K(x:y) \ := \ \beta_K(x:f)$$

and call $\mathbf{h}_K(x:y)$ the **relative approximation degree of** y **in** x (over K).

Lemma 10.3. *The integers $\mathbf{h}_K(x:y)$ and $\beta_K(x:y)$ are well-defined, i.e., they do not depend on the choice of $f(x)$ as long as $v(y-f(x)) \geq \mathrm{dist}(y,K)$ is satisfied.*

Proof. If $g(x)$ is another polynomial in $K[x]$ such that $v(y - g(x)) \geq \mathrm{dist}(y,K)$, then by Lemma 3.1, we have that

$$\mathrm{appr}(g(x),K) = \mathrm{appr}(y,K) = \mathrm{appr}(f(x),K)\ ,$$

whence $\mathbf{h}_K(x:g) = \mathbf{h}_K(x:f)$ and $\beta_K(x:g) = \beta_K(x:f)$ by Lemma 7.2 since $\mathrm{appr}(x,K)$ is transcendental. □

In the situation described in (10.1), we can prove Theorem 9.1 also for y in place of $f(x)$ provided that the extension $K(x)^h|K(y)^h$ is separable. For the proof, we need the following lemma:

Lemma 10.4. *Assume (10.1) and let $v(y - f(x)) \geq \mathrm{dist}(y,K)$. Then there exists an element z in the algebraic closure $\widetilde{K(y)}$ of $K(y)$ such that*

$$[K(y,z)^h : K(y)^h] \leq \mathbf{h} = \mathbf{h}_K(x:y)$$

and

$$v(x-z) \geq \frac{1}{\mathbf{h}}\left(v(y-f(x)) - \beta_K(x:f)\right)\ .$$

Proof. Recall that $\mathbf{h} = \mathbf{h}_K(x:y) = \mathbf{h}_K(x:f)$. We put $r := y - f(x)$. We choose $c,d \in K$, x_0 and $F(Z)$ as in the proof of Theorem 9.1. Then

$$\begin{aligned} vr \ \geq \ \mathrm{dist}(y,K) \ &> \ v(y-f(c)) \ = \ v(f(x)-f(c)) \\ &= \ v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) \ = \ v(f_{\mathbf{h}}(c)d^{-\mathbf{h}})\ . \end{aligned}$$

This shows that

$$F^\circ(Z) := F(Z) - \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)}\cdot r = \frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)}\cdot(\tilde{f}(Z)-y) \ \in \ \mathcal{O}_{K(y)}[Z]$$

has the same reduction as $F(Z)$. We find, as for $F(Z)$, that $F^\circ(Z)$ admits a factorization

$$F^\circ(Z) = G^\circ(Z)H^\circ(Z)$$

over $K(y)^h$ with $G^\circ(Z)v = Z^{\mathbf{h}} - (x_0v)^{\mathbf{h}}$, G° monic, $\deg G^\circ(Z) = \deg G^\circ(Z)v = \mathbf{h}$ and $H^\circ(Z)v = 1$. Note that $vF^\circ(x_0) = vG^\circ(x_0)$ since $x_0 \in \mathcal{O}_{K(x)}$. Recall that $F(x_0) = 0$. Consequently, from

$$F^\circ(x_0) = -\frac{d^{\mathbf{h}}}{f_{\mathbf{h}}(c)}\cdot r$$

it follows that, with $\beta_{\mathbf{h}} = vf_{\mathbf{h}}(c) = \beta_K(x:f)$,

$$v(d^{\mathbf{h}}r) - \beta_{\mathbf{h}} = vF^\circ(x_0) = vG^\circ(x_0)\ .$$

Hence there must exist a root z_{j_0} of

$$G^\circ(Z) = \prod_{1\le j\le \mathbf{h}} (Z - z_j)\ ,\quad z_j \in \widetilde{K(y)}$$

with

$$v(x_0 - z_{j_0}) \ge \frac{1}{\mathbf{h}}\left(v(d^{\mathbf{h}}r) - \beta_{\mathbf{h}}\right)\ ,$$

which is equivalent to

$$v(x - (d^{-1}z_{j_0} + c)) \ge \frac{1}{\mathbf{h}}(vr - \beta_{\mathbf{h}}) = \frac{1}{\mathbf{h}}(v(y - f(x)) - \beta_K(x:f))\ .$$

Now $z := d^{-1}z_{j_0} + c$ is the element of our assertion, since it satisfies $K(y,z) = K(y, z_{j_0})$ and thus $[K(y,z)^h : K(y)^h] \le \mathbf{h}$. □

Proposition 10.5. *Assume (10.1). If $K(x)^h|K(y)^h$ is separable, then*

$$[K(x)^h : K(y)^h]\ \le\ \mathbf{h}_K(x:y)\ .$$

Proof. Set

$$\alpha := \max\{v(\sigma x - x) \mid \sigma \in \mathrm{Gal}(\widetilde{K(y)}|K(y)^h) \text{ with } \sigma x \ne x\}\ .$$

Then $\alpha < \infty$ since $K(x)^h|K(y)^h$ is separable. Now, by Lemma 10.2 we can choose $f(x) \in K[x]$ such that $v(y - f(x)) \ge \mathrm{dist}(y,K) = \mathrm{dist}(f(x),K)$ as well as

$$v(y - f(x))\ >\ \beta_K(x:y) + \mathbf{h}\alpha\ =\ \beta_K(x:f) + \mathbf{h}\alpha\ ,$$

where $\mathbf{h} = \mathbf{h}_K(x:y)$. Using the foregoing lemma, we choose $z \in \widetilde{K(y)}$ such that

$$v(x - z)\ \ge\ \frac{1}{\mathbf{h}}(v(y - f(x)) - \beta_K(x:f))\ >\ \alpha\ ,$$

and $[K(y,z)^h : K(y)^h] \le \mathbf{h}$. In view of our separability condition, we can deduce by Krasner's Lemma (see [2], Theorem 4.1.7) that $x \in K(y)^h(z)$. This yields that $[K(x,y)^h : K(y)^h] \le [K(y,z)^h : K(y)^h] \le \mathbf{h}$. Since $y \in K(x)^h$ by assumption, $K(x,y)^h = K(x)^h$ and thus $[K(x)^h : K(y)^h] \le \mathbf{h}$, as asserted. □

In order to prove the assertion of the proposition without the separability condition, we need the following tool.

Lemma 10.6. *Assume that (10.1) holds. Then it also holds for y in place of x. So if $z \in K(y)^h$ is transcendental over K, then $\mathbf{h}_K(y:z)$ is defined. In this situation, $\mathbf{h}_K(x:z) = \mathbf{h}_K(x:y)\cdot\mathbf{h}_K(y:z)$.*

Proof. Recall that from Lemma 10.2 we have that $y \notin K^c$. Moreover, as $K(y)|K$ is a subextension of the immediate extension $K(x)^h|K$, it is also immediate. For the definition of $\mathbf{h}_K(x:y)$ we have already used the fact that there exists some polynomial $f(x)$ such that $\mathrm{appr}(y,K) = \mathrm{appr}(f(x),K)$; by Corollary 8.3, this approximation type is transcendental since $\mathrm{appr}(x,K)$ is. We have proved that (10.1) holds for y in place of x.

Let us now prove the multiplicativity. Since $\mathbf{h}_K(y:z) = \mathbf{h}_K(y:g(y))$ whenever $v(z - g(y)) \ge \mathrm{dist}(z,K)$, it suffices to show our assertion under the additional assumption $z = g(y) \in K[y]$. Furthermore, because of $y \in K[x]^c\backslash K^c$ we may choose $f(x) \in K[x]$ so that $v(y - f(x)) \ge \mathrm{dist}(y,K)$ and $v(g(y) - g(f(x))) \ge \mathrm{dist}(g(y),K)$;

hence it suffices to show our assertion under the assumption that $y = f(x) \in K[x]$ and $z = g(f(x)) \in K[x]$. Since by hypothesis, $\mathrm{appr}(x, K)$ is transcendental, it fixes the value of every polynomial over K, and thus we know from Lemma 8.2 that $f(c) \nearrow f(x)$ whenever $c \nearrow x$. Also since $\mathrm{appr}(f(x), K)$ is transcendental, it fixes the value of every polynomial over K, and thus for $f(c) \nearrow f(x)$,

$$\begin{aligned} v(g(f(x)) - g(f(c))) &= vg_{\mathbf{h}_1}(f(c)) + \mathbf{h}_1 \cdot v(f(x) - f(c)) \\ &= vg_{\mathbf{h}_1}(f(c)) + \mathbf{h}_1 \cdot (vf_{\mathbf{h}_2}(c) + \mathbf{h}_2 \cdot v(x - c)) \\ &= \beta + \mathbf{h}_1 \cdot \mathbf{h}_2 \cdot v(x - c) \end{aligned}$$

where $\mathbf{h}_1 = \mathbf{h}_K(f(x) : g(f(x)))$, $\mathbf{h}_2 = \mathbf{h}_K(x : f)$ and $\beta = vg_{\mathbf{h}_1}(f(c)) + \mathbf{h}_1 \cdot vf_{\mathbf{h}_2}(c)$. This shows that

$$\mathbf{h}_K(x : g(f(x))) = \mathbf{h}_1 \cdot \mathbf{h}_2 = \mathbf{h}_2 \cdot \mathbf{h}_1 = \mathbf{h}_K(x : f) \cdot \mathbf{h}_K(f(x) : g(f(x))) \;,$$

as asserted. □

Theorem 10.7. *Assume (10.1). Then*

$$[K(x)^h : K(y)^h] \;\leq\; \mathbf{h}_K(x : y) \;.$$

Proof. Take p^n to be the inseparable degree of $K(x)^h|K(y)^h$ and $L|K(y)^h$ to be the maximal separable subextension of $K(x)^h|K(y)^h$. Then $[K(x)^h : L] = p^n$. Further, x^{p^n} is separable over $K(y)^h$, so $x^{p^n} \in L$ and $K(x^{p^n})^h \subseteq L$. As $K(x)^h = K(x^{p^n})^h(x)$, we find that

$$p^n \;\geq\; [K(x)^h : K(x^{p^n})^h] \;=\; [K(x)^h : L] \cdot [L : K(x^{p^n})^h] \;=\; p^n \cdot [L : K(x^{p^n})^h] \,,$$

which shows that $[L : K(x^{p^n})^h] = 1$ and in particular, $y \in K(x^{p^n})^h$. So we are able to apply Lemma 10.6 to obtain that $\mathbf{h}_K(x : y) = \mathbf{h}_K(x : x^{p^n}) \cdot \mathbf{h}_K(x^{p^n} : y) = p^n \cdot \mathbf{h}_K(x^{p^n} : y)$.

As x^{p^n} is separable over $K(y)^h$, we can infer from Proposition 10.5 that $[K(x^{p^n})^h : K(y)^h] \leq \mathbf{h}_K(x^{p^n} : y)$. On the other hand, $[K(x)^h : K(x^{p^n})^h] = [K(x)^h : L] = p^n$. So we get

$$[K(x)^h : K(y)^h] \;=\; p^n \cdot [K(x^{p^n})^h : K(y)^h] \;\leq\; p^n \cdot \mathbf{h}_K(x^{p^n} : y) \;=\; \mathbf{h}_K(x : y) \,,$$

as desired. □

Corollary 10.8. *Assume that (10.1) holds. Then*

$$K(x)^h = K(y)^h \iff \mathbf{h}_K(x : y) = 1 \;.$$

Proof. If $K(x)^h = K(y)^h$, then $x \in K(y)^h$ and $y \in K(x)^h$, and by Lemma 10.6 we have that

$$\mathbf{h}_K(x : y) \cdot \mathbf{h}_K(y : x) \;=\; \mathbf{h}_K(x : x) \;=\; 1 \,,$$

which yields $\mathbf{h}_K(x : y) = 1$. The reverse implication follows from Theorem 10.7. □

11. An application to henselian rationality

In this section we will apply Theorem 10.7 to immediate valued function fields which are the henselization of a rational function field.

Theorem 11.1. *Take a valued field (K, v) of rank 1 and an immediate function field $(F|K, v)$ of transcendence degree 1. Suppose there is some $x \in F^h \setminus K^c$ with transcendental approximation type over K such that $F^h = K(x)^h$. Then there is already some $y \in F$ such that $F^h = K(y)^h$. In fact, there is some $\gamma \in vK$ such that $K(x)^h = K(y)^h$ holds for every $y \in F$ with $v(x - y) \geq \gamma$.*

Proof. Since $x \notin K^c$ there is $\gamma \in vK$ such that $\gamma > \mathrm{dist}(x,K)$. By assumption, the rank of (K,v) is 1, and since $(F|K,v)$ is immediate, also (F,v) has rank 1. Thus, the element x lies in the completion of F. So we may take some $y \in F$ such that $v(x-y) \geq \gamma > \mathrm{dist}(x,K)$. For every such y, $[K(x)^h : K(y)^h] \leq \mathbf{h}_K(x:y)$ holds by Theorem 10.7, and $\mathbf{h}_K(x:y) = \mathbf{h}_K(x:x) = 1$ holds by Lemma 10.3. This yields that $K(x)^h = K(y)^h$. □

12. Approximation coefficients

Throughout this section, we will assume the situation as described in (10.1). As before, take $f(x) \in K[x]$ such that $v(y - f(x)) \geq \mathrm{dist}(y,K)$. An element $d \in K$ will be called an **approximation coefficient of** y **in** x (over K), if

$$v(f(x) - f(c)) < v(f(x) - f(c) - d \cdot (x-c)^{\mathbf{h}}) \tag{12.1}$$

for $c \nearrow x$, where $\mathbf{h} = \mathbf{h}_K(x:y)$.

Lemma 12.1. *If d satisfies (12.1) for some $f(x)$ with $v(y-f(x)) \geq \mathrm{dist}(y,K)$, then it satisfies (12.1) for every such $f(x)$; in other words: approximation coefficients are independent of the choice of $f(x)$. If d satisfies (12.1), then it satisfies*

$$v(y - f(c)) < v(y - f(c) - d \cdot (x-c)^{\mathbf{h}}) \quad \textit{for } c \nearrow x\,. \tag{12.2}$$

Proof. If $g(x)$ is another element of $K[x]$ with $v(y - g(x)) \geq \mathrm{dist}(y,K)$, then

$$v(f(x) - g(x)) \geq \mathrm{dist}(y,K) = \mathrm{dist}(f(x),K) > v(f(x) - f(c))$$

for all $c \in K$. Since $\mathrm{appr}(x,K)$ is transcendental, it fixes the value of the polynomial $f - g$, whence

$$v(f(c) - g(c)) = v(f(x) - g(x)) > v(f(x) - f(c)) \quad \text{for } c \nearrow x\,.$$

Hence by the ultrametric triangle law,

$$\begin{aligned} v(g(x) - g(c)) &= \min\{v(g(x) - f(x)), v(f(x) - f(c)), v(f(c) - g(c))\} \\ &= v(f(x) - f(c)) \end{aligned}$$

and

$$\begin{aligned} & v(g(x) - g(c) - d \cdot (x-c)^{\mathbf{h}}) \\ & \quad \geq \min\{v(f(x) - f(c) - d \cdot (x-c)^{\mathbf{h}})\,,\ v(f(x) - g(x))\,,\ v(f(c) - g(c))\} \\ & \quad > v(f(x) - f(c)) = v(g(x) - g(c)) \end{aligned}$$

for $c \nearrow x$, which shows that d fulfills equation (12.1) also with g in place of f. Replacing $g(x)$ by y and $g(c)$ by $f(c)$ in the above deduction, one obtains a proof of (12.2). □

The following lemma proves the existence of approximation coefficients:

Lemma 12.2. *The element $d \in K$ is an approximation coefficient of y in x if and only if*

$$vd = vf_{\mathbf{h}}(c) < v(f_{\mathbf{h}}(c) - d) \quad \textit{for} \quad c \nearrow x\,.$$

In particular, there exists an approximation coefficient of y in x. Furthermore,

$$\mathrm{dist}(y,K) = vd + \mathbf{h} \cdot \mathrm{dist}(x,K) \tag{12.3}$$

Proof. By definition of $\mathbf{h} = \mathbf{h}_K(x:y) = \mathbf{h}_K(x:f)$, we have that

$$v(f(x) - f(c) - f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) > v(f(x) - f(c)) = v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}})$$

for $c \nearrow x$. Hence (12.1) holds for $c \nearrow x$ if and only if

$$v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}} - d \cdot (x-c)^{\mathbf{h}}) > v(f_{\mathbf{h}}(c)(x-c)^{\mathbf{h}}) \, ,$$

which is equivalent to

$$vf_{\mathbf{h}}(c) < v(f_{\mathbf{h}}(c) - d) \quad \text{for} \quad c \nearrow x \, .$$

Since $K(x)|K$ is assumed to be an immediate extension, by Lemma 4.1 a) there exists some $d \in K$ such that $v(f_{\mathbf{h}}(x) - d) > vf_{\mathbf{h}}(x)$. Since $\operatorname{appr}(x,K)$ is transcendental, for $c \nearrow x$ we have that $v(f_{\mathbf{h}}(c) - d) = v(f_{\mathbf{h}}(x) - d)$ and $vf_{\mathbf{h}}(c) = vf_{\mathbf{h}}(x)$ and thus,

$$v(f_{\mathbf{h}}(c) - d) = v(f_{\mathbf{h}}(x) - d) > vf_{\mathbf{h}}(x) = vf_{\mathbf{h}}(c) = vd \, .$$

Hence d is an approximation coefficient for y in x by the first part of our proof.

In view of the hypothesis that $\operatorname{appr}(x,K)$ is transcendental, $f(x)$ satisfies equation (8.3) of Lemma 8.2. From this we obtain:

$$\begin{aligned} \operatorname{dist}(y,K) &= \operatorname{dist}(f(x),K) = vf_{\mathbf{h}}(c) + \mathbf{h} \cdot \operatorname{dist}(x,K) \\ &= vd + \mathbf{h} \cdot \operatorname{dist}(x,K) \, . \end{aligned}$$

□

Lemma 12.3. *Take elements $y_i \in K[x]^c \setminus K^c$ with common approximation degree $\mathbf{h} = \mathbf{h}_K(x:y_i)$, $1 \le i \le m$. Assume that $d_i \in K$ is an approximation coefficient of y_i in x and let k_i be elements in K such that*

$$v\sum_{i=1}^{m} k_i d_i \;=\; \min_{1 \le i \le m} vk_i d_i \;<\infty \, . \tag{12.4}$$

Then the following will hold:

$$\mathbf{h}_K\left(x : \sum_{i=1}^{m} k_i y_i\right) = \mathbf{h} \, .$$

Proof. We choose polynomials $f^{[i]}(X) \in K[X]$ with $v(y_i - f^{[i]}(x)) \ge \operatorname{dist}(y_i, K)$. Then by Lemma 3.1 b), we have that $\operatorname{dist}(f^{[i]}(x), K) = \operatorname{dist}(y_i, K)$. We set

$$g(X) := \sum_{i=1}^{m} k_i f^{[i]}(X) \in K[X]$$

and show that $\mathbf{h}_K(x:g) = \mathbf{h}$.

First, we observe that by the previous lemma together with (12.4),

$$\begin{aligned} vg_{\mathbf{h}}(c) &= v\sum_{i=1}^{m} k_i f_{\mathbf{h}}^{[i]}(c) \;=\; \min\left\{ v\sum_{i=1}^{m} k_i d_i \, , \, v\left(\sum_{i=1}^{m} (k_i f_h^{[i]}(c) - k_i d_i)\right)\right\} \\ &= v\sum_{i=1}^{m} k_i d_i \;=\; \min_{1 \le i \le m} vk_i d_i \;=\; \min_{1 \le i \le m} v\, k_i f_{\mathbf{h}}^{[i]}(c) \end{aligned}$$

for $c \nearrow x$ (in particular, $vg_{\mathbf{h}}(c) < \infty$ which implies that g is nonconstant); with $1 \le j \ne \mathbf{h}$ we obtain:

$$\begin{aligned} v\, g_{\mathbf{h}}(c)(x-c)^{\mathbf{h}} &= vg_{\mathbf{h}}(c) + \mathbf{h}\cdot v(x-c) \;=\; \left(\min_{1\le i\le m} v\, k_i f_{\mathbf{h}}^{[i]}(c)\right) + \mathbf{h}\cdot v(x-c) \\ &= \min_{1\le i\le m} v\, k_i f_{\mathbf{h}}^{[i]}(c)(x-c)^{\mathbf{h}} \\ &< \min_{1\le i\le m} v\, k_i f_j^{[i]}(c)(x-c)^j \\ &\le v\sum_{i=1}^m k_i f_j^{[i]}(c)(x-c)^j \;=\; v\, g_j(c)(x-c)^j \;. \end{aligned}$$

This proves that $\mathbf{h}_K(x:g) = \mathbf{h}$. It also follows that

$$\begin{aligned} \operatorname{dist}(g(x),K) &= vg_{\mathbf{h}}(c) + \mathbf{h}\cdot \operatorname{dist}(x,K) \;=\; \left(\min_{1\le i\le m} vk_i f_{\mathbf{h}}^{[i]}(c)\right) + \mathbf{h}\cdot\operatorname{dist}(x,K) \\ &= \min_{1\le i\le m} v\, k_i f_{\mathbf{h}}^{[i]}(c) + \mathbf{h}\cdot\operatorname{dist}(x,K) \\ &= \min_{1\le i\le m} vk_i + vf_{\mathbf{h}}^{[i]}(c) + \mathbf{h}\cdot\operatorname{dist}(x,K) \\ &= \min_{1\le i\le m} vk_i + \operatorname{dist}(f^{[i]}(x),K) \;=\; \min_{1\le i\le m} vk_i + \operatorname{dist}(y_i,K) \\ &\le \min_{1\le i\le m} vk_i + v(y_i - f^{[i]}(x)) \le \min_{1\le i\le m} v(k_i y_i - k_i f^{[i]}(x)) \\ &\le v\sum_{i=1}^m (k_i y_i - k_i f^{[i]}(x)) = v\left(\sum_{i=1}^m k_i y_i - g(x)\right)\;, \end{aligned}$$

where the first equality follows from Lemma 8.2 as $\operatorname{appr}(x,K)$ is transcendental. By Lemma 3.1 b), this shows that

$$\operatorname{dist}\left(\sum_{i=1}^m k_i y_i, K\right) \;=\; \operatorname{dist}(g(x),K) \le v\left(\sum_{i=1}^m k_i y_i - g(x)\right)\;.$$

Consequently,

$$\mathbf{h}_K(x : \sum_{i=1}^m k_i y_i) = \mathbf{h}_K(x:g) = \mathbf{h}\;.$$

□

13. Valuation independence of Galois groups

In this section, we will introduce a valuation theoretical property that characterizes the Galois groups of tame Galois extensions. Take a Galois extension $(L|K,v)$ of henselian fields. Its Galois group $\operatorname{Gal} L|K$ will be called **valuation independent** if for every choice of elements $d_1,\dots,d_n \in \tilde{L}$ and automorphisms $\sigma_1,\dots,\sigma_n \in \operatorname{Gal} L|K$ there exists an element $d \in L$ such that (for the unique extension of the valuation v from L to $\tilde{L}$):

$$v\sum_{i=1}^n \sigma_i(d)\, d_i \;=\; \min_{1\le i\le n} v\,\sigma_i(d)\, d_i\;. \tag{13.1}$$

Since (K,v) is assumed to be henselian, we have that $v\sigma(d) = vd$ for all $\sigma \in \mathrm{Gal}\, L|K$ and therefore, $v\sigma_i(d)\, d_i = vd + vd_i$. Suppose that $vd_{i_0} = \min_i vd_i$; then (13.1) will hold if and only if

$$v\sum_{i=1}^{n} \frac{\sigma_i(d)}{d}\,\frac{d_i}{d_{i_0}} \;=\; 0\;.$$

In this sum, the terms with $v(d_i/d_{i_0}) > 0$ have no influence, and we can delete the corresponding σ_i from the list. So we see:

Lemma 13.1. *Assume that $(L|K,v)$ is a Galois extension of henselian fields. Then $\mathrm{Gal}\, L|K$ is valuation independent if and only if for every choice of elements $d_i \in \tilde{L}$ with $vd_i = 0$ for $1 \le i \le n$, and automorphisms $\sigma_1, \ldots, \sigma_n \in \mathrm{Gal}\, L|K$, there exists an element $d \in L$ such that*

$$v\sum_{i=1}^{n} \frac{\sigma_i(d)}{d}\, d_i \;=\; 0\;. \tag{13.2}$$

Theorem 13.2. *A Galois extension of henselian fields is tame if and only if its Galois group is valuation independent.*

Proof. Take a Galois extension $(L|K,v)$ of henselian fields, elements $d_i \in \tilde{L}$ with $vd_i = 0$ for $1 \le i \le n$, and automorphisms $\sigma_1, \ldots, \sigma_n \in \mathrm{Gal}\, L|K$. For $\sigma \in \mathrm{Gal}\, L|K$ and $d \in L^\times$, we set

$$\chi_\sigma(d) \;:=\; \frac{\sigma(d)}{d}\, v\;.$$

Since $v\sigma(d) = vd$, the right hand side is a nonzero element in Lv. Now equation (13.2) is equivalent to

$$\sum_{i=1}^{n} d_i v \cdot \chi_{\sigma_i}(d) \;\neq\; 0\,; \tag{13.3}$$

note that $d_i v \neq 0$ since $vd_i = 0$.

We extend the homomorphism

$$G^i(L|K,v) \ni \sigma \;\mapsto\; \chi_\sigma \in \mathrm{Hom}(L^\times, (Lv)^\times)\;,$$

which is well known from ramification theory (see [2], Lemma 5.2.6), to a crossed homomorphism from $\mathrm{Gal}\, L|K$ to $\mathrm{Hom}(L^\times, (Lv)^\times)$. For the definition and an application of crossed homomorphisms, see [6, §6]. As in the case of $\sigma \in G^i(L|K,v)$, it is shown that $\chi_\sigma \in \mathrm{Hom}(L^\times, \overline{L}^\times)$. This group is a right $\mathrm{Gal}\, L|K$-module under the scalar multiplication

$$\chi^\rho \;:=\; \chi \circ \rho\;.$$

We compute:

$$\chi_{\sigma\tau}(d) = \frac{\sigma\tau(d)}{d}\, v = \frac{\sigma\tau(d)}{\tau(d)}\, v \cdot \frac{\tau(d)}{d}\, v = (\chi_\sigma \circ \tau)(d) \cdot \chi_\tau(d)\;.$$

Thus,

$$\chi_{\sigma\tau} \;=\; \chi_\sigma^\tau \cdot \chi_\tau\;.$$

In other words, the map

$$\mathrm{Gal}\, L|K \ni \sigma \;\mapsto\; \chi_\sigma \in \mathrm{Hom}(L^\times, (Lv)^\times) \tag{13.4}$$

is a crossed homomorphism. Hence, it is injective if and only if its kernel is trivial. This kernel consists of all $\sigma \in \mathrm{Gal}\, L|K$ for which $\frac{\sigma(d)}{d}\, v = 1$ for all $d \in L^\times$. So the kernel is the ramification group $G^r(L|K,v)$.

The theorem of Artin on linear independence of characters (see [11], VI, §4, Theorem 4.1) tells us that if the χ_{σ_i} are distinct characters, then an element d satisfying (13.3) will exist. This shows that G is valuation independent if the map in (13.4) is injective. The converse is also true: if $\sigma_1 \neq \sigma_2$ but $\chi_{\sigma_1} = \chi_{\sigma_2}$, then with $n = 2$ and $d_1 = -d_2 = 1$, (13.2) does not hold for any d.

Since the kernel is the ramification group of $(L|K,v)$, we conclude that $\mathrm{Gal}\, L|K$ is valuation independent if and only if the ramification group is trivial. This is equivalent to $(L|K,v)$ being a tame extension. □

Note that we could give the above definition and the result of the theorem also for extensions which are not Galois, replacing automorphisms by embeddings; however, the normal hull of an algebraic extension $L|K$ of a henselian field K is a tame extension of K if and only if $L|K$ is a tame extension, so there is no loss of generality in restricting our scope to Galois extensions.

14. A pull down principle for henselian rationality through tame extensions

Take a tame extension $(L|K,v)$ of fields of rank 1 and an immediate function field $(F|K,v)$ of transcendence degree 1 with F not contained in K^c. By Lemma 2.2, the extension $(F^h.L|L,v)$ is again immediate. Since $L|K$ is algebraic, so is $F^h.L|F^h$ and therefore, $F^h.L$ is henselian, so $F^h.L = (F.L)^h$. We consider the following question:

If $F^h.L|L$ is a henselian rational function field, does this imply the same for $F^h|K$?

To start with, we observe that w.l.o.g. we may assume the extension $L|K$ to be finite and Galois. Indeed, if $x \in F^h.L$ such that $F^h.L = L(x)^h$, then x lies already in $F^h.L_1$ for some finite subextension $L_1|K$ of $L|K$. Since x must be transcendental over L_1, the extension $F^h.L_1|L_1(x)^h$ is finite, generated by finitely many elements that lie in $L(x)^h$. So we can choose a finite subextension $L_2|L_1$ of $L|L_1$ such that these elements already lie in $L_2(x)^h$. Since the normal hull of a tame extension is a tame extension as well, we may replace L_2 by its normal hull L_3 over K because also $L_3(x)^h$ will contain these elements.

From now on we assume that $L|K$ is a finite tame Galois extension and that $F^h.L = L(x)^h$ for some $x \in F^h.L$. In addition, we assume that $\mathrm{appr}(x,L)$ is transcendental.

We show that hypothesis (10.1) holds with K replaced by L. First, since $(F.L|L,v)$ is an immediate function field, so is $(L(x)|L,v)$. Second, $\mathrm{appr}(x,L)$ is transcendental by assumption. Third, we have:

Lemma 14.1. *The condition $F \not\subset K^c$ implies that $F.L \not\subset L^c$, hence $x \notin L^c$.*

Proof. Since $F \not\subset K^c$, there exists some $z \in F$ with $z \notin K^c$. By assumption, $(L|K,v)$ is a tame extension, and as remarked in Section 2, is therefore defectless. Hence by Lemma 4.5, $\mathrm{dist}(z,L) = \mathrm{dist}(z,K) < \infty$. Consequently, $F.L \not\subset L^c$, as asserted.

Furthermore, $x \in L^c$ would imply that $L(x) \subset L^c$; since the rank of (K,v) is 1 by assumption, the same is true for $(L(x),v)$ and $L(x)$ is thus dense in $L(x)^h$, so we would get that $F.L \subset F^h.L = L(x)^h \subset L^c$, a contradiction. □

Lemma 14.2. *If there exists an element $y \in F^h$ such that $L(y)^h = L(x)^h$, then $F^h = K(y)^h$.*

Proof. Since $(F^h|K,v)$ and hence also its subextension $(K(y)^h|K,v)$ are immediate and $(L|K,v)$ is defectless and finite, we obtain from Lemma 2.2 that $[F^h.L : F^h] = [L : K] = [K(y)^h.L : K(y)^h]$. On the other hand, $F^h.L = L(x)^h = L(y)^h = K(y)^h.L$, so $F^h = K(y)^h$ must hold, because by assumption on y, $K(y)^h \subseteq F^h$. □

Since $L|K$ is a finite tame Galois extension, also the extension $F^h.L|F^h$ is a finite tame Galois extension. As shown in the preceding proof, it is of degree $n := [L : K]$. We write

$$\mathrm{Gal}\,(F^h.L|F^h) = \{\rho_i \mid 1 \leq i \leq n\} \,.$$

Then $\mathrm{Gal}\,(L|K) = \{\rho_i|_L \mid 1 \leq i \leq n\}$.

The next lemma will help us to determine the relative approximation degrees of the conjugates $\rho_i(x)$.

Lemma 14.3. *Assume that ρ is a valuation preserving automorphism of $L(x)^h$ such that $\rho(L) = L$. Then*

$$L(x)^h = L(\rho x)^h \,.$$

Proof. Since $\rho x \in \rho(L(x)^h) = L(x)^h$, we have that $L(\rho x)^h \subseteq L(x)^h$. Further, $L \subseteq \rho^{-1}(L(\rho x)^h) \subseteq L(x)^h$ and $x \in \rho^{-1}(L(\rho x)^h)$. Thus, $L(x) \subseteq \rho^{-1}(L(\rho x)^h)$. Since ρ is valuation preserving and induces an isomorphism from $\rho^{-1}(L(\rho x)^h)$ to the henselian field $L(\rho x)^h$, also $\rho^{-1}(L(\rho x)^h)$ is henselian; it is therefore equal to $L(x)^h$. This shows that its image $L(\rho x)^h$ under the automorphism ρ is also equal to $L(x)^h$. □

The following lemma and theorem make essential use of the valuation independence of Galois groups of tame Galois extensions. Let Tr denote the trace.

Lemma 14.4. *There is an element $d \in L$ such that*

$$\mathbf{h}_K(x : \mathrm{Tr}_{F^h.L|F^h}(d \cdot x)) = 1 \,.$$

Proof. From the preceding lemma it follows that every $\rho_i(x)$ is transcendental over L and hence over K. Hence by Lemma 12.2 we can choose approximation coefficients d_i of $\rho_i(x)$ in x over K for $1 \leq i \leq n$. By Theorem 13.2, we have that $\mathrm{Gal}\,(L|K)$ is valuation independent. This means we can choose an element $d \in L$ such that (13.1) holds with $\sigma_i = \rho_i|_L$. Then for $k_i := \sigma_i(d) = \rho_i(d)$, the hypothesis (12.4) of Lemma 12.3 holds. In view of the previous lemma and Corollary 10.8 we have that $\mathbf{h}_K(x : \rho_i(x)) = 1$. From Lemma 12.3 we can now infer that

$$\begin{aligned}\mathbf{h}_K\left(x : \mathrm{Tr}_{F^h.L|F^h}(d \cdot x)\right) &= \mathbf{h}_K\left(x : \sum_i \rho_i(d \cdot x)\right)\\ &= \mathbf{h}_K\left(x : \sum_i \rho_i(d) \cdot \rho_i(x)\right) = 1 \,.\end{aligned}$$

□

Now we are able to answer our question:

Theorem 14.5. *Let (K, v) be an algebraically maximal field of rank 1, and let (F, v) be an immediate function field of transcendence degree 1 over (K, v), with $F \not\subset K^c$. If $F^h.L$ is a henselian rational function field over L for some tame extension $(L|K, v)$, then F^h is a henselian rational function field over K.*

Proof. As shown in the beginning of this section, we may assume that $L|K$ is finite and Galois. Now the foregoing lemma shows that there is some $d \in L$ such that for $y := \mathrm{Tr}_{F^h.L|F^h}(d \cdot x) \in F^h$ we have $\mathbf{h}_K(x : y) = 1$. By virtue of Corollary 10.8, $L(y)^h = L(x)^h$. From Lemma 14.2, we can now infer that F^h is henselian rational over K, as asserted. □

References

[1] Endler, O.: *Valuation theory*, Springer, Berlin 1972

[2] Engler, A.J. – Prestel, A.: *Valued fields*, Springer Monographs in Mathematics. Springer-Verlag, Berlin 2005

[3] Kaplansky, I.: *Maximal fields with valuations I*, Duke Math. Journ. **9** (1942), 303–321

[4] Knaf, H. - Kuhlmann, F.-V.: *Every place admits local uniformization in a finite extension of the function field*, Advances in Math. 221 (2009), 428-453

[5] Kuhlmann, F.–V.: *Henselian function fields and tame fields*, extended version of Ph.D. thesis, Heidelberg 1990; available at `http://math.usask.ca/~fvk/Fvkpaper.html`

[6] Kuhlmann, F.-V.: *Additive Polynomials and Their Role in the Model Theory of Valued Fields*, Proceedings of the Workshop and Conference on Logic, Algebra, and Arithmetic, 2003. Lecture Notes in Logic 26 (2006), 160–203

[7] Kuhlmann, F.-V.: *A classification of Artin Schreier defect extensions and characterizations of defectless fields*, Illinois J. Math. **54** (2010), 397–448

[8] Kuhlmann, F.-V.: *The defect*, in: Commutative Algebra - Noetherian and non-Noetherian perspectives, 277-318, Springer, New York, 2011

[9] Kuhlmann, F.-V.: *The algebra and model theory of tame valued fields*, submitted

[10] Kuhlmann, F.-V.: *Elimination of Ramification II: Henselian Rationality of Valued Function Fields*, in preparation

[11] Lang, S.: *Algebra*, Graduate Texts in Mathematics. Springer-Verlag, New York, 2002

[12] Ostrowski, A.: *Untersuchungen zur arithmetischen Theorie der Körper*, Math. Z. **39** (1935), 269–404

[13] Prestel, A. – Ziegler, M.: *Model theoretic methods in the theory of topological fields*, J. reine angew. Math. **299/300** (1978), 318–341

[14] Ribenboim, P.: *Théorie des valuations*, Les Presses de l'Université de Montréal, Montréal 1968

[15] Temkin, M.: Inseparable local uniformization. preprint, arXiv:[0804.1554]

[16] Warner, S.: *Topological fields*, Mathematics studies **157**, North Holland, Amsterdam 1989

[17] Zariski, O. – Samuel, P.: *Commutative Algebra*, Vol. II, New York–Heidelberg–Berlin 1960

Department of Mathematics and Statistics, University of Saskatchewan, 106 Wiggins Road, Saskatoon, Saskatchewan, Canada S7N 5E6

E-mail address: `fvk@math.usask.ca, izabelavlahu@gmail.com`