ON THE STRUCTURE OF CERTAIN VALUED FIELDS

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Abstract. For any two complete discrete valued fields $K_1$ and $K_2$ of mixed characteristic with perfect residue fields, we show that if each pair of $n$-th residue rings is isomorphic for each $n \geq 1$, then $K_1$ and $K_2$ are isometric and isomorphic. More generally, for $n_1, n_2 \geq 1$, if $n_2$ is large enough, then any homomorphism from the $n_1$-th residue ring of $K_1$ to the $n_2$-th residue ring of $K_2$ can be lifted to a homomorphism between the valuation rings. We can find a lower bound for $n_2$ depending only on $K_2$. Moreover, we get a functor from a category of certain principal Artinian local rings of length $n$ to a category of certain complete discrete valuation rings of mixed characteristic with perfect residue fields, which naturally generalizes the functorial property of unramified complete discrete valuation rings. The result improves Basarab’s generalization of the AKE-principle for finitely ramified henselian valued fields, which solves a question posed by Basarab, in the case of perfect residue fields.

1. Introduction

In this paper, we consider some problems on valued fields arising from the interaction of number-theoretic approaches and model-theoretic approaches. In number theory, it is well-known that the following are equivalent.

• There is an isometric isomorphism between two complete unramified discrete valued fields $K_1$ and $K_2$ of mixed characteristic $(0,p)$ with perfect residue fields.

• There is an isomorphism between residue fields of $K_1$ and $K_2$.

In model theory, as a counterpart, there is a principle called Ax-Kochen-Ershov-principle (briefly, AKE-principle) which states the following are equivalent.

• Two absolutely unramified henselian valued fields $K_1$ and $K_2$ of the same type of mixed characteristic $(0,p)$ with value groups as $\mathbb{Z}$-groups are elementarily equivalent.

• Residue fields of $K_1$ and $K_2$ are elementarily equivalent.

We introduce more elementary classes of valued fields satisfying the AKE-principle:

a) Algebraically closed valued fields by Robinson in [28].

b) Henselian fields with residue fields of characteristic 0 by Ax and Kochen in [4] and independently by Ershov in [14].

c) $p$-adically closed fields by Ax and Kochen in [4] and independently by Ershov in [14].

d) Algebraically maximal Kaplansky fields by Ershov in [15] and independently by Ziegler in [31].

e) Tame fields of equal characteristic by Kuhlmann in [22].

f) Separably tame fields of equal characteristic by Kuhlmann and Pal in [23].

Some elementary classes of valued fields with additional structures are also known to satisfy the AKE-principle.
g) The ring of Witt vectors after adding a predicate for a unique multiplicative set of representatives for the residue field by van den Dries in [12].

h) Some valued difference fields by Bélair, Macintyre, and Scanlon in [9], by Azgin and van den Dries in [1], and by Pal in [26].

Most of all, in this paper, we are interested in finitely ramified valued fields. Prestel and Roquette in [27] considered the class of $\wp$-closed fields which are finite extensions of $p$-closed fields so that the residue fields are finite. They showed that the theory of $\wp$-closed fields of a fixed $p$-rank is model complete. Basarab in [6] extended this result and generalized the AKE-principle for the case of finitely ramified valued fields. Actually, he showed that for any two finitely ramified henselian valued fields of mixed characteristic, they are elementarily equivalent if and only if their value groups are elementarily equivalent, and their $n$-th residue rings are elementarily equivalent for each $n \geq 1$, where the $n$-th residue ring is the quotient of the valuation ring by the $n$-th power of the maximal ideal. And the theory of a finitely ramified henselian valued field is model complete if and only if each theory of its $n$-th residue ring and its value group are model complete. Motivated from the relation between number theory and model theory for the unramified case, we ask whether there is a number-theoretic part which corresponds to Basarab’s result on the AKE-principle:

**Question 1.1.** Are two finitely ramified complete discrete valued fields $K_1$ and $K_2$ of mixed characteristic with perfect residue fields isomorphic if the $n$-th residue rings of $K_1$ and $K_2$ are isomorphic for each $n \geq 1$?

We report some known necessary and sufficient conditions for certain valued fields to be isomorphic. For a $p$-valued field $(K, \nu)$ and any two $\wp$-closed fields $(L_1, \nu)$ and $(L_2, \nu)$ of the same $p$-rank as $(K, \nu)$, Prestel and Roquette in [27] showed that $L_1$ and $L_2$ are $K$-isomorphic as valued fields if and only if the $n$-th powers of $L_1$ and $L_2$ contained in $K$ are the same for each $n$. Basarab and Kuhlmann in [7] introduced some structure called the mixed $\delta$-structure for each $\delta$ in the value group of a valued field. By using these mixed structures, for any two henselian algebraic extensions $(L_1, \nu)$ and $(L_2, \nu)$ of a given valued field $(K, \nu)$, they gave a criterion for $(L_1, \nu)$ and $(L_2, \nu)$ to be $K$-isomorphic as valued fields under a certain condition with respect to tame extensions.

We return to the unramified case. The previous equivalence for the unramified case in number theory is a corollary of the following well-known theorem([29]).

- For any perfect field $k$ of characteristic $p$, there exists a unique unramified complete discrete valuation ring $R$, called the ring of Witt vectors of $k$, of characteristic 0 which has $k$ as its residue field.
- For any two unramified complete discrete valuation rings $R_1$ and $R_2$ of mixed characteristic with perfect residue fields $k_1$ and $k_2$ respectively, suppose that there is a homomorphism $\phi : k_1 \rightarrow k_2$. Then there is a unique lifting homomorphism $g : R_1 \rightarrow R_2$ such that $g$ induces $\phi$.

In categorical setting, the theorem above is equivalent to the following statement.

- Let $C_p$ be a category of complete unramified discrete valuation rings of mixed characteristic $(0, p)$ with perfect residue fields and $R_p$ a category of perfect fields of characteristic $p$. Then $C_p$ is equivalent to $R_p$. More precisely, there is a functor $L' : R_p \rightarrow C_p$ which satisfies:
In this paper, the main result shows that for sufficiently large $R$, Question 1.2.(2) is not true in general, that is, there is a homomorphism $\phi$ such that $L \circ \phi$ induces $\phi$.

Based on Question 1.1 and the statements above, we raise generalized questions.

**Question 1.2.**

1. For a principal Artinian local ring $R$ of length $n$ with certain conditions, is there a unique complete discrete valuation ring $\hat{R}$ which has $R$ as its residue ring?

2. For any two finitely ramified complete discrete valuation rings $R_1$ and $R_2$ of mixed characteristic with perfect residue fields, let $R_{1,n_1}$ and $R_{2,n_2}$ be the $n_1$-th residue ring of $R_1$ and the $n_2$-th residue ring of $R_2$ respectively. Under certain conditions on $n_1$ and $n_2$, given a homomorphism $\phi : R_{1,n_1} \to R_{2,n_2}$, is there a unique lifting homomorphism $g : R_1 \to R_2$ such that $g$ induces $\phi$?

**Question 1.3.** Let $C_{p,e}$ be a category of complete discrete valuation rings of mixed characteristic $(0,p)$ with perfect residue fields and absolute ramification index $e$. Let $R_{p,e}^n$ be a category of principal Artinian local rings of length $n$ with certain conditions. Let $Pr_n : C_{p,e} \to R_{p,e}^n$ be the natural projection functor. Is there a functor $L : R_{p,e}^n \to C_{p,e}$ which satisfies:

- $Pr_n \circ L$ is equivalent to the identity functor $\text{Id}_{C_{p,e}}$.
- $L \circ Pr_n$ is equivalent to $\text{Id}_{R_{p,e}^n}$.

Question 1.2.(2) is not true in general, that is, there is a homomorphism $\phi : R_{1,n_1} \to R_{2,n_2}$ such that any homomorphism from $R_1$ into $R_2$ does not induce $\phi$. In this paper, the main result shows that for sufficiently large $n_2$, if there is a given homomorphism $\phi : R_{1,n_1} \to R_{2,n_2}$, then there is a homomorphism $g : R_1 \to R_2$ rather naturally related with $\phi$. In the beginning of Section 2, we show that Question 1.1 is true in a special case of local fields. The main ingredient in the proof is to use the compactness of the valuation rings of local fields. In order to extend the result to the case of infinite perfect residue fields, we need the Witt subring. Since valuation rings are not compact in general, we use Krasner’s lemma instead. More precisely, the main result shows that for sufficiently large $n_2$, if there is a given homomorphism $\phi : R_{1,n_1} \to R_{2,n_2}$, then there is a homomorphism $L(\phi) : R_1 \to R_2$ satisfying a lifting property similar to that of the unramified case. Even though the construction of $L(\phi)$ depends on the choice of uniformizer, it turns out that $L(\phi)$ does not depend on the choice of uniformizer. Moreover, when $\phi$ is an isomorphism, so is $L(\phi)$. This provides an answer for Question 1.1. We define $L(\phi)$ as the lifting of $\phi$ even though $L(\phi)$ does not induce $\phi$. The lifting map $L$ provides an answer for Question 1.2.(2) and Question 1.2.(1) where the latter follows from $L$ and the Cohen structure theorem for complete local ring([19]).

In Section 3, we concentrate on Question 1.3. By using the fact that the definition of the lifting map $L$ is independent of the choice of uniformizer, we can show that $L$ is compatible with composition of homomorphisms between residue rings. More precisely, $L(\phi_2 \circ \phi_1) = L(\phi_2) \circ L(\phi_1)$ for any $\phi_1 : R_{1,n_1} \to R_{2,n_2}$ and $\phi_2 : R_{2,n_2} \to R_{3,n_3}$. This defines a functor $L : R_{p,e}^n \to C_{p,e}$ for sufficiently large $n$. We prove that a lower bound for $n$ depends only on the ramification index $e$ and the prime number $p$. Even though $L$ does not give an equivalence between $R_{p,e}^n$ and $C_{p,e}$, it turns out that $L$ satisfies a similar functorial property to $L' : R_p \to C_p$. This provides an answer for Question 1.3.
We define the lifting number for $\mathcal{C}_{p,e}$ as the least number $n$ such that there is a lifting functor $L: \mathcal{R}_{p,e}^n \to \mathcal{C}_{p,e}$. For the tamely ramified case, we prove that the lifting number for $\mathcal{C}_{p,e}$ is $e + 1$ when $e \geq 2$. For the wildly ramified case, we have that the lifting number for $\mathcal{C}_{p,e}$ is at least $e + 1$. Finally, we conclude that the lifting number for $\mathcal{C}_{p,e}$ is either $1$ or $\geq 3$ for any case. We note that the lifting number for $\mathcal{C}_{p,e}$ is $1$ if and only if $e = 1$.

In [6], Basarab posed the following question:

**Question 1.4.** Given a finitely ramified henselian valued field $K$ of ramification index $e \geq 2$, is there a finite integer $N' \geq 1$ depending on $K$ such that any other finitely ramified henselian valued field of the same ramification index $e$ is elementarily equivalent to $K$ if and only if their $N'$-th residue rings are elementarily equivalent and their value groups are elementarily equivalent?

In Section 4, for given valued fields, each of whose value groups has a least positive element, we reduce the problem determining elementary equivalence between them to the problem determining whether certain complete discrete valued fields related with them are isomorphic. Using results in Section 2, we improve Basarab’s result on the AKE-principle which gives a positive answer for Question 1.4 when the residue fields are perfect.

Given a finitely ramified henselian valued field $K$, Basarab([6]) denoted the minimal number $N'$ which satisfies the equivalence in Question 1.4 by $\lambda(T)$ for a complete theory $T$ of $K$. $\lambda(T)$ can be $1$ even when $K$ is not unramified. Under certain conditions, we calculate $\lambda(T)$ explicitly for the tame case and get a lower bound of $\lambda(T)$ for the wild case. As a special case, we conclude that $\lambda(T)$ is $1$ or $e + 1$ if $p \not| e$, and $\lambda(T) \geq e + 1$ if $p|e$ when $K$ is a finitely ramified henselian subfield of $\mathbb{C}_p$ with ramification index $e$.

We introduce basic notations and terminologies which will be used in this paper. We denote a valued field by a tuple $(K, R, m, \nu, k, \Gamma)$ consisting of the following data: $K$ is the underlying field, $R$ is the valuation ring, $m$ is the maximal ideal of $R$, $\nu$ is the valuation map, $k$ is the residue field, and $\Gamma$ is the value group. Hereafter, the full tuple $(K, R, m, \nu, k, \Gamma)$ will be abbreviated in accordance with the situational need for the components.

**Definition 1.5.** Let $(K, \nu, k, \Gamma)$ be a valued field of characteristic zero. We say $(K, \nu)$ is absolutely unramified if $\text{char}(k) = 0$, or $\text{char}(k) = p$ and $\nu(p)$ is the minimal positive element in $\Gamma$ for $p > 0$. We say $(K, \nu)$ is absolutely ramified if it is not absolutely unramified.

**Definition 1.6.** Let $(K, \nu, k, \Gamma, R)$ be a valued field whose residue field has prime characteristic $p$.

1. We say $(K, \nu, k, \Gamma, R)$ is absolutely finitely ramified if the set $\{\gamma \in \Gamma \mid 0 < \gamma \leq \nu(p)\}$ is finite. The cardinality of $\{\gamma \in \Gamma \mid 0 < \gamma \leq \nu(p)\}$ is called the absolute ramification index of $(K, \nu)$, denoted by $e(K, \nu)$ or $e(R)$. If $K$ or $\nu$ is clear from context, we write $e(K)$ or $e$ for $e(K, \nu)$. For $x \in R$, we write $e_{\nu}(x) := |\{\gamma \in \Gamma \mid 0 < \gamma \leq \nu(x)\}|$. If there is no confusion, we write $e(x)$ for $e_{\nu}(x)$.

2. Let $(K, \nu, k, \Gamma, R)$ be finitely ramified. If $p$ does not divide $e_{\nu}(p)$, we say $(K, \nu)$ is absolutely tamely ramified. Otherwise, we say $(K, \nu)$ is absolutely wildly ramified.
Note that if a valued field of mixed characteristic has the absolute finite ramification index, then its value group has the minimum positive element.

**Definition 1.7.** Let $(K_1, \nu_1)$ and $(K_2, \nu_2)$ be valued fields. Let $R_1$ and $R_2$ be subrings of $K_1$ and $K_2$ respectively. Let $f : R_1 \to R_2$ be an injective ring homomorphism. We say $f$ is an isometry if for $a, b \in R_1$,

$$\nu_1(a) > \nu_1(b) \iff \nu_2(f(a)) > \nu_2(f(b)).$$

**Definition 1.8.** For a local ring $R$ with maximal ideal $m$, we denote $R/m^n$ by $R_n$, and we call $R_n$ the $n$-th residue ring of $R$. In particular, $R_1$ is the residue field of $R$.

For each $m > n$, let $pr_m : R \to R_n$ and $pr_m^n : R_m \to R_n$ be the canonical projection maps respectively. For $R$-algebras $S_1$ and $S_2$, we denote the set of $R$-algebra homomorphisms from $S_1$ to $S_2$ by $\text{Hom}_R(S_1, S_2)$, and we briefly write $\text{Hom}(S_1, S_2)$ for $\text{Hom}_R(S_1, S_2)$. We denote the set of $R$-algebra isomorphisms by $\text{Iso}_R(S_1, S_2)$, and we write $\text{Iso}(S_1, S_2)$ for $\text{Iso}(R, R)$. We denote a primitive $n$-th root of unity by $\zeta_n$.

### 2. Lifting Homomorphisms

We start from the following proposition.

**Proposition 2.1.** Let $K_1$ and $K_2$ be finite extensions of $\mathbb{Q}_p$ for some prime $p$. Let $R_{1,n}$ and $R_{2,n}$ be the $n$-th residue rings of $K_1$ and $K_2$ respectively. Suppose that there is an isomorphism $\iota_n : R_{1,n} \to R_{2,n}$ for each $n > 0$. Then there is an isomorphism $\iota : K_1 \to K_2$ over $\mathbb{Q}_p$.

**Proof.** (1) First method: Let $\text{Iso}(R_n)$ be the set of isomorphisms from $R_{1,n}$ onto $R_{2,n}$, and $\xi_{n+1,n}$ be the natural reduction map from $\text{Iso}(R_{n+1})$ to $\text{Iso}(R_n)$. Then $\{\text{Iso}(R_n), \xi_{n+1,n}\}$ forms an inverse system. Since each residue ring $R_{i,n}$ is finite, $\text{Iso}(R_n)$ is finite, in particular compact for each $n$. By the theory of topological algebra, $\lim \text{Iso}(R_n)$ is not empty([25]). This shows there exists an isomorphism $\iota : K_1 \to K_2$. $\iota$ is defined over $\mathbb{Q}_p$ since all elements of $\text{Iso}(R_n)$ are continuous.

(2) Second method: Let $R_1$ and $R_2$ be valuation rings of $K_1$ and $K_2$ respectively. Take an element $a$ in $R_1$ satisfying $K_1 = \mathbb{Q}_p(a)$. Let $f$ be the monic irreducible polynomial of $a$ over $\mathbb{Z}_p$. Consider a sequence $(a'_n \in R_2)_{n \geq 1}$ such that $pr_{2,n}(a'_n) = \iota_n(pr_{1,n}(a))$ where $pr_{i,n}$ denotes a $n$-th natural projection from $R_i$ to $R_{i,n}$. We note that each $\iota_n$ is an $\mathbb{Z}_p$-algebra isomorphism since $\iota_n$ is continuous. Since $f(a) = 0$, $f(\iota_n(pr_{1,n}(a))) = \iota_n(f(pr_{1,n}(a))) = \iota_n(pr_{1,n}(f(a))) = 0$ in $R_{2,n}$. First equality follows from the fact that $\iota_n$ is an $\mathbb{Z}_p$-algebra homomorphism. Hence, $f(pr_{2,n}(a'_n)) = pr_{2,n}(f(a'_n)) = 0$ in $R_{2,n}$, that is, $f(a'_n) \in m_2^n$ where $m_2$ is the maximal ideal of $R_2$. Since $R_2$ is compact, there is a subsequence $(a'_n)$ which converges to $a' \in R_2$, and since $f$ is continuous, $f(a') = \lim_{n \to \infty} f(a'_n) = 0$ in $R_2$. Thus $K_2$ contains a zero $a'$ of $f$. Therefore, there is an injection $\iota : K_1 \to K_2$, $a \to a'$ over $\mathbb{Q}_p$, and hence, we obtain an inequality $[K_1 : \mathbb{Q}_p] \leq [K_2 : \mathbb{Q}_p]$ between the field extension degrees. Similarly, one can show $[K_1 : \mathbb{Q}_p] \geq [K_2 : \mathbb{Q}_p]$. Hence, $\iota$ is an isomorphism over $\mathbb{Q}_p$.

Since the proof of the fact that the inverse limit of $\text{Iso}(R_n)$ is not empty uses Zorn’s lemma, we can only prove the existence of an isomorphism in the first method.
second method does not use Zorn’s lemma and the given isomorphism is more easier to construct. But both methods use the fact that the homomorphisms are defined over \( \mathbb{Q}_p \) or \( \mathbb{Z}_p \) crucially. For the case of infinite perfect residue fields, we need the absolutely unramified discrete valuation rings called the ring of Witt vectors. By Krasner’s lemma, it suffices to consider a single \( n \)-th residue ring for sufficiently large \( n \).

The following theorem is well-known.

**Theorem 2.2.**

1. Let \( k \) be a perfect field of characteristic \( p \). Then there exists a complete discrete valuation ring of characteristic 0 which is absolutely unramified and has \( k \) as its residue field. Such a ring is unique up to isomorphism. This unique ring is called the ring of Witt vectors of \( k \), denoted by \( W(k) \).

2. Let \( R_1 \) and \( R_2 \) be complete discrete valuation rings of mixed characteristic with perfect residue fields \( k_1 \) and \( k_2 \) respectively. Suppose \( R_1 \) is absolutely unramified. Then for every homomorphism \( \phi : k_1 \to k_2 \), there exists a unique homomorphism \( g : R_1 \to R_2 \) making the following diagram commutative:

\[
\begin{array}{ccc}
R_1 & \xrightarrow{g} & R_2 \\
pr_{1,1} & & \downarrow pr_{2,1} \\
k_1 & \xrightarrow{\phi} & k_2
\end{array}
\]

**Proof.** Chapter 2, section 5 of [29].

Before stating main theorems, we need some lemmas.

**Lemma 2.3.** Let \( R \) be a complete discrete valuation ring of characteristic 0 with perfect residue field \( k \) of characteristic \( p \) and corresponding valuation \( \nu \). Then \( W(k) \) can be embedded as a subring of \( R \) and \( R \) is a free \( W(k) \)-module of rank \( \nu(p) \). Moreover, \( R = W(k)[\pi] \) where \( \pi \) is a uniformizer of \( R \).

**Proof.** Chapter 2, Section 5 of [29]

**Lemma 2.4.** Let \( A \) be a ring that is Hausdorff and complete for a topology defined by a decreasing sequence \( a_1 \supset a_2 \supset \ldots \) of ideals such that \( a_n \cdot a_m \subset a_{n+m} \). Assume that the residue ring \( A_1 = A/a_1 \) is a perfect field of characteristic \( p \). Then:

1. There exists one and only one system of representatives \( h : A_1 \to A \) which commutes with \( p \)-th powers: \( h(\lambda^p) = h(\lambda)^p \). This system of representatives is called the set of Teichmüller representatives.

2. In order that \( a \in A \) belong to \( S = h(A_1) \), it is necessary and sufficient that \( a \) be a \( p^n \)-th power for all \( n \geq 0 \).

3. This system of representatives is multiplicative which means

\[
h(\lambda \mu) = h(\lambda)h(\mu)
\]

for all \( \lambda, \mu \in A_1 \).

4. \( S \) contains 0 and 1.

5. \( S \setminus \{0\} \) is a subgroup of the unit group of \( A \).

**Proof.** (1)(2)(3): Chapter 2, Section 4 of [29]

(4): 0 and 1 satisfy (2).

(5): (3) and (4) show that \( S \setminus \{0\} \) is a subgroup of the unit group of \( A \).
Lemma 2.5. Let $R_1$ and $R_2$ be discrete valuation rings of characteristic 0 with residue characteristic $p$. Let $m_i$ be the maximal ideal of $R_i$ generated by $\pi_i$ and $\nu_i$ corresponding discrete valuation of $R_i$ for $i = 1, 2$. Suppose there is a homomorphism $i : R_1 \rightarrow R_{2,n}$. If $n > \nu_2(p)$ for some real number $a \geq 1$, kernel of $i$ is equal to $m_i^m$ for some $m > \nu_i(p)$.

Proof. Let $\overline{m}_2 = m_2/m_2^2$ be the maximal of $R_{2,n}$. If we write $i(\pi_1)R_{2,n} = \overline{m}_2^2$,
\[
\overline{m}_2^2(p) = pR_{2,n} = i(p)R_{2,n} = \nu_i(p) \overline{m}_2^\nu_i(p).
\]
In particular $x = \nu_2(p)/\nu_1(p)$ and $\nu_2(p)/\nu_1(p)$ is a positive integer. Suppose
\[
i(\pi_1^m)R_{2,n} = \overline{m}_2^\nu_2(p) = 0
\]
in $R_{2,n}$ for some $m$. Then we obtain
\[
m\nu_2(p)/\nu_1(p) \geq n > \nu_2(p),
\]
and hence $m > \nu_1(p)$. \hfill $\Box$

For any field $L$, $L^{alg}$ denotes a fixed algebraic closure of $L$. Let $(L, \nu)$ be a valued field whose value group is contained in $\mathbb{R}$. If $L$ is of characteristic 0 and of residue characteristic $p$, we define a normalized valuation $\nu$ on $L$ by the property $\nu(p) = 1$, that is, $\nu(p)\nu = \nu$. We denote an extended valuation of $\nu$ on $L^{alg}$ by $\tilde{\nu}$. When $L$ is henselian, $\tilde{\nu}$ is unique.

Lemma 2.6. Let $(K_1, \nu_1)$ and $(K_2, \nu_2)$ be valued fields whose value groups are contained in $\mathbb{R}$. Let $f : K_1 \rightarrow K_2$ be an isometric homomorphism. Suppose $K_1$ is henselian. Let $\tilde{f} : K_1^{alg} \rightarrow K_2^{alg}$ be an extended homomorphism of $f$. Then $\tilde{f}$ is an isometry.

Proof. There are two valuations on $\tilde{f}(K_1^{alg}), \tilde{\nu}_1 \circ \tilde{f}^{-1}$ and $\tilde{\nu}_2 |_{\tilde{f}(K_1^{alg})}$ where $\tilde{\nu}_2 |_{\tilde{f}(K_1^{alg})}$ is the restriction of $\tilde{\nu}_2$ to $\tilde{f}(K_1^{alg})$. Since $f$ is an isometry, the restrictions of $\tilde{\nu}_1 \circ \tilde{f}^{-1}$ and $\tilde{\nu}_2 |_{\tilde{f}(K_1^{alg})}$ to $f(K_1)$ are equivalent, in fact, they are equal since $(\tilde{\nu}_1 \circ \tilde{f}^{-1})(p) = \tilde{\nu}_2 |_{\tilde{f}(K_1^{alg})}(p) = 1$. Since $K_1$ is henselian, $f(K_1)$ is Henselian. Hence, $\tilde{\nu}_1 \circ \tilde{f}^{-1}$ is equal to $\tilde{\nu}_2 |_{\tilde{f}(K_1^{alg})}$ by the henselian property. This shows that $\tilde{f}$ is an isometry. \hfill $\Box$

Let $R$ be a complete discrete valuation ring of mixed characteristic with perfect residue field. Let $\pi$ be a uniformizer of $R$ and $\nu$ corresponding valuation of $R$. Let $L$ and $K$ be the fraction fields of $R$ and $W(k)$ respectively. We denote the maximal number
\[
\max \{ \tilde{\nu}(\pi - \sigma(\pi)) : \sigma \in \text{Hom}_K(L, L^{alg}), \sigma(\pi) \neq \pi \}
\]
by $M(R)_{\pi}$ or $M(L)_{\pi}$.

Definition 2.7. Let $R_1$ and $R_2$ be complete discrete valuation rings of characteristic 0 with perfect residue fields $k_1$ and $k_2$ of characteristic $p$ respectively. Let $m_i$ be the maximal ideal of $R_i$ generated by $\pi_i$ and $\nu_i$ corresponding valuation of $R_i$ for $i = 1, 2$. Let $L_i$ and $K_i$ be the fraction fields of $R_i$ and $W(k_i)$ for $i = 1, 2$ respectively. For any homomorphism $\phi : R_{1,n_1} \rightarrow R_{2,n_2}$, we say that a homomorphism $g : R_1 \rightarrow R_2$ is a $(n_1, n_2)$-lifting of $\phi$ at $\pi_i$ if $g$ satisfies the following:
• There exists a representative \( \beta \) of \( \phi(\pi_1 + m_1^{n_1}) \) which satisfies
\[
\nu_2(g(\pi_1) - \beta) > \max \left\{ \nu_2 \left( \sigma(g(\pi_1)) - \beta : \sigma(g(\pi_1)) \neq g(\pi_1) \right) \right\}
\]
where \( \sigma \) runs through all of \( \text{Hom}_{K_2}(L_2, L_2^{alg}) \).

• \( \phi_{red,1} \circ \text{pr}_{1,1} = \text{pr}_{2,1} \circ g \) where \( \phi_{red,1} : k_1 \rightarrow k_2 \) denotes the natural reduction map of \( \phi \).

When such \( g \) is unique, we denote \( g \) by \( L_{\pi_1, n_1, n_2}(\phi) \). When \( L_{\pi_1, n_1, n_2}(\phi) \) exists for all \( \phi : R_{1, n_1} \rightarrow R_{2, n_2} \), we write \( L_{\pi_1, n_1, n_2} : \text{Hom}(R_{1, n_1}, R_{2, n_2}) \rightarrow \text{Hom}(R_1, R_2) \).

When \( n_1 = n_2 = n \), we briefly write \( L_{\pi_1, n_1, n_2} = L_{\pi_1, n} \) and say that \( L_{\pi_1, n} \) is the \( n \)-lifting at \( \pi_1 \).

If \( \nu_2(\nu_1(p)/\nu_2(p)) \leq n_1 \), there is a natural projection map \( \text{pr}_{n_1, n_2} : \text{Hom}(R_1, R_2) \rightarrow \text{Hom}(R_{1, n_1}, R_{2, n_2}) \) such that for any \( g \) in \( \text{Hom}(R_1, R_2) \), \( \text{pr}_{n_1, n_2} \circ g = \text{pr}_{n_1, n_2}(g) \circ \text{pr}_{1, n_1} \). In particular, \( g \) is a \( (n_1, n_2) \)-lifting of \( \text{pr}_{n_1, n_2}(g) \) at \( \pi_1 \).

Note that when \( n_2 > \nu_2(p) \) and \( \text{Hom}(R_{1, n_1}, R_{2, n_2}) \) is not empty, \( \nu_2(\nu_1(p)/\nu_2(p)) \leq n_1 \) by Lemma 2.5.

The definition of liftings does not depend on the choice of uniformizer. In order to prove this, we need the following lemmas.

**Lemma 2.8.** Let \( R_i \) be a complete discrete valuation ring of characteristic 0 with perfect residue field \( k_i \) of characteristic \( p \) for \( i = 1, 2 \). Let \( m_i \) be the maximal ideal of \( R_i \) generated by \( \pi_i \) and \( S_i \) the set of Teichmüller representatives of \( R_i \) for \( i = 1, 2 \).

1. For any homomorphism \( \phi : R_{1, n_1} \rightarrow R_{2, n_2} \), \( \phi(S_1 + m_1^{n_1}) \) is contained in \( S_2 + m_2^{n_2} \). Similarly, for any homomorphism \( g : R_1 \rightarrow R_2 \), \( g(S_1) \) is contained in \( S_2 \).

2. For any homomorphism \( \phi : R_{1, n_1} \rightarrow R_{2, n_2} \), \( \phi((W(k_1) + m_1^{n_1})/m_1^{n_1}) \) is contained in \( (W(k_2) + m_2^{n_2})/m_2^{n_2} \). Similarly, for any homomorphism \( g : R_1 \rightarrow R_2 \), \( g(W(k_1)) \) is contained in \( W(k_2) \).

**Proof.** (1) Since \( W(k_1)/pW(k_1) \cong R_1/m_1 \cong k_1 \), \( S_1 \) is contained in \( W(k_1) \) by Lemma 2.4. For any \( \lambda \in S_1 \), let \( \eta_\lambda \) be any representative of \( \phi(\lambda^{1/p^r} + m_1^{n_1}) \). We note that \( \lambda^{1/p^r} \) exists in \( S_1 \) by Lemma 2.4 and \( \eta_\lambda^{p^r} + m_2^{n_2} = \phi(\lambda + m_1^{n_1}) \). If \( \theta_\lambda \) is any other representative of \( \phi(\lambda^{1/p^r} + m_1^{n_1}) \), then \( \eta_\lambda - \theta_\lambda \in m_2^{n_2} \). Hence, if we write \( \eta_\lambda = \theta_\lambda + \pi_2^{n_2}a \) for some \( a \) in \( R_2 \), the following binomial expansion
\[
\eta_\lambda^{p^r} = (\theta_\lambda + \pi_2^{n_2}a)^{p^r} = \theta_\lambda^{p^r} + p^r \theta_\lambda^{p^r-1} \pi_2^{n_2}a + \ldots + (\pi_2^{n_2}a)^{p^r}
\]
shows \( \eta_\lambda^{p^r} - \theta_\lambda^{p^r} \in m_2^{n_2} \). Since \( \eta_\lambda^{p^r} \) is a representative of \( \phi(\lambda^{1/p^r} + m_1^{n_1}) \), the calculation above shows that \( \{\eta_\lambda^{p^r}\} \) is a Cauchy sequence and \( \lim_{s \to \infty} \eta_\lambda^{p^r} \) is well-defined in \( R_2 \).

Since \( \eta_\lambda^{p^r} + m_2^{n_2} = \phi(\lambda + m_1^{n_1}) \) and \( m_2^{n_2} \) is topologically closed in \( R_2 \),
\[
\phi(\lambda + m_1^{n_1}) = \left( \lim_{s \to \infty} \eta_\lambda^{p^r} \right) + m_2^{n_2}.
\]

Similarly, we have
\[
\phi(\lambda^{1/p} + m_1^{n_1}) = \left( \lim_{s \to \infty} \eta_\lambda^{p^{r-1}} \right) + m_2^{n_2}.
\]
by Lemma 2.4.

Since \( g(\lambda)^{1/p} = g(\lambda^{1/p}) \), \( g(S_1) \) is contained in \( S_2 \) by Lemma 2.4.

(2) For any element \( a \) in \( W(k_1) \), we can write \( a = \sum_{r=0}^{\infty} \lambda_r p^r \) uniquely where \( \lambda_r \) is in \( S_1 \) by Lemma 2.4. Then by Lemma 2.8.(1), \( g(a) = \sum_{r=0}^{\infty} g(\lambda_r)p^r \) is in \( W(k_2) \).

Let \( \phi_{\text{res}} : (W(k_1) + m_1^{n_1})/m_1^{n_1} \rightarrow R_{2,n_2} \) be the restriction map of \( \phi \) to the domain \( W(k_1)/(W(k_1) \cap m_1^{n_1}) \cong (W(k_1) + m_1^{n_1})/m_1^{n_1} \). By Theorem 2.2, we define \( g_{\text{res}} \) to be the \((1,1)\)-lifting of \( \phi_{\text{res,red,1}} \) where \( \phi_{\text{res,red,1}} : k_1 \rightarrow k_2 \) denotes the natural reduction map of \( \phi_{\text{res}} \). We claim that \( g_{\text{res}} \) induces \( \phi_{\text{res}} \). For any \( \lambda \) in \( S_1 \), \( g_{\text{res}}(\lambda) = \tau \) where \( \tau \) is a unique representative of \( \phi(\lambda + m_1^{n_1}) \) contained in \( S_2 \) by Lemma 2.4. Since \( g_{\text{res}} \) is a ring homomorphism, \( g_{\text{res}}(a) = \sum_{r=0}^{\infty} \tau_r p^r \) where \( \tau_r \) is a unique representative of \( \phi_{\text{res}}(\lambda_r + m_1^{n_1}) \) which is contained in \( S_2 \). This shows

$$g_{\text{res}}(a) + m_2^{n_2} = \left( \sum_{r=0}^{\infty} \tau_r p^r \right) + m_2^{n_2} = \sum_{r=0}^{\infty} p^r \phi_{\text{res}}(\lambda_r + m_1^{n_1}) = \phi_{\text{res}}(a + m_1^{n_1}),$$

and hence, \( g_{\text{res}} \) induces \( \phi_{\text{res}} \). Since the image of \( g_{\text{res}} \) is contained in \( W(k_2) \), the image of \( \phi_{\text{res}} \) is contained in \( (W(k_2) + m_2^{n_2})/m_2^{n_2} \).

\[\square\]

**Lemma 2.9.** Let \( R_i \) and \( R_2 \) be complete discrete valuation rings of characteristic 0 with perfect residue fields \( k_i \) and \( k_2 \) of characteristic \( p \) respectively. Let \( m_i \) be the maximal ideal of \( R_i \) generated by \( \pi_i \) and \( n_i \) corresponding valuation of \( R_i \) for \( i = 1, 2 \). Let \( L_i \) and \( K_i \) be the fraction fields of \( R_i \) and \( W(k_i) \) for \( i = 1, 2 \) respectively.

1. Let \( \alpha \) be a uniformizer of \( R_1 \) other than \( \pi_1 \). Then \( M(R_1)_{\pi_1} = M(R_1)_{\alpha} \).

   We briefly write \( M(R_1)_{\pi_1} = M(R_1) \).

2. Suppose \([L_1 : K_1] = [L_2 : K_2] = e \), that is, \( \nu_1(p) = \nu_2(p) = e \). Suppose there is an isometry \( g : L_1 \rightarrow L_2 \). Then \( M(R_1) = M(R_2) \).

**Proof.** (1) By Lemma 2.4, we can write \( \alpha = \sum_{r=1}^{\infty} \lambda_r \pi_1^r \) where \( \lambda_r \) is a Teichmüller representative of \( R_1 \) for each \( r \) and \( \lambda_1 \neq 0 \). Since \( R_1/m_1 = k_1 \), \( \lambda_r \) is in \( W(k_1) \) for each \( r \) by Lemma 2.4. For any \( \sigma \) in \( \text{Hom}_{K_i}(L_1, K_i^{alg}) \),

\[
\alpha - \sigma(\alpha) = \sum_{r=1}^{\infty} \lambda_r \pi_1^r - \sigma \left( \sum_{r=1}^{\infty} \lambda_r \pi_1^r \right) = \sum_{r=1}^{\infty} \lambda_r \left( \pi_1^r - \sigma(\pi_1^r) \right) = \left( \pi_1 - \sigma(\pi_1) \right) \sum_{r=1}^{\infty} \lambda_r \sum_{j=0}^{r-1} \pi_1^{r-1-j} \sigma(\pi_1^j)
\]

shows \( \tilde{\nu}_1(\alpha - \sigma(\alpha)) = \tilde{\nu}_1(\pi_1 - \sigma(\pi_1)) \) since

\[
\tilde{\nu}_1 \left( \sum_{r=1}^{\infty} \lambda_r \left( \sum_{j=0}^{r-1} \pi_1^{r-1-j} \sigma(\pi_1^j) \right) \right) = 0.
\]

We have \( M(R_1)_{\pi_1} = M(R_1)_{\alpha} \).
(2) By Lemma 2.8.(2), \( g(K_1) \) is contained in \( K_2 \). Let \( f_1 \) be the monic irreducible polynomial of \( \pi_1 \) over \( W(k_1) \). Since \( g \) is an isometry, we have \( \nu_2(g(\pi_1)) = \nu_1(\pi_1) = 1/v \), and hence, \( g(\pi_1) \) is a uniformizer of \( L_2 \). Let \( \bar{g} : L_1^{alg} \to L_2^{alg} \) be an extended homomorphism of \( g \). Let \( g(f_1) \) be the monic irreducible polynomial of \( g(\pi_1) \) over \( K_2 \). If we write \( f_1 = x^n + \cdots + a_1x + a_0 \), we have

\[
g(f_1) = x^n + \cdots + g(a_1)x + g(a_0)
\]

since \( g(K_1) \) is contained in \( K_2 \). Then by Lemma 2.9.(1) and Lemma 2.6,

\[
M(R_2) = \max \left\{ \tilde{\nu}_2(g(\pi_1) - \eta) : g(f_1)(\eta) = 0, \eta \neq g(\pi_1) \right\} = \max \left\{ \tilde{\nu}_2(g(\pi_1) - \bar{g}(\pi_1')) : f_1(\pi_1') = 0, \pi_1' \neq \pi_1 \right\} = \max \left\{ \tilde{\nu}_1(\pi_1 - \pi_1') : f_1(\pi_1') = 0, \pi_1' \neq \pi_1 \right\} = M(R_1)
\]

\( \square \)

**Proposition 2.10.** Let \( R_1 \) and \( R_2 \) be complete discrete valuation rings of characteristic 0 with perfect residue fields of characteristic \( p \). Let \( \mathfrak{m}_i \) be the maximal ideal of \( R_i \) generated by \( \pi_i \) and \( \nu_i \) corresponding valuation of \( R_i \) for \( i = 1, 2 \). Let \( L_i \) and \( K_i \) be the fraction fields of \( R_i \) and \( W(k_i) \) for \( i = 1, 2 \) respectively.

(1) Let \( g : R_1 \to R_2 \) be a \((n_1, n_2)\)-lifting of \( \phi : R_{1,n_1} \to R_{2,n_2} \) at \( \pi_1 \) which satisfies

\[
\tilde{\nu}_2(g(\pi_1) - \beta) > \max \left\{ \tilde{\nu}_2(\sigma(g(\pi_1)) - \beta) : \sigma(g(\pi_1)) \neq g(\pi_1) \right\}
\]

where \( \sigma \) runs through all of \( \text{Hom}_{K_2}(L_2, L_2^{alg}) \) and \( \beta \) is a representative of \( \phi(\pi_1 + \mathfrak{m}_1^{n_1}) \). Then

\[
\max \left\{ \tilde{\nu}_2(\sigma(g(\pi_1)) - \beta) : \sigma(g(\pi_1)) \neq g(\pi_1) \right\} = M(R_1).
\]

(2) The definition of liftings is independent of the choice of uniformizer of \( R_1 \). More precisely, saying that \( g : R_1 \to R_2 \) is a \((n_1, n_2)\)-lifting of \( \phi : R_{1,n_1} \to R_{2,n_2} \) at \( \pi_1 \) is equivalent to the following:

- For any \( x \) in \( R_1 \), there exists a representative \( \beta_x \) of \( \phi(x + \mathfrak{m}_1^{n_1}) \) which satisfies

\[
\tilde{\nu}_2(g(x) - \beta_x) > M(R_1)
\]

- \( \phi_{\text{red}, 1} \circ \text{pr}_{1,1} = \text{pr}_{2,1} \circ g \)

We write \( L_{\pi_1,n_1,n_2} = L_{n_1,n_2} \) and say that \( L_{n_1,n_2} \) is the \((n_1, n_2)\)-lifting.

**Proof.** (1) For \( \sigma \in \text{Hom}_{K_2}(L_2, L_2^{alg}) \) with \( \sigma(g(\pi_1)) \neq g(\pi_1) \),

\[
\tilde{\nu}_2(\sigma(g(\pi_1)) - \beta) = \tilde{\nu}_2(\sigma(g(\pi_1)) - g(\pi_1) + g(\pi_1) - \beta) = \min \left\{ \tilde{\nu}_2(\sigma(g(\pi_1)) - g(\pi_1)), \tilde{\nu}_2(g(\pi_1) - \beta) \right\}
\]

\[
= \tilde{\nu}_2(\sigma(g(\pi_1)) - g(\pi_1))
\]
Suppose that for all embeddings $\sigma \in \mathbb{R}$ since Lemma 2.11 (Krasner’s lemma). Proof. Chapter 2 of [24].

Then $K$ where the second equality follows from Lemma 2.9.(2) since $[K_2(g(\pi_1)) : K_2]$ is equal to $[L_1 : K_1]$ and $g(\pi_1)$ is a uniformizer of $K_2(g(\pi_1))$.

(2) Let $g : R_1 \rightarrow R_2$ be a $(n_1, n_2)$-lifting of $\phi : R_{1,n_1} \rightarrow R_{2,n_2}$ at $\pi_1$ which satisfies

$$\tilde{\nu}_2(g(\pi_1) - \beta) > \max_{\sigma} \left\{ \tilde{\nu}_2(\sigma(g(\pi_1)) - \beta) : \sigma(g(\pi_1)) \neq g(\pi_1) \right\}$$

where $\sigma$ runs through all of $\text{Hom}_{K_2}(L_2, L_2^{alg})$ and $\beta$ is a representative of $\phi(\pi_1 + m_1^{\nu_1})$. For any $x$ in $R_1$, we can write $x = \sum_{r=0}^{\infty} \lambda_r \pi_1^r$ where $\lambda_r$ is in the set $S_1$ of Teichmüller representatives for each $r$. Then

$$\phi(x + m_1^{\nu_1}) = \phi \left( \sum_{r=0}^{\infty} \lambda_r \pi_1^r + m_1^{\nu_1} \right) = \sum_{r=0}^{\infty} \tau_r \beta^r + m_2^{\nu_2}$$

where $\tau_r$ is a representative of $\phi(\lambda_r + m_1^{\nu_1})$ contained in $S_2$ guaranteed by Lemma 2.8.(1). In particular $\sum_{r=0}^{\infty} \tau_r \beta^r$ is a representative of $\phi(x + m_1^{\nu_1})$, say $\beta_x$. By the second condition of the definition of liftings and Lemma 2.4, we have $g(\lambda_r) = \tau_r$, and hence,

$$g(x) = g \left( \sum_{r=0}^{\infty} \lambda_r \pi_1^r \right) = \sum_{r=0}^{\infty} \tau_r g(\pi_1)^r.$$  

We obtain

$$\tilde{\nu}_2(g(x) - \beta_x) = \tilde{\nu}_2 \left( \sum_{r=0}^{\infty} \tau_r g(\pi_1)^r - \sum_{r=0}^{\infty} \tau_r \beta^r \right)$$

$$= \tilde{\nu}_2 \left( \left( g(\pi_1) - \beta \right) \sum_{r=1}^{\infty} \tau_r \left( \sum_{j=0}^{r-1} g(\pi_1)^{r-1-j} \beta^j \right) \right)$$

$$> M(R_1)$$

since

$$\tilde{\nu}_2(\sigma(\pi_1) - \beta) > \max_{\sigma} \left\{ \tilde{\nu}_2(\sigma(g(\pi_1)) - \beta) : \sigma(g(\pi_1)) \neq g(\pi_1) \right\}$$

$$= M(R_1).$$

Lemma 2.11 (Krasner’s lemma). Let $(K, \nu)$ be henselian valued field whose value group is contained in $\mathbb{R}$ and let $a, b \in K^{alg}$. Suppose $a$ is separable over $K(b)$. Suppose that for all embeddings $\sigma(\neq id)$ of $K(a)$ over $K$, we have

$$\tilde{\nu}(b - a) > \tilde{\nu}(\sigma(a) - a).$$

Then $K(a) \subset K(b)$.

Proof. Chapter 2 of [24].
The following theorem shows that there is a unique lifting if we enlarge the lengths of residue rings.

**Theorem 2.12.** Let $R_1$ and $R_2$ be complete discrete valuation rings of characteristic $0$ with perfect residue fields $k_1$ and $k_2$ of characteristic $p$ respectively. Let $m_i$ be the maximal ideal of $R_i$ generated by $x_i$ and $ν_i$ corresponding valuation of $R_i$ for $i = 1, 2$. Let $L_i$ and $K_i$ be the fraction fields of $R_i$ and $W(k_i)$ for $i = 1, 2$ respectively. Suppose $ν_2 > M(R_1)ν_1(p)ν_2(p)$ and $\text{Hom}(R_1, R_2)$ is not empty. Then there exists a unique $(ν_1, ν_2)$-lifting $L_{ν_1, ν_2}: \text{Hom}(R_1, R_2) → \text{Hom}(R_1, R_2)$. $L_{ν_1, ν_2}(φ)$ is also an isomorphism when $φ$ is an isomorphism.

**Proof.** Let $S_i$ be the set of Teichmüller representatives of $R_i$. By Lemma 2.8.(2), let $φ_{res}: (W(k_1) + m_1^{ν_1})/m_1^{2ν_1} → (W(k_2) + m_2^{ν_2})/m_2^{2ν_2}$ be the restriction map of $φ$. For an element $a = \sum_{r=0}^{∞} λ_r p^r$ in $W(k_1)$, as in the proof of Lemma 2.8.(2), we define $g_{res}: W(k_1) → W(k_2)$ by $g_{res}(a) = \sum_{r=0}^{∞} τ_r p^r$ where $τ_r$ is a unique representative of $φ_{res}(λ_r + m_1^{ν_1})$ which is contained in $S_2$. Then $g_{res}$ induces $φ_{res}$. By Lemma 2.3, $L_1 = K_1(α)$ is totally ramified of degree $ν_1(p)$ over $K_1$ where $α = π_1$. Let $f$ be the monic irreducible polynomial of $α$ over $K_1$. The ring homomorphism $g_{res}$ induces the field homomorphism from $K_1$ into $K_2$. We still denote the fraction field homomorphism by $g_{res}$ if there is no confusion. Then $g_{res}: K_1 → K_2$ is an isometry. Let $g_{res}: K_1^{alg} → K_2^{alg}$ be an extended field homomorphism of $g_{res}$. Then $g_{res}$ is an isometry by Lemma 2.6. Let $g_{res}(f)$ be the monic irreducible polynomial of $g_{res}(α)$ over $K_2$. If we write $f = x^{ν_1(p)} + ... + a_1 x + a_0 = (x - α_1) ... (x - α_{ν_1(p)})$, where $α = α_1$, then

$$g_{res}(f) = x^{ν_1(p)} + ... + g_{res}(a_1)x + g_{res}(a_0) = (x - g_{res}(α_1)) ... (x - g_{res}(α_{ν_1(p)}))$$

since $[K_2(g_{res}(α)) : K_2] ≤ ν_1(p)$ and $\tilde{ν}_2(g_{res}(α)) = 1/ν_1(p)$. Let $β$ be any representative of $φ(α + m_1^{ν_1})$. Since $g_{res}$ induces $φ_{res}$, we can write

$$0 + m_2^{ν_2} = φ(f(α) + m_1^{ν_1})$$

$$= φ(α + m_1^{ν_1})^{ν_1(p)} + ... + φ(a_1 + m_1^{ν_1})φ(α + m_1^{ν_1}) + φ(a_0 + m_1^{ν_1})$$

$$= g_{res}(f)(β) + m_2^{ν_2}.$$ 

This shows that $g_{res}(f)(β)$ is in $m_2^{ν_2}$ and

$$ν_2(g_{res}(f)(β)) ≥ ν_2 > M(R_1)ν_1(p)ν_2(p).$$

We claim that there exists an index $i_0$ satisfying $\tilde{ν}_2(β - g_{res}(α_{i_0})) > M(R_1)$. If $\tilde{ν}_2(β - g_{res}(α_{i_0})) ≤ M(R_1)$ for all $i$, then

$$\tilde{ν}_2(g_{res}(f)(β)) = \tilde{ν}_2\left(\prod_i (β - g_{res}(α_i))\right) ≤ M(R_1)ν_1(p).$$

This shows

$$ν_2(g_{res}(f)(β)) = ν_2(p)\tilde{ν}_2(g_{res}(f)(β)) ≤ M(R_1)ν_1(p)ν_2(p).$$

Thus there is an index $i_0$ satisfying $\tilde{ν}_2(β - g_{res}(α_{i_0})) > M(R_1) = \max\{\tilde{ν}_2(g_{res}(α_1) - g_{res}(α_j)) : j = 2, ..., ν_1(p)\}$.
Lemma 2.11 shows $K_2(\overline{g}_{\text{res}}(\alpha_{i_0})) \subseteq K_2(\beta) \subseteq L_2$. We define an extended homomorphism $g : L_1 \rightarrow L_2$ of $g_{\text{res}} : K_1 \rightarrow K_2$ by the rule $\pi_1 \mapsto g(\pi_1) = \overline{g}_{\text{res}}(\alpha_{i_0})$. $g$ induces the restricted homomorphism from $R_1$ to $R_2$ which is still denoted by $g$. Since $g_{\text{res}}$ induces $\phi_{\text{res}}$ and

$$M(R_1) = \max_{\sigma} \left\{ \tilde{\nu}_2 \left( \sigma(g(\pi_1)) - \beta \right) : \sigma(g(\pi_1)) \neq g(\pi_1) \right\}$$

by Lemma 2.10,(1), $g$ is a $(n_1, n_2)$-lifting of $\phi$. Suppose that $g_1 : R_1 \rightarrow R_2$ is a $(n_1, n_2)$-lifting of $\phi$ other than $g$. Then we have

$$\tilde{\nu}_2(g_1(\pi_1) - \beta) > \max_{\sigma} \left\{ \tilde{\nu}_2 \left( \sigma(g_1(\pi_1)) - \beta \right) : \sigma(g(\pi_1)) \neq g(\pi_1) \right\}$$

by the first condition of the definition of liftings. By the second condition of the definition of liftings and by Theorem 2.2, we obtain the restriction $g_1|_{W(k_1)}$ of $g_1$ to $W(k_1)$ is equal to $g|_{W(k_1)}$. This shows that the monic irreducible polynomial $g_1(f)$ of $g_1(\pi_1)$ is equal to the monic irreducible polynomial $g(f)$ of $g(\pi_1)$ and

$$\left\{ \sigma(g_1(\pi_1)) : \sigma \in \text{Hom}_{K_2}(L_2, L_2^{alg}) \right\} = \left\{ \sigma(g(\pi_1)) : \sigma \in \text{Hom}_{K_2}(L_2, L_2^{alg}) \right\}.$$

In particular $g_1(\pi_1) = \sigma(g(\pi_1))$ for some $\sigma \in \text{Hom}_{K_2}(L_2, L_2^{alg})$. But since $g_1(\pi_1) \neq g(\pi_1)$, we have the inequalities $\tilde{\nu}_2(g_1(\pi_1) - \beta) > \tilde{\nu}_2(g(\pi_1) - \beta)$ and $\tilde{\nu}_2(g_1(\pi_1) - \beta) < \tilde{\nu}_2(g(\pi_1) - \beta)$ simultaneously. This gives a contradiction. Hence we obtain the uniqueness of the lifting.

When $\phi$ is an isomorphism, so are $\phi_{\text{res}}$ and $g_{\text{res}}$. We obtain $[L_2 : K_2] = [L_1 : K_1]$, and hence, $L_{n_1, n_2}(\phi)$ is also an isomorphism.

We note that the proof of Theorem 2.12 works for any representative $\beta$ of $\phi(\pi_1 + m_1^{n_1})$.

**Example 2.13.**

(1) Let $R_1 = \mathbb{Z}_3[\sqrt{3}]$ and $R_2 = \mathbb{Z}_3[\sqrt{-3}]$. There is no homomorphism between $R_1$ and $R_2$ by Kummer theory. But there is an isomorphism

$$\phi : R_{1,2} = \frac{\mathbb{Z}_3[\sqrt{3}]}{2\mathbb{Z}_3[\sqrt{3}]} \rightarrow R_{2,2} = \frac{\mathbb{Z}_3[\sqrt{-3}]}{2\mathbb{Z}_3[\sqrt{-3}]}$$

given by the rule $a + b\sqrt{3} \mapsto a + b\sqrt{-3}$. Since $\nu_1(3) = \nu_2(3) = 2$ and $M(R_1) = \tilde{\nu}_1(\sqrt{3} - (-\sqrt{3})) = 1/2$, we obtain $M(R_1)\nu_1(3)\nu_2(3) = 2$. Hence the lower bound for $n_2$ in Theorem 2.12 is the best possible in this case. This phenomenon will be generalized in Proposition 3.11.

(2) If we take $R_1 = R_2 = \mathbb{Z}_3[\sqrt{3}]$ and $n_1 = n_2 = 2n$, then $R_1, 2n = R_2, 2n \cong (\mathbb{Z}_3/3^{n}\mathbb{Z}_3)[x]/(x^2 - 3)$. Then $\phi : a + bx \mapsto a + (1 + 3^{n-1})bx = \phi(a + bx)$ defines an isomorphism between $R_1$ and $R_2$. But when $n > 1$, there is no homomorphism $g : R_1 \rightarrow R_2$ which induces $\phi$ since Galois conjugates of $\sqrt{3}$ are $\pm \sqrt{3}$. This shows that in Theorem 2.12, we can not guarantee that the following diagram is commutative:
Remark 2.14. (1) We regard Theorem 2.12 as a generalization of Theorem 2.2.(2). We can restate Theorem 2.2.(2) as follows. For \( \phi : k_1 \rightarrow k_2 \), there exists a unique homomorphism \( g : W(k_1) \rightarrow W(k_2) \) which is characterized by the following property:  
- For any \( x \) in \( W(k_1) \), there exists a representative \( \beta_x \) of \( \phi(x + pW(k_1)) \) which satisfies \( \nu_2(g(x) - \beta_x) = \infty \).

In general, the property above does not hold as is seen in Example 2.13.(2). But we can restate Theorem 2.12 as follows. For \( \phi : R_{1,n_1} \rightarrow R_{2,n_2} \), there exists a unique homomorphism \( g : R_1 \rightarrow R_2 \) which is characterized by the following property:  
- There exists \( N \) depending on \( R_1 \) only such that for any \( x \) in \( R_1 \), there exists a representative \( \beta_x \) of \( \phi(x + m_1^{n_1}) \) which satisfies \( \nu_2(g(x) - \beta_x) > N \).

This follows from Proposition 2.10.(2).

(2) Suppose \( R_1 = R_2 = R \), \( k_1 = k_2 = k \), \( n_1 = n_2 = n \), and \( \nu_1 = \nu_2 = \nu \). When \( k \) is finite and \( \phi \) is an isomorphism, Basarab([6]) claimed that if \( n > M(R)\nu(p) \), Theorem 2.12 should hold by Krasner’s lemma. But it is not correct by Example 2.13.(1). Moreover, there is a gap in the argument with respect to the choice of uniformizer in [6].

In Theorem 2.12, \( M(R)e(R) \) can vary when \( R \) changes. The following lemma will play an important role for bounding \( M(R)e(R) \).

Lemma 2.15. Let \( R \subset S \) be discrete valuation rings and \( S \) a finitely generated \( R \)-module. Suppose \( S = R[\alpha] \) for some \( \alpha \) in \( S \). Let \( f(x) \) in \( R[x] \) be the monic irreducible polynomial of \( \alpha \) over \( R \).

(1) The different \( D_{S/R} \) of \( S/R \) is a principal ideal generated by \( f'(\alpha) \).

(2) Let \( \mathfrak{B} \) be the maximal ideal of \( S \). Let \( v \) be the ramification index of \( S \) over \( R \) and \( \nu_s \) the valuation corresponding to \( S \). Let \( s \) be the power which satisfies \( \mathfrak{B}^s = D_{S/R} \). Then one has  
\[
s = e - 1 \quad \text{if } S \text{ is tamely ramified},
\]
\[
e \leq s \leq e - 1 + \nu_s(e) \quad \text{if } S \text{ is wildly ramified}.
\]

Proof. Chapter 3, Section 2 of [25].

The following theorem can be regarded as a generalized version of Theorem 2.2.(1) for the ramified case.

Theorem 2.16. Let \( \overline{R} \) be a principal Artinian local ring of length \( n \) with perfect residue field \( k \) of characteristic \( p \) and maximal ideal \( \overline{m} \). Here length \( n \) means \( \overline{m}^n = 0 \) and \( \overline{m}^{n-1} \neq 0 \) which is denoted by \( l(\overline{R}) = n \). Suppose that \( \overline{R} \) has no finite subfield as a subring. For any positive integer \( a \), if \( a \) generates an ideal \( \overline{m}^k \), we denote \( k \) by \( \nu(a) \). Suppose \( l(\overline{R}) = n > \nu(p) + \nu(p)\nu(\nu(p)) \).

Then there exists a complete discrete valuation ring of characteristic 0 which has \( \overline{R} \) as its \( n \)-th residue ring. Such a ring is unique up to isomorphism.

Proof. Any principal Artinian local ring is a homomorphic image of a discrete valuation ring. This can be proved by Cohen structure theorem for complete local rings([19]) or, more directly, by the property of CPU-rings([17]). Since the completion of a discrete valuation ring \( R \) has the same \( n \)-th residue ring as that of \( R \),
we may assume that there are complete discrete valuation rings $R_i$ and $R_2$ which have $\overline{R}$ as isomorphic copies of $R_1, n$ and $R_2, n$ respectively. We note that $R_i$ is of characteristic 0 for $i = 1, 2$ since $\overline{R}$ has no finite subfield as a subring. Let $L_i$ and $K_i$ be the fraction fields of $R_i$ and $W(k_i)$ for $i = 1, 2$ respectively. Then by Lemma 2.3, $L_1 = K_1(\alpha)$ where $\alpha = \pi_1$ is a uniformizer of $R_1$. Let $f$ be the monic irreducible polynomial of $\alpha$ over $K_1$. Then one can write

$$f = x^{\nu(p)} + \ldots + a_1 x + a_0 = (x - a_1)(x - a_{\nu(p)})$$

where $\alpha = a_1$. Let $\nu_i$ be the corresponding valuation of $R_i$. We note that $\nu_1(p) = \nu_2(p) = \nu(p)$ since $\overline{R}$ has no finite subfield as a subring. We consider the differentiation $f'$ of $f$. There are two cases.

- **Tame case:** Suppose $L_1/K_1$ is tamely ramified. Hence, $\nu(\nu(p)) = 0$. For all distinct $i$ and $j$, $\nu_i(\alpha_j) = 1/\nu(p)$ and hence $\nu_1(\alpha_i - \alpha_j) \geq 1/\nu(p)$. We obtain

$$\nu_1(f' (\alpha_1)) = \nu_1 \left( \prod_{j \neq 1} (\alpha_1 - \alpha_j) \right) = \sum_{j \neq 1} \nu_1(\alpha_1 - \alpha_j) \geq \frac{\nu(p) - 1}{\nu(p)}.$$

Since

$$\nu_1(f' (\alpha_1)) = \frac{\nu(p) - 1}{\nu(p)}$$

by Lemma 2.15, $\nu_1(\alpha_1 - \alpha_j) = 1/\nu(p) = M(R_1)$ for $j \neq 1$. Hence we have

$$\nu(p) + \nu(p)\nu(\nu(p)) = \nu(p) = M(R_1)\nu(p)^2$$

and Theorem 2.12 finishes the proof.

- **Wild case:** Suppose $L_1/K_1$ is wildly ramified. Since $\nu_i(\alpha_i - \alpha_j) \geq 1/\nu(p)$ for all distinct $i$ and $j$,

$$M(R_1) \leq \nu_1(f' (\alpha_1)) = \frac{\nu(p) - 2}{\nu(p)}$$

by Lemma 2.15, and hence, $M(R_1)\nu(p)^2 \leq \nu(p) + \nu(p)\nu(\nu(p))$. Again Theorem 2.12 finishes the proof.

Note that the notation $\nu(p)$ in Theorem 2.16 is compatible with the previously defined valuation. Suppose that a discrete valuation ring $R$ with valuation $\nu$ and maximal ideal $m$ has $\overline{R}$ as its residue ring. Then $\nu(p)$ is equal to the power of
the maximal ideal generated by $p$, that is, $Rp = \mathfrak{m}^e(p)$ as we noted in the proof of Theorem 2.16.

3. Functoriality

For a prime number $p$, let $\mathcal{C}_p$ be a category consisting of the following data:

- $\text{Ob}(\mathcal{C}_p)$ is the family of absolutely unramified complete discrete valuation rings of mixed characteristic having perfect residue fields of characteristic $p$.
- $\text{Mor}_{\mathcal{C}_p}(R_1, R_2) := \text{Hom}(R_1, R_2)$ for $R_1$ and $R_2$ in $\text{Ob}(\mathcal{C}_p)$.

Let $\mathcal{R}_p$ be a category consisting of the following data:

- $\text{Ob}(\mathcal{R}_p)$ is the family of perfect fields of characteristic $p$.
- $\text{Mor}_{\mathcal{R}_p}(k_1, k_2) := \text{Hom}(k_1, k_2)$ for $k_1$ and $k_2$ in $\text{Ob}(\mathcal{R}_p)$.

Let $\text{Pr} : \mathcal{C}_p \rightarrow \mathcal{R}_p$ be the canonical projection functor. We restate Theorem 2.2 categorically as follows:

**Theorem 3.1.** There exists a functor $L : \mathcal{R}_p \rightarrow \mathcal{C}_p$ which satisfies:

- The composite functor $\text{Pr} \circ L$ is equivalent to the identity functor $\text{Id}_{\mathcal{R}_p}$.
- The composite functor $L \circ \text{Pr}$ is equivalent to the identity functor $\text{Id}_{\mathcal{C}_p}$.

The main purpose of this section is to give a generalized version of Theorem 3.1 for the ramified case. For a prime number $p$ and a positive integer $e$, let $\mathcal{C}_{p,e}$ be a category consisting of the following data:

- $\text{Ob}(\mathcal{C}_{p,e})$ is the family of complete discrete valuation rings of mixed characteristic having perfect residue fields of characteristic $p$ and the ramification index $e$; and
- $\text{Mor}_{\mathcal{C}_{p,e}}(R_1, R_2) := \text{Hom}(R_1, R_2)$ for $R_1$ and $R_2$ in $\text{Ob}(\mathcal{C}_{p,e})$.

Let $\mathcal{R}_{p,e}^n$ be a category consisting of the following data:

- For $n \leq e$, $\text{Ob}(\mathcal{R}_{p,e}^n)$ is the family of principal Artinian local rings $\mathcal{R}$ of length $n$ with perfect residue fields of characteristic $p$, and for $n > e$, $\text{Ob}(\mathcal{R}_{p,e}^n)$ is the family of principal Artinian local rings $\mathcal{R}$ of length $n$ with perfect residue fields of characteristic $p$ such that $p \in \mathfrak{m}^e \setminus \mathfrak{m}^{e+1}$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{R}$; and
- $\text{Mor}_{\mathcal{R}_{p,e}^n}(\mathcal{R}_1, \mathcal{R}_2) := \text{Hom}(\mathcal{R}_1, \mathcal{R}_2)$ for $\mathcal{R}_1$ and $\mathcal{R}_2$ in $\text{Ob}(\mathcal{R}_{p,e}^n)$.

Note that for $e_1, e_2 \geq 1$ and for $n \leq e_1, e_2$, two categories $\mathcal{R}_{p,e_1}^n, \mathcal{R}_{p,e_2}^n$ are the same. For each $m > n$, let $\text{Pr}_n : \mathcal{C}_{p,e} \rightarrow \mathcal{R}_{p,e}^n$ and $\text{Pr}_m : \mathcal{R}_{p,e}^n \rightarrow \mathcal{R}_{p,e}^m$ be the canonical projection functors respectively.

**Definition 3.2.** Fix a prime number $p$ and a positive integer $e$.

1. We say that the category $\mathcal{C}_{p,e}$ is $n$-liftable if there is a functor $L : \mathcal{R}_{p,e}^n \rightarrow \mathcal{C}_{p,e}$ which satisfies the following:
   - $(\text{Pr}_n \circ L)(\mathcal{R}) \cong \mathcal{R}$ for each $\mathcal{R}$ in $\text{Ob}(\mathcal{R}_{p,e})$.
   - $\text{Pr}_1 \circ L$ is equivalent to $\text{Pr}_1^+$.
   - $L \circ \text{Pr}_n$ is equivalent to $\text{Id}_{\mathcal{C}_{p,e}}$.

   We say that $L$ is a $n$-th lifting functor of $\mathcal{C}_{p,e}$.

2. The lifting number for $\mathcal{C}_{p,e}$ is the smallest positive integer $n$ such that $\mathcal{C}_{p,e}$ is $n$-liftable. If there is no such $n$, we define the lifting number for $\mathcal{C}_{p,e}$ to be $\infty$. 

Remark 3.3.  (1) In Definition 3.2, the restriction of \( L \) to \( \text{Iso}(R_n) \) is a surjective group homomorphism from \( \text{Iso}(R_n) \) to \( \text{Iso}(R) \) for each \( R \in \text{Ob}(C_{p_e}) \).

(2) Suppose that there is a \( n \)-th lifting functor \( L : \mathcal{R}_{p,e}^n \to C_{p,e} \). For any \( \mathcal{R} \) in \( \text{Ob}(\mathcal{R}_{p,e}) \), up to isomorphism, \( L(\mathcal{R}) \) is a unique object in \( \text{Ob}(C_{p,e}) \) which has \( \mathcal{R} \) as its \( n \)-th residue ring. Suppose that \( R \in \text{Ob}(C_{p,e}) \) has \( \mathcal{R} \) as its \( n \)-th residue ring. Since \( L \circ \text{Pr}_n \) is equivalent to the identity functor \( \text{Id}_{C_{p,e}} \), \( R = \text{Id}_{C_{p,e}}(R) \) is isomorphic to \( (L \circ \text{Pr}_n)(R) = L(\mathcal{R}) \).

(3) The lifting number for \( C_p \) is 1 by Theorem 3.1. We will see that the lifting number for \( C_{p,e} \) is always larger than \( e \) whenever \( e > 1 \) in Corollary 3.17. For \( n \geq e \), we have that a functor \( L_{n+1} = L_n \circ \text{Pr}^{n+1}_n \) is a \((n+1)\)-th lifting functor of \( C_{p,e} \) for any \( n \)-th lifting functor \( L_n : \mathcal{R}_{p,e}^n \to C_{p,e} \). For \( \mathcal{R} \) in \( \text{Ob}(\mathcal{R}_{p,e}^{n+1}) \), there exists a ring \( R \in \text{Ob}(C_{p,e}) \) which satisfies \( \text{Pr}_{n+1}(R) = \mathcal{R} \) as noted in the proof of Theorem 2.16. Since there is a unique object in \( \text{Ob}(C_{p,e}) \) which has \( \text{Pr}_n(R) \) as its \( n \)-th residue ring by Remark 3.3(2), we have

\[
(\text{Pr}_n \circ L_{n+1})(\mathcal{R}) = \text{Pr}_n \circ (L_n \circ \text{Pr}^{n+1}_n)(\mathcal{R}) = \text{Pr}_{n+1}(R) = \mathcal{R}.
\]

\[
\text{Pr}_1 \circ L_{n+1} = (\text{Pr}_1 \circ L_n) \circ \text{Pr}^{n+1}_n \text{ is equivalent to } \text{Pr}_1 \circ \text{Pr}^{n+1}_n = \text{Pr}^{n+1}_n \text{ and } \text{Pr}_{n+1} \circ L_n = (L_n \circ \text{Pr}^{n+1}_n) \circ \text{Pr}_{n+1} = L_n \circ \text{Pr}_n \text{ is equivalent to } \text{Id}_{C_{p,e}}.
\]

Proposition 3.4. For \( 1 \leq i \leq 3 \), let \( R_i \) be a complete discrete valuation ring of characteristic 0 with perfect residue field of characteristic \( p \). Let \( m_i \) be the maximal ideal of \( R_i \) generated by \( \pi_i \) and \( v_i \) corresponding valuation of \( R_i \). For \( \phi^{1,2} : R_{1,n_1} \to R_{2,n_2} \) and \( \phi^{2,3} : R_{2,n_2} \to R_{3,n_3} \), suppose that there are liftings \( g^{1,2} : R_1 \to R_2 \) and \( g^{2,3} : R_2 \to R_3 \) of \( \phi^{1,2} \) and \( \phi^{2,3} \) respectively. If \( v_1(p) = v_2(p) \), then \( g = g^{2,3} \circ g^{1,2} \) is a lifting of \( \phi^{2,3} \circ \phi^{1,2} \). Moreover, \( g \) is a unique lifting of \( \phi^{2,3} \circ \phi^{1,2} \) when \( n_3 > M(R_2) \circ v_2(p) v_3(p) \) and \( n_2 > M(R_1) \circ v_1(p) v_2(p) \).

Proof. By Lemma 2.9, \( M(R_1) \) is equal to \( M(R_2) \), say \( M \). Since \( g^{1,2} \) is a lifting of \( \phi^{1,2} \), there is a representative \( \beta_1 \) of \( \phi^{1,2}(\pi_1 + m_1^{n_1}) \) such that \( \bar{v}_2(g^{1,2}(\pi_1)) = \beta_1 > M \). We note that \( \beta_1 \) is a uniformizer of \( R_2 \). Since \( g^{2,3} \) is a lifting of \( \phi^{2,3} \), there is a representative \( \beta_2 \) of \( \phi^{2,3}(\pi_1 + m_1^{n_1}) = \phi^{2,3}(\beta_1 + m_2^{n_2}) \) such that \( \bar{v}_2(g^{2,3}(\beta_1)) = \beta_2 > M \). If we write \( g^{1,2}(\pi_1) = \beta_1 + x_M \) where \( \bar{v}_2(x_M) > M \), then \( g(\pi_1) = g^{2,3}(g^{1,2}(\pi_1)) = g^{2,3}(\beta_1 + x_M) \). Since \( \bar{v}_3(g^{2,3}(\beta_1)) = \beta_2 > M \) and \( \bar{v}_3(g^{2,3}(x_M)) = \bar{v}_2(x_M) > M \),

\[
\bar{v}_3(g(\pi_1) - \beta_2) = \bar{v}_3(g^{2,3}(\beta_1) + \beta_2 + g^{2,3}(x_M)) > M.
\]

The equality \( \phi^{2,3} \circ \phi^{1,2} \circ \text{pr}_{1,1} = \text{pr}_{3,1} \circ g \) follows directly from \( g = g^{2,3} \circ g^{1,2} \).

By Definition 2.7 and Proposition 2.10, \( g \) is a lifting of \( \phi^{2,3} \circ \phi^{1,2} \).

When \( n_3 > M(R_2) \circ v_2(p) v_3(p) = M(R_1) \circ v_1(p) v_2(p) \) and \( n_2 > M(R_1) \circ v_1(p) v_2(p) \), \( g \) is a unique lifting of \( \phi^{2,3} \circ \phi^{1,2} \) by Theorem 2.12.

\[ \square \]

Corollary 3.5. Let \( e \geq 1 \) and \( R \in \text{Ob}(C_{p,e}) \). Suppose \( n > M(R) \circ v(p)^2 \). Let \( \text{pr}^{n,n} \circ \text{Iso}(R) : \text{Iso}(R) \to \text{Iso}(R_n) \) be the natural projection map. Then there exists a surjective group homomorphism

\[
L_n : \text{Iso}(R_n) \to \text{Iso}(R)
\]

which satisfies \( L_n \circ (\text{pr}^{n,n} \circ \text{Iso}(R)) = \text{Id}_{\text{Iso}(R)} \).
Corollary 3.6. Let $R_1$ be in $\text{Ob}(\mathcal{C}_{p,e_1})$ and $R_2$ in $\text{Ob}(\mathcal{C}_{p,e_2})$. Suppose $n_2 > M(R_1)\nu_1(p)\nu_2(p)$ and $\text{Hom}(R_{1,n_1}, R_{2,n_2})$ is not empty. Then $\text{Hom}(R_{1,n_1}, R_{2,n_2})$ and $\text{Hom}(R_1, R_2)$ are right ISO($R_{1,n_1}$)-sets and there exists a surjective ISO($R_{1,n_1}$)-map
\[
L_{n_1,n_2} : \text{Hom}(R_{1,n_1}, R_{2,n_2}) \longrightarrow \text{Hom}(R_1, R_2)
\]
such that
\[
L_{n_1,n_2} \circ \text{pr}^{n_1,n_2} = \text{Id}_{\text{Hom}(R_1, R_2)}
\]
where $\text{pr}^{n_1,n_2} : \text{Hom}(R_1, R_2) \longrightarrow \text{Hom}(R_{1,n_1}, R_{2,n_2})$ is the natural projection map.

Proof. It is clear that $\text{Hom}(R_{1,n_1}, R_{2,n_2})$ is a right ISO($R_{1,n_1}$)-set. By Lemma 2.5, we have $n_1 > M(R_1)\nu_1(p)^2$. Since $\text{Hom}(R_1, R_2)$ is a right ISO($R_1$)-set, $\text{Hom}(R_1, R_2)$ is a right ISO($R_{1,n_1}$)-set via $L_n : \text{ISO}(R_n) \longrightarrow \text{ISO}(R)$ by Corollary 3.5. Moreover, by Proposition 3.4, the lifting map $L_{n_1,n_2}$ is a ISO($R_{1,n_1}$)-map. 

Lemma 3.7. Let $R \subset S$ be discrete valuation rings and $S$ a finitely generated $R$-module. The discriminant $D_{S/R}$ of $S/R$ is equal to the norm $\text{Norm}(D_{S/R})$ of the different $D_{S/R}$ of $S/R$.

Proof. Chapter 3, Section 2 of [25].

Even though next corollary directly follows from Corollary 3.6, we state here because it is useful for numerical calculations.

Corollary 3.8. Let $R_1$ and $R_2$ be complete discrete valuation rings of characteristic 0 with perfect residue fields of characteristic $p$. Let $\mathfrak{m}_i$ be the maximal ideal of $R_i$ generated by $\pi_i$ and $\nu_i$ corresponding valuation of $R_i$ for $i = 1, 2$. Let $L_i$ and $K_i$ be the fraction fields of $R_i$ and $W(k_i)$ for $i = 1, 2$ respectively. Suppose $\text{Hom}(R_{1,n_1}, R_{2,n_2})$ is not empty. Then there is a surjective ISO($R_{1,n_1}$)-map
\[
L_{n_1,n_2} : \text{Hom}(R_{1,n_1}, R_{2,n_2}) \longrightarrow \text{Hom}(R_1, R_2)
\]
such that
\[
L_{n_1,n_2} \circ \text{pr}^{n_1,n_2} = \text{Id}_{\text{Hom}(R_1, R_2)}
\]
if one of the following holds:

- $L_1/K_1$ is a Galois extension.
  - $i$ is the least number such that the $i$-th ramification group $G_i$ of $\text{Gal}(L_1/K_1)$ vanishes.
  - $n_2 > \nu_2(p)i$.
- $n_2 > \nu_2(p) + \nu_2(p)\nu_1(\nu_1(p))$.
- $n_2 > \nu_2(D_{R_i/W(k_i)})$ where $D_{R_i/W(k_i)}$ is the discriminant of $R_i/W(k_i)$.

Proof. We recall that $G_i$ is defined by $G_i = \{ \sigma \in \text{Gal}(L_1/K_1) : \nu_i(\sigma(\pi_i) - \pi_1) \geq i + 1 \}$. Then $i$ is equal to $M(R_1)\nu_1(p)$.

- By Lemma 2.15, one can obtain
  \[
  M(R_1) \leq \tilde{\nu}_1(f'(\alpha)) - \frac{\nu_1(p) - 2}{\nu_1(p)}
  \leq \frac{\nu_1(p) - 1 + \nu_1(\nu_1(p))}{\nu_1(p)} - \frac{\nu_1(p) - 2}{\nu_1(p)}
  \leq \frac{1 + \nu_1(\nu_1(p))}{\nu_1(p)}.
  \]
as in the proof of Theorem 2.16. Hence,
\[ M(R_1)\nu_1(p)\nu_2(p) \leq \nu_2(p) + \nu_2(p)\nu_1(\nu_1(p)). \]

- Let \( D_{R_1/W(k_1)} \) be the different of \( R_1/W(k_1) \). Then one can obtain that
\[ \nu_2(D_{R_1/W(k_1)}) = \nu_2(p)\tilde{\nu}_1(D_{R_1/W(k_1)}) \]
\[ = \nu_2(p)\nu_1(p)\tilde{\nu}_1(D_{R_1/W(k_1)}) \]
\[ = \nu_2(p)\nu_1(p)\tilde{\nu}_1(f'(\pi_1)) \]
\[ \geq \nu_2(p)\nu_1(p)M(R_1). \]

The second equality follows from the fact that \( \tilde{\nu}_i \) is normalized, the third equality follows from Lemma 3.7 and the fourth equality follows from Lemma 2.15 where \( f \) is the monic irreducible polynomial of \( \pi_1 \) over \( K_1 \).

\[ \square \]

**Theorem 3.9.** The lifting number for \( C_{p,e} \) is finite. More precisely, \( C_{p,e} \) is \( (e + ev(e) + 1) \)-liftable. Here \( \nu(e) \) denotes the exponent \( n \) such that \( e \) generates an ideal \( m^n \) of \( R \) in \( \text{Ob}(C_{p,e}) \) where \( m \) denotes the maximal ideal of \( R \). \( \nu(e) \) depends only on the prime number \( p \) and the ramification index \( e \), in particular \( \nu(e) \) is independent of the choice of \( R \) in \( \text{Ob}(C_{p,e}) \).

**Proof.** Suppose \( n \) is bigger than \( e + ev(e) \). For any \( \overline{R}, \overline{R}_1 \) and \( \overline{R}_2 \) in \( \text{Ob}(R^n_{p,e}) \), by Theorem 2.16, we define \( L_n(\overline{R}) \) to be a unique ring \( R \) in \( \text{Ob}(C_{p,e}) \) which satisfies \( \text{Pr}_n(R) = \overline{R} \). As in the proof of Theorem 2.16, \( e + ev(e) \geq M(R)e^2 \). By Theorem 2.12, for any \( \phi : \overline{R}_1 \rightarrow \overline{R}_2 \), there exists a unique \( n \)-th lifting map \( L(\phi) : L(\overline{R}_1) \rightarrow L(\overline{R}_2) \), and hence we obtain a functor \( L_n : R^n_{p,e} \rightarrow C_{p,e} \) by Proposition 3.4. By Definition 2.7, \( L_n \) is a lifting functor.

\[ \square \]

**Remark 3.10.** For a fixed absolutely unramified valued field \( K \), \( M(L)e(L) \) can be arbitrarily large when extension degrees \( [L : K] \) vary. For example, we can take \( L = \mathbb{Q}_p(\zeta_{p^{-n}}) \) and \( K = \mathbb{Q}_p \). More generally, if \( L \) runs through subfields of a deeply ramified extension of a local field \( K \) (see [10] for the definition of deeply ramified extensions), then \( M(L)e(L) \) can be arbitrarily large. But Lemma 2.15 and the proof of Theorem 2.16 show that \( M(L)e(L) \) must be bounded if we fix \( [L : K] \). Hence we deduce the finiteness of the lifting number for \( C_{p,e} \).

Example 2.13.(1) can be generalized as follows.

**Proposition 3.11.** Let \( R_1/W(k) \) and \( R_2/W(k) \) be totally ramified extensions of degree \( e \). Then \( R_{1,e} \) is isomorphic to \( R_{2,e} \) as \( W(k) \)-algebras.

**Proof.** Let \( \pi_i \) be a uniformizer of \( R_i \) and \( \nu_i \) the valuation corresponding to \( R_i \) for \( i = 1, 2 \). By the theory of totally ramified extensions (see Chapter 2 of [24] for example), the monic irreducible polynomial \( f_i \) of \( \pi_i \) over \( W(k) \) is an Eisenstein polynomial for \( i = 1, 2 \). If we write \( f_i = x^e + a_{i,e-1}x^{e-1} + \ldots + a_{i,1}x + a_{i,0} \), then
\[ \nu_i(p) = \nu_i(a_{i,0}) = e \text{ and } \nu_i(a_{i,j}) \geq e \text{ for } i = 1, 2 \text{ and } j = 1, 2, \ldots, e - 1. \] This shows

\[
R_{i,e} = \frac{W(k)[\pi_i]}{(\pi_i)^e} = \frac{W(k)[x]}{(p_j, f_i)} = k[x] = \frac{k[x]}{(x^e + \ldots + a_{i,1}x + a_{i,0})} = \frac{k[x]}{(x^e)},
\]

and hence, \( R_{1,e} \) is isomorphic to \( R_{2,e} \) as \( W(k) \)-algebras.

Now we focus on tamely ramified extensions. For the tame case, we can calculate the lifting number.

**Lemma 3.12.** For a perfect field \( k \) of characteristic \( p \), let \( K \) be the fraction field of the Witt ring \( W(k) \) of \( k \). For a positive integer \( e \) prime to \( p \), suppose that there is a prime divisor \( l \) of \( e \) such that \( \zeta_n \) is in \( k^\times \) and \( \zeta_{n+1} \) is not in \( k^\times \) for some \( n \). Then there are two totally ramified extensions \( L_1 \) and \( L_2 \) of degree \( e \) over \( K \) which are not isomorphic over \( \mathbb{Q} \).

**Proof.** \( \zeta_n \) is in \( W(k)^\times \) and \( \zeta_{n+1} \) is not in \( W(k)^\times \) by Hensel’s lemma. Then \( L_1 = K(\sqrt[p]{\theta}) \) and \( L_2 = K(\sqrt[p]{\theta^{[n]}}) \) are totally ramified extensions of degree \( e \) over \( K \). Suppose that there is an isomorphism \( \sigma : L_2 \to L_1 \). Since Galois conjugates of \( \sqrt[p]{\theta} \) and \( \zeta_{n+1} \) over \( \mathbb{Q} \) are \( \sqrt[p]{\theta_{i,j}} \) and \( \zeta_{n+1}^j \) for each \( i \) and \( j \) where \( j \) is prime to \( e \) respectively,

\[
\sigma \left( \sqrt[p]{\theta_{i,j}} \right) = \sigma \left( \sqrt[p]{\theta_{i,j}^{n+1}} \right) = \sqrt[p]{\theta_{i,j}^{k}}
\]

for some \( k \) prime to \( l \). In particular, \( L_1 \) contains both \( \sqrt[p]{\theta} \) and \( \sqrt[p]{\theta_{i,j}^{n+1}} \), and hence, \( \zeta_{n+1} \) is in \( L_1 \). This is a contradiction since \( L_1/K \) is totally ramified.

**Corollary 3.13.** Suppose that \( p \) does not divide \( e \) and \( e > 1 \). Then \( e + 1 \) is the lifting number for \( C_{p,e} \).

**Proof.** Since \( \nu(p) = 0, e + 1 = e + 1 \) by Theorem 3.9, \( C_{p,e} \) is \( (e+1) \)-liftable. Let \( \mathbb{F}_p \) be the prime field of \( p \) elements. Let \( K \) be the fraction field of the Witt ring \( W(k) \) of \( k = \mathbb{F}_p(\zeta_e) \). By Lemma 3.12, there are two totally ramified extensions \( L_1 \) and \( L_2 \) of degree \( e \) over \( K \) such that there is no isomorphism between \( L_1 \) and \( L_2 \). If \( C_{p,e} \) is \( e \)-liftable, \( L_1 \) and \( L_2 \) are isomorphic over \( K \) by Proposition 3.11 and it is a contradiction.

**Remark 3.14.** Proposition 3.11 and Corollary 3.13 show the difference between the unramified case and the tamely ramified case. We can regard the absolutely unramified valued fields of mixed characteristic as the absolutely tamely ramified valued fields having the ramification index \( e = 1 \). If we apply the formula \( e + 1 \) in Corollary 3.13 to \( C_p \), the lifting number for \( C_p \) should be \( 1 + 1 = 2 \). But the argument in the proof of Corollary 3.13 does not work for \( C_p \). For an absolutely unramified complete discrete valued field \( K \), there is a unique totally ramified extension of
degree 1 over \( K \), that is, \( K \) itself. Hence the fact that the lifting number for \( C_p \) is 1 does not disagree with Corollary 3.13.

For the wild case, we have the following example. Let \( R_1 = \mathbb{Z}_2[\sqrt{2}] \) and \( R_2 = \mathbb{Z}_2[\sqrt{10}] \). There is no homomorphism between \( R_1 \) and \( R_2 \) by Kummer theory. But there is an isomorphism between \( R_{1,6} \) and \( R_{2,6} \) since

\[
R_{1,6} = \frac{\mathbb{Z}_2[\sqrt{2}]}{(\sqrt{2})} \cong \frac{\mathbb{Z}_2[x]}{(x^2 - 2, 8)}
\]

\[
\cong \frac{\mathbb{Z}_2[\sqrt{10}]}{(\sqrt{10})} = R_{2,6}.
\]

This shows that the lifting number for \( C_{2,2} \) is \( 2 + 2\nu(2) + 1 = 7 > \nu(2) \) by Theorem 3.9. In general, we have the lower bound \( e + 1 \) of the lifting number for the wild case. For proving this, we need the following lemma.

**Lemma 3.15.** For a perfect field \( k \) of characteristic \( p \), let \( K \) be the fraction field of the Witt ring \( W(k) \) of \( k \). Let \( e \) be a positive integer divided by \( p \). Then there are two totally ramified extensions \( L_1 \) and \( L_2 \) of degree \( e \) over \( K \) which are not isomorphic over \( \mathbb{Q} \).

**Proof.** We write \( e = sp^r \) for some positive integers \( s \) and \( r \) where \( s \) is prime to \( p \). Let \( \mathbb{Q}_\infty/\mathbb{Q} \) be the cyclotomic \( \mathbb{Z}_p \)-extension, in particular \( \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p \).

Let \( M_r \) be a unique subfield of \( \mathbb{Q}_\infty \) such that \( [M_r : \mathbb{Q}] = p^r \). By the theory of cyclotomic fields (see for example Chapter 1 of [25]), the Galois extension \( M_r/\mathbb{Q} \) is totally ramified at the place above \( p \). Let \( \alpha \) be a uniformizer of \( M_r \) corresponding to the place above \( p \). Since, \( M_r/\mathbb{Q} \) is a Galois extension, \( M_r = \mathbb{Q}(\alpha) = \mathbb{Q}(\sigma(\alpha)) \) for any embedding \( \sigma \). We fix an embedding \( \mathbb{Q}_p^{alg} \subset K^{alg} \).

Let \( L_1 = K(p^{1/p^r}) = K(p^{1/p^r}, p^{1/p^r}) \) and \( L_2 = K(p^{1/p^r}, \alpha) \). Then \( L_1 \) and \( L_2 \) are totally ramified extensions of degree \( e \) over \( K \). If there is an isomorphism \( \sigma : L_2 \rightarrow L_1 \), \( L_1 \) contains both \( \sigma(\alpha) \) and \( p^{1/p^r} \). Since \( \mathbb{Q}(\alpha) = \mathbb{Q}(\sigma(\alpha)) \), \( K(\sigma(\alpha)) = K(\alpha) \) is contained in \( L_1 \). We note that \([K(p^{1/p^r}, \alpha) : K(p^{1/p^r})]\) divides \([K(\alpha) : K] = p^r\) since \( K(\alpha)/K \) is a Galois extension. Since

\[
s = [L_1 : K(p^{1/p^r})] = [L_1 : K(p^{1/p^r}, \alpha)] [K(p^{1/p^r}, \alpha) : K(p^{1/p^r})],
\]

\([K(p^{1/p^r}, \alpha) : K(p^{1/p^r})]\) divides \( s \). Hence we obtain \([K(p^{1/p^r}, \alpha) : K(p^{1/p^r})] = \gcd(s, p^r) = 1\). This shows \( K(p^{1/p^r}) = K(\alpha) \) since \([K(p^{1/p^r}) : K] = [K(\alpha) : K] = 1\).

This is a contradiction, and hence, \( L_1 \) and \( L_2 \) are not isomorphic.

**Proposition 3.16.** Let \( p \) be a prime number and \( e \) be a natural number divided by \( p \). Then the lifting number for \( C_{p,e} \) is bigger than \( e \).

**Proof.** By Lemma 3.15, there are two totally ramified extensions \( L_1 \) and \( L_2 \) of degree \( e \) over \( \mathbb{Q}_p \) such that there is no isomorphism over \( \mathbb{Q}_p \) between \( L_1 \) and \( L_2 \). If \( C_{p,e} \) is \( e \)-liftable, \( L_1 \) and \( L_2 \) are isomorphic over \( \mathbb{Q}_p \) by Proposition 3.11 and it is a contradiction. Hence, the lifting number for \( C_{p,e} \) is bigger than \( e \).

**Corollary 3.17.** The lifting number for \( C_{p,e} \) is bigger than \( e \) whenever \( e > 1 \).
Although we have the lower bound \( e + 1 \) and the upper bound \( e + e \nu(e) + 1 \) of the lifting number for \( \mathcal{C}_{p,e} \), we have no clue to calculate the lifting number explicitly for the wild case.

**Question 3.18.** What is the lifting number for the wild case?

**Remark 3.19.** When \( e > 1 \), for a lifting functor \( L_n : \mathcal{R}_{p,e}^n \longrightarrow \mathcal{C}_{p,e} \) and \( \mathcal{R} \) in \( \mathcal{R}_{p,e}^n \), any complete discrete valuation ring \( R \) which has \( \mathcal{R} \) as its \( n \)-th residue ring necessarily has ramification index \( e \) and equal to \( L_n(\mathcal{R}) \) by Corollary 3.17. But for the lifting functor \( L : \mathcal{R}_p \longrightarrow \mathcal{C}_p \) in Theorem 3.1, there is no information on ramification indices which have the same residue field. For example, \( R_1 = \mathbb{Z}_3[\sqrt{3}] \) and \( R_2 = \mathbb{Z}_3[\sqrt{3}] \) have the same residue field, but their ramification indices are different.

For a fixed set \( \mathcal{N} = \{ n_e \in \mathbb{N} \}_{e \in \mathcal{N}} \), if we try to make a unified lifting functor from \( \mathcal{R} \mathcal{N} := \bigcup_e \mathcal{R}_{p,e}^n \) to \( \mathcal{C} := \bigcup_e \mathcal{C}_{p,e} \), we can not apply our method to get such a functor since \( M(R_e) \) is unbounded as varying \( e \) and \( R_e \in \text{Ob}(\mathcal{C}_{p,e}) \). But we have a unified functor for a finite set of ramification indices.

**Corollary 3.20.** For a finite set \( \{ e_1, \ldots, e_s \} \) of ramification indices, there is a finite set of natural numbers \( \{ n_1, \ldots, n_s \} \) such that there exists a lifting functor

\[
L : \bigcup_{1 \leq k \leq s} \mathcal{R}_{p,e_k}^{n_k} \longrightarrow \bigcup_{1 \leq k \leq s} \mathcal{C}_{p,e_k}^{n_k}.
\]

**Proof.** Follows from Lemma 2.5, Proposition 3.4, Corollary 3.8 and Theorem 3.9. \( \square \)

### 4. Ax-Kochen-Ershov Principle for Finitely Ramified Valued Fields

Our main goal in this section is to prove a strengthened version of Basarab’s result on the AKE-principle for finitely ramified valued fields in the case of perfect residue fields. Firstly, we quickly review the basic results in model theory of valued fields, concentrating on the AKE-principle. We take the language of valued fields, which consists of three types of sorts for valuation fields, residue fields, and value groups. Let \( \mathcal{L}_K = \{ +, -; 0, 1 \} \) be the ring language for valued fields, \( \mathcal{L}_k = \{ +', -, '; 0', 1' \} \) be the ring language for residue fields, and \( \mathcal{L}_T = \{ +; 0^*; < \} \) be the ordered group language for value groups. Let \( \mathcal{L}_{\text{val}} = \mathcal{L}_R \cup \mathcal{L}_k \cup \mathcal{L}_T \) be the language of valued fields. Next, we consider an extended language of \( \mathcal{L}_{\text{val}} \) by adding the ring languages for the \( n \)-th residue rings. For each \( n \leq 1 \), let \( \mathcal{L}_{R_n} = \{ +^n, -^n; 0_n, 1_n \} \) be the ring language for the \( n \)-th residue ring. For \( n = 1 \), we identify \( \mathcal{L}_{R_1} = \mathcal{L}_k \).

We get an extended language \( \mathcal{L}_{\text{val},R} = \mathcal{L}_{\text{val}} \cup \bigcup_{n \geq 1} \mathcal{L}_{R_n} \) for valued fields. Let \( (K_1, \nu_1, k_1, \Gamma_1) \) and \( (K_2, \nu_2, k_2, \Gamma_2) \) be valued fields, and let \( R_{1,n} \) and \( R_{2,n} \) be the \( n \)-th residue rings of \( (K_1, \nu_1) \) and \( (K_2, \nu_2) \) respectively. We say \( (K_1, \nu_1) \) and \( (K_2, \nu_2) \) are elementarily equivalent if they are elementarily equivalent in \( \mathcal{L}_{\text{val}} \). If \( (K_1, \nu_1) \) and \( (K_2, \nu_2) \) are elementarily equivalent, then they are elementarily equivalent in \( \mathcal{L}_{\text{val},R} \) because the \( n \)-th residue rings are interpretable in \( \mathcal{L}_{\text{val}} \). For \( (K_1, \nu_1) \) and \( (K_2, \nu_2) \) which are elementarily equivalent, it necessarily implies that

- \( k_1 \) and \( k_2 \) are elementarily equivalent in \( \mathcal{L}_k \);
- \( \Gamma_1 \) and \( \Gamma_2 \) are elementarily equivalent in \( \mathcal{L}_T \); and
- \( R_{1,n} \) and \( R_{2,n} \) are elementarily equivalent in \( \mathcal{L}_{R_n} \) for each \( n \leq 1 \).
Ax and Kochen in [4], and Ershov in [14] proved the fact that these conditions on the residue fields and the value groups imply elementary equivalence for unramified valued fields:

**Theorem 4.1.** [4, 14](The Ax-Kochen-Ershov principle) Let \((K_1, \nu_1, k_1, \Gamma_1)\) and \((K_2, \nu_2, k_2, \Gamma_2)\) be unramified henselian valued fields of characteristic zero. 

\[ K_1 \equiv K_2 \text{ if and only if } k_1 \equiv k_2 \text{ and } \Gamma_1 \equiv \Gamma_2. \]


**Theorem 4.2.** [6] Let \((K_1, \nu_1, k_1, \Gamma_1)\) and \((K_2, \nu_2, k_2, \Gamma_2)\) be henselian valued fields of mixed characteristic having finite absolute ramification indices. The following are equivalent:

1. \(K_1 \equiv K_2.\)
2. \(R_{1,n} \equiv R_{2,n}\) for each \(n \leq 1\) and \(\Gamma_1 \equiv \Gamma_2.\)

Next we review on the coarse valuations. For the coarse valuations, we refer to [13, 20, 27, 30].

**Remark/Definition 4.3.** [27] Suppose \((K, \nu, k, \Gamma)\) has the finite absolute ramification index so that the value group has the minimum positive element, and let \(\pi\) be a uniformizer so that \(\nu(\pi)\) is the smallest positive element in \(\Gamma.\) Let \(\Gamma^o\) be the convex subgroup of \(\Gamma\) generated by \(\nu(\pi)\) and \(\nu: K \setminus \{0\} \to \Gamma^o\) be a map sending \(x(\neq 0) \in K\) to \(\nu(x) + \Gamma^o \in \Gamma/\Gamma^o.\) The map \(\nu\) is a valuation, called the coarse valuation. The residue field \(K^o,\) called the core field of \((K, \nu),\) of \((K, \nu)\) forms a valued field equipped with a valuation \(\nu^o\) induced from \(\nu\) and the value groups \(\Gamma^o.\)

More precisely, the valuation \(\nu^o\) is defined as follows: Let \(\text{pr}_\nu: R_\nu \to K^o\) be the canonical projection map and let \(x \in R_\nu.\) If \(x^o := \text{pr}_\nu(x) \in K^o \setminus \{0\},\) then \(\nu^o(x^o) := \nu(x).\) And \(x^o = 0 \in K^o\) if and only if \(\nu(x) > \gamma\) for all \(\gamma \in \Gamma^o.\)

**Lemma 4.4.**

1. Let \(R_\nu, R_\nu^o,\) and \(R_\nu^o\) be the valuation rings of \((K, \nu),\) \((K, \nu^o),\) and \((K^o, \nu^o)\) respectively. Then \((\text{pr}_\nu)^{-1}(R_\nu^o) = R_\nu.\)
2. Let \(R_n\) and \(R_n^o\) be the \(n\)-th residue rings of \((K, \nu)\) and \((K^o, \nu^o)\) respectively. Then there is a canonical isomorphism \(\theta_n : R_n \to R_n^o\) such that \(\text{pr}_n^o \circ (\text{pr}_n |_{R_n}) = \theta_n \circ \text{pr}_n,\) where \(\text{pr}_n : R_n \to R_n\) and \(\text{pr}_n^o : R_n^o \to R_n^o\) are the canonical projection map.
3. If \((K, \nu)\) is henselian, then \((K, \nu^o)\) is henselian.
4. If \((K, \nu)\) is \(k_1\)-saturated, then \((K^o, \nu^o)\) is complete.

**Proof.** (1) Note that \(R_\nu := \{x \in K | \nu(x) \geq 0\} = \{x \in K | \nu(x) \geq \gamma\) for some \(\gamma \in \Gamma^o.\})\) Let \(x \in R_\nu\) be such that \(\text{pr}_\nu(x) := x^o \in R_\nu^o,\) that is, \(\nu^o(x^o) \in \Gamma^o \geq 0.\) If \(x^o = 0,\) \(\nu(x) > \gamma\) for all \(\gamma \in \Gamma^o\) and \(x \in R_\nu.\) If \(x^o \neq 0,\) then \(\nu^o(x^o) = \nu(x) \geq \gamma\) in \(\Gamma^o,\) and hence \(\nu(x) \geq 0\) in \(\Gamma.\) Thus \(x \in R_\nu.\) Therefore, for \(x \in R_\nu, x \in R_\nu\) if and only if \(x^o \in R_\nu^o.\)

(2) Note that each \(\theta_n\) is induced from \(\text{pr}_\nu |_{R_n} : R_\nu \to R_\nu^o.\) It is easy to see that each \(\theta_n\) is surjective. To show that \(\theta_n\) is injective, it is enough to show that \(\nu(x) \geq n\) if and only if \(\nu^o(x^o) \geq n\) for \(x \in R_\nu.\) It clearly comes from the definition of \(\nu^o\) in (1).

(3)-(4) Section 5 of [20].

**Proposition 4.5.** Let \((K_1, \nu_1, \Gamma_1)\) and \((K_2, \nu_2, \Gamma_2)\) be valued fields. Let \(R_{1,n}\) and \(R_{2,n}\) be the \(n\)-th residue rings of \(K_1\) and \(K_2\) respectively. Suppose
Then there are $\aleph_1$-saturated elementary extensions $(K'_1, \nu'_1, \Gamma'_1)$ and $(K'_2, \nu'_2, \Gamma'_2)$ of $K_1$ and $K_2$ such that

- $R'_{1,n} \cong R'_{2,n}$ for $n \geq 1$;
- $\Gamma'_1 \cong \Gamma'_2$.

Then there are $\aleph_1$-saturated elementary extensions $(K'_1, \nu'_1, \Gamma'_1)$ and $(K'_2, \nu'_2, \Gamma'_2)$ of $K_1$ and $K_2$ such that

- $R'_{1,n} \cong R'_{2,n}$ for $n \geq 1$;
- $\Gamma'_1 \cong \Gamma'_2$.

, where $R'_{1,n}$ and $R'_{2,n}$ are the $n$-th residue rings of $K'_1$ and $K'_2$ respectively.

Proof. We inductively construct chains of valued fields $(K'_1, \Gamma'_1)_{i \in \omega}$ and $(K'_2, \Gamma'_2)_{i \in \omega}$, and isomorphisms $\xi^i_j : R'_{1,j} \to R'_{2,j}$ for $0 < i$ and $1 \leq j \leq i$, where $R'_{1,j}$ and $R'_{2,j}$ are the $j$-th residue rings of $K'_1$ and $K'_2$ respectively such that for $i \in \omega$,

1. $K'_1 \preceq K'_{i+1}$ and $K'_2 \prec K'_{i+1}$;
2. $\xi^i_j \subset \xi^{i+1}_j$ for $1 \leq j \leq i$;
3. $\Gamma'_1 \cong \Gamma'_2$.

Recall the Keisler-Shelah isomorphism theorem:

**Theorem 4.6.** (Keisler-Shelah Isomorphism Theorem) Let $\mathcal{M}$ and $\mathcal{N}$ be two first order structures. If $\mathcal{M} \cong \mathcal{N}$, then there is a ultrafilter $\mathcal{U}$ on an infinite set $I$ such that

$$\mathcal{M}^\mathcal{U} \cong \mathcal{N}^\mathcal{U},$$

where $\mathcal{M}^\mathcal{U}$ and $\mathcal{N}^\mathcal{U}$ are the ultrapowers of $\mathcal{M}$ and $\mathcal{N}$ with respect to $\mathcal{U}$.


Since $\Gamma_1 \cong \Gamma_2$, by Theorem 4.6, there is an ultrafilter $\mathcal{U}_0$ such that $\Gamma_1^{\mathcal{U}_0} \cong \Gamma_2^{\mathcal{U}_0}$. Set $(K_1^{\mathcal{U}_0}, \nu_1^{\mathcal{U}_0}) = (K_{1,0}, \nu_{1,0})$ and $(K_2^{\mathcal{U}_0}, \nu_2^{\mathcal{U}_0}) = (K_{2,0}, \nu_{2,0})$.

Assume we construct sequences of valued fields $(K'_i, \nu'_i, \Gamma'_i)_{i \leq m}$ and $(K'_2, \nu'_2, \Gamma'_2)_{i \leq m}$ with isomorphisms $\xi^i_j : R'_{i,j} \to R'_{2,j}$ for $1 \leq j \leq i \leq m$ satisfying the conditions (1), (2), and (3), for $i \leq m$. Since $R'_{i,m+1} \equiv R'_{2,m+1}$, from Theorem 4.6, there is an ultrafilter $\mathcal{U}$ such that $\xi^{i+1}_j : R'_{i+1,j} \equiv R'_{2,j}$ for each $j \leq m$, where $R'_{i,j} \equiv (R'_{i,j})^\mathcal{U}$. Then the sequences of valued fields $(K'_1, \nu'_1, \Gamma'_1)_{i \leq m+1}$ and $(K'_2, \nu'_2, \Gamma'_2)_{i \leq m+1}$ with isomorphisms $\xi^i_j : R'_{i,j} \to R'_{2,j}$ for $1 \leq j \leq i \leq m+1$ satisfying (1), (2), and (3), for $i \leq m+1$. By induction, we get chains of valued fields, $(K'_i)_{i \geq 0}$ and $(K'_2)_{i \geq 0}$ with isomorphisms $\xi^i_n : R'_{i,n} \to R'_{2,n}$ for $1 \leq n \leq i$. At last to get $\aleph_1$-saturated valued fields, consider an ultrapower $K_1^{\omega : \mathcal{U}} := (K_1^\omega)^{\mathcal{U}'}$ with respect to a nonprincipal $\mathcal{U}'$ on $\omega$ for $k = 1, 2$, and $K_2^{\omega : \mathcal{U}} := (K_2^\omega)^{\mathcal{U}'}$ are desired valued fields.

By combining Theorem 4.1 and Lemma 4.4, Proposition 4.5, we reduce the problem on elementary equivalence between henselian valued fields of mixed characteristic having finite ramification indices to the problem on isometricity between complete discrete valued fields of mixed characteristic whose the $n$-th residue rings are isomorphic for each $n \geq 1$. Now we improve Theorem 4.2.
Theorem 4.7. Let \((K_1, \nu_1, k_1, \Gamma_1)\) and \((K_2, \nu_2, k_2, \Gamma_2)\) be henselian valued fields of mixed characteristic with finite ramification indices. Suppose \(k_1\) and \(k_2\) are perfect fields of characteristic \(p > 0\). For \(n \geq 1\), let \(R_{1,n}\) and \(R_{2,n}\) be the \(n\)-th residue rings of \(K_1\) and \(K_2\) respectively. Let \(n_0 > \max\{e_{\nu_1}(p)(1 + e_{\nu_1}(e_{\nu_1}(p)), e_{\nu_2}(p)(1 + e_{\nu_2}(e_{\nu_2}(p)))\}. The following are equivalent:

1. \(K_1 \equiv K_2;\)
2. \(\Gamma_1 \equiv \Gamma_2\) and \(R_{1,n} \equiv R_{2,n}\) for each \(n \leq 1;\) and
3. \(\Gamma_1 \equiv \Gamma_2\) and \(R_{1,n} \equiv R_{2,n}\).

Proof. Let \((K_1, \nu_1, k_1, \Gamma_1)\) and \((K_2, \nu_2, k_2, \Gamma_2)\) be henselian valued fields of characteristic zero with the finite ramification indices so that \(\Gamma_1\) and \(\Gamma_2\) have the minimum positive elements. Suppose \(k_1\) and \(k_2\) are perfect fields of characteristic \(p > 0\). It is easy to check \(1 \Rightarrow 2 \Rightarrow 3\). We show \(3 \Rightarrow 1\).

\(3 \Rightarrow 1\). Suppose \(R_{1,n_0} \equiv R_{2,n_0}\) and \(\Gamma_1 \equiv \Gamma_2\). By the proof of Proposition 4.5, we may assume that \(R_{1,n_0} \equiv R_{2,n_0}\) and \(\Gamma_1 \equiv \Gamma_2\), and that \((K_1, \nu_1, \Gamma_1)\) and \((K_2, \nu_2, \Gamma_2)\) are \(\xi_1\)-saturated. Consider the coarse valuations \(\nu_1\) and \(\nu_2\) of \(\nu_1\) and \(\nu_2\) respectively and the valued fields \((K_1, \nu_1, \Gamma_1/\Gamma_1^n)\) and \((K_2, \nu_2, \Gamma_2/\Gamma_2^n)\), where \(\Gamma_i^n\) is the convex subgroup of \(\Gamma_i\) generated by the minimum positive element in \(\Gamma_i\) for \(i = 1, 2\). Since \((K_1, \nu_1)\) and \((K_2, \nu_2)\) are \(\xi_1\)-saturated, by Lemma 4.4.(4), the core fields \((K_1^\circ, \nu_1^\circ)\) and \((K_2^\circ, \nu_2^\circ)\) are complete discrete valued fields, where \(\nu_1^\circ\) and \(\nu_2^\circ\) are the valuation induced from \(\nu_1\) and \(\nu_2\) respectively. Since the \(n_0\)-th residue rings of \((K_1, \nu_1)\) and \((K_2, \nu_2)\) are isomorphic, by Lemma 4.4.(2), the \(n_0\)-th residue rings of \((K_1^\circ, \nu_1^\circ)\) and \((K_2^\circ, \nu_2^\circ)\) are isomorphic.

By Theorem 2.12 and the proof of Theorem 2.16, \(K_1^\circ\) and \(K_2^\circ\) are isomorphic. Since \(\Gamma_1^\circ \equiv \Gamma_2^\circ, \Gamma_1/\Gamma_1^n \equiv \Gamma_2/\Gamma_2^n\). Therefore by Theorem 4.1, \((K_1, \nu_1) \equiv (K_2, \nu_2)\). To get that \((K_1, \nu_1) \equiv (K_2, \nu_2)\), it is enough to show that the valuation rings \(R_{1\nu_1}\) of \((K_1, \nu_1)\) and \(R_{2\nu_2}\) of \((K_2, \nu_2)\) are definable in \((K_1, \nu_1)\) and \((K_2, \nu_2)\) by the same formula. We need the following lemma on a definability of valuation ring in the ring language.

Lemma 4.8. Let \((K, \nu)\) be a complete field of characteristic zero. Suppose the residue field \(k\) is perfect and has prime characteristic \(p\). Then the valuation ring \(R_{\nu}\) of \((K, \nu)\) is definable by the formula

\[
\phi_q(x) \equiv \exists y \ y^q = 1 + px^q
\]

for some \(q > 0\) such that \(p \ n \ q > e_{\nu}(p)\). For example, we can take \(q\) as \(p^l + 1\) for sufficiently large \(l > 0\).

Proof. See [8].

Take \(l > 0\) large enough so that \(q := p^l + 1 > \max\{e_{\nu_1}(p), e_{\nu_2}(p)\}\). By Lemma 4.8, \(\phi_q(x)\) defines the residue rings \(R_{\nu_1^\circ}\) and \(R_{\nu_2^\circ}\) of \((K_1^\circ, \nu_1^\circ)\) and \((K_2^\circ, \nu_2^\circ)\). By Lemma 4.4.(1), the valuation rings \(R_{\nu_1}\) and \(R_{\nu_2}\) are definable by the same formula in \((K_1, \nu_1)\) and \((K_2, \nu_2)\) so that \((K_1, \nu_1) \equiv (K_2, \nu_2)\).

We give some corollaries of Theorem 4.7. At first, we improve the result in [5] on a decidability of henselian valued fields of finite absolute ramification indices in the case of perfect residue fields.

Corollary 4.9. Let \((K, \nu, \Gamma)\) be a henselian valued field of mixed characteristic having finite absolute ramification index and the perfect residue field. Let \(R_{n}\) be the \(n\)-th residue ring of \((K, \nu)\) for each \(n \geq 1\). Let \(n_0 > e_{\nu}(p)(1 + e_{\nu}(e_{\nu}(p)))\). Let
Th($K,ν$) be the theory of ($K,ν$), Th($Γ$) be the theory of $Γ$, and Th($R_n$) be the theory of $R_n$. The following are equivalent:

1. Th($K,ν$) is decidable.
2. Th($Γ$) is decidable, and Th($R_n$) is decidable for each $n \geq 1$.
3. Th($Γ$) is decidable, and Th($R_{n_0}$) is decidable.

Proof. (1) ⇔ (2) It was already given by Basarab in [5].

(1) ⇔ (3) Let ($K,ν,Γ,k$) be a henselian valued field of mixed characteristic having a perfect residue field $k$. Let $p > 0$ be the characteristic of $k$ and let $e := e_ν(p)$ be the absolute ramification index of ($K,ν$). Suppose $e$ is finite. Consider the following theory $T_{p,e}$ consisting of the following statements, which can be expressed by the first order logic:

- ($K,ν$) is a henselian valued field of characteristic zero;
- $Γ$ is an abelian ordered group having the minimum positive element;
- $k$ is a perfect field of characteristic $p > 0$;
- ($K,ν$) has the absolute ramification index $e$.

By Theorem 4.7, the theory $T_{p,e} \cup$ Th($Γ$) $\cup$ Th($R_{n_0}$) is complete. Thus Th($K,ν$) is decidable if and only if Th($Γ$) and Th($R_{n_0}$) are decidable. □

Thus we get the following results on local fields of mixed characteristic.

Corollary 4.10. [5][6] Let ($K_1,ν_1$) and ($K_2,ν_2$) be local fields of mixed characteristic.

1. ($K_1,ν_1)$ $≡$ ($K_2,ν_2)$ $⇔$ $f : K_1 \cong K_2$.
2. Th($K_1,ν_1$) is decidable.

Next we recall the following definition introduced in [6]:

Definition 4.11. [6] Let $T$ be the theory of a henselian valued field ($K,ν,Γ$) of mixed characteristic having finite absolute ramification index $e$. Let $λ(T) \in \mathbb{N}∪\{∞\}$ be defined as the smallest positive integer $n$ (if such a number exists) such that for every henselian valued field ($K',ν',Γ'$) of mixed characteristic having the same absolute ramification index as ($K,ν,Γ$), the following are equivalent:

1. ($K',ν',Γ'$) $|$ $T$.
2. $Γ$ $≡$ $Γ'$ and the $n$-th residue rings of ($K,ν$) and ($K',ν'$) are elementarily equivalent.

Otherwise, $λ(T) = ∞$. 

Question 4.12. [6] Let $T$ be the theory of a henselian valued field of mixed characteristic having finite absolute ramification index. Is $λ(T)$ finite?

It was proved that $λ(T) < ∞$ for the theories $T$ of local fields of mixed characteristic in [6](but the statement and its proof are incorrect as we remarked in Section 2). We give a positive answer when residue fields are perfect.

Corollary 4.13. Let ($K,ν$) be a henselian valued field of mixed characteristic having finite absolute ramification index with perfect residue field. Let $T$ be the theory of ($K,ν$). Then $λ(T) ≤ e_ν(p)(1 + e_ν(e_ν(p)) + 1$.

We compute explicitly $λ(T)$ for the theories $T$ of some tamely ramified valued fields. We say that an abelian group $G$ is $e$-divisible (respectively, uniquely $e$-divisible) when the multiplication by $e$ map, $e : G \rightarrow G$ is surjective (respectively, bijective). We denote the unit group of a ring $R$ by $R^×$.
**Lemma 4.14.** Let \((K, W(k), m, k)\) be an absolutely unramified complete discrete valued field of mixed characteristic \((0, p)\) with perfect residue field \(k\). Suppose that \(k^\times\) is \(e\)-divisible for a positive integer \(e\) prime to \(p\).

1. If \(\zeta_e\) is contained in \(W(k)\), then there exists a unique totally tamely ramified extension \(L\) of degree \(e\) over \(K\).

2. If \(\zeta_e\) is not contained in \(W(k)\), then there exists a unique totally tamely ramified extension \(L\) of degree \(e\) over \(K\) up to \(K\)-isomorphism.

**Proof.** Let \(S'\) be the group of nonzero Teichmüller representatives of \(W(k)\) and \(U(n) = 1 + m^n\) the \(n\)-th principal unit group of \(W(k)\) for each \(n \geq 1\). Since

\[
U(n) = \ker \left( W(k)^\times \to (\frac{W(k)}{m^n})^\times \right)
\]

and

\[
W(k) = \lim_{\substack{\longrightarrow \\\ \ \\ n}} \left( \frac{W(k)}{m^n} \right),
\]

we have

\[
W(k)^\times = \lim_{\substack{\longrightarrow \\\ \ \\ n}} \left( \frac{W(k)}{m^n} \right)^\times = \lim_{\substack{\longrightarrow \\\ \ \\ n}} \left( \frac{W(k)^\times}{U(n)} \right).
\]

Since \(W(k)^\times = S' \times U(1)\), we obtain

\[
U(1) = \lim_{\substack{\longrightarrow \\\ \ \\ n}} \left( \frac{U(1)}{U(n)} \right).
\]

Since \(U(n)/U(n+1) \cong k\) for each \(n \geq 1\), a short exact sequence

\[
0 \to U(n+1) \to U(n) \to U(n)/U(n+1) \to 0
\]

shows that \(U(1)/U(n)\) is a \(p\)-group, and hence, uniquely \(e\)-divisible for each \(n\). Hence, \(U(1)\) is uniquely \(e\)-divisible and \(W(k)^\times = S' \times U(1)\) is \(e\)-divisible since \(k^\times \cong S'\).

1. Suppose that \(\zeta_e\) is contained in \(S'\). Then there is a unique totally tamely ramified extension of degree \(e\) over \(K\) by Kummer theory since

\[
\frac{K^\times}{(K^\times)^e} = \frac{p^{pe} \times W(k)^\times}{p^{pe} \times (W(k)^\times)} \cong \frac{\mathbb{Z}}{e\mathbb{Z}}.
\]

2. Suppose that \(\zeta_e\) is not contained in \(S'\). For a totally tamely ramified extension \(L\) of degree \(e\) over \(K\), there is \(u\) in \(W(k)^\times\) such that \(L = K(\sqrt[p]{u})\) by the theory of tamely ramified extensions (see Chapter 2 of [24] for example). Since \(W(k)^\times\) is \(e\)-divisible, there is \(v\) in \(W(k)^\times\) such that \(v^e = u\). Hence, \(\sqrt[p]{u} = \sqrt[p]{v^e} = \sqrt[p]{v^e}\zeta_e^i\) for some \(i\). This shows that \(L = K(\sqrt[p]{u}) = K(\sqrt[p]{v^e})\) is isomorphic to \(K(\sqrt[p]{v})\) over \(K\) since the irreducible polynomial of \(\sqrt[p]{v}\) over \(K\) is \(x^p - p\).

\(\square\)

**Proposition 4.15.** Let \((K, \nu, \Gamma, k)\) be a finitely tamely ramified henselian valued field of mixed characteristic with perfect residue field. Let \(e \geq 2\) be the absolute ramification index of \((K, \nu)\). Let \(T\) be the theory of \((K, \nu)\).

1. If \(k^\times\) is \(e\)-divisible, then \(\lambda(T) = 1\).

2. If there is a prime divisor \(t\) of \(e\) such that \(\zeta_n^\nu \in k^\times\) and \(\zeta_{n+1}^\nu \notin k^\times\) for some \(n\), and \(\Gamma\) is a \(\mathbb{Z}\)-group, then \(\lambda(T) = e + 1\).
Proof. (1) Suppose $k^\infty$ is $e$-divisible. Let $(K', \nu', \Gamma', k')$ be a henselian valued field of mixed characteristic with a perfect residue field having absolute ramification index $e$. Suppose $k \equiv k'$ and $\Gamma \equiv \Gamma'$. By the proof of Proposition 4.5, we may assume that $k \equiv k', \Gamma \equiv \Gamma'$, and both $K$ and $K'$ are $\mathbb{N}_1$-saturated. Consider the core fields $(K^\circ, \nu^\circ, k^\circ)$ and $((K')^\circ, (\nu')^\circ, (k')^\circ)$ of $(K, \nu)$ and $(K', \nu')$ respectively. Since $k^\infty$ is $e$-divisible, so is $(k^\circ)^\infty$. Then by Lemma 4.14, $(K^\circ, \nu^\circ) \cong ((K')^\circ, (\nu')^\circ)$. By the proof of Theorem 4.7, we have $(K, \nu) \equiv (K', \nu')$. Thus $\lambda(T) = 1$.

(2) Suppose there is a prime divisor $l$ of $e$ and a natural number $n$ such that $\zeta_n \in k^\infty$ and $\zeta_{n+1} \notin k^\infty$, and $\Gamma \equiv \mathbb{Z}$. Let $p$ be the characteristic of $k$ and $e$ be the absolute ramification index of $(K, \nu)$. Let $T_{p,e}$ be the theory introduced in the proof of Corollary 4.9. Set $T_0 = T_{p,e} \cup \text{Th}(\Gamma) \cup \text{Th}(R_\nu)$. Consider the following theories:

- $T_1 = T_0 \cup \{\exists x(x^e - p = 0)\}$;
- $T_2 = T_0 \cup \{\exists y((x^e - py = 0) \land \Phi_{p'}(y) = 0)\}$;
- $T_3 = T_0 \cup \{\neg\exists x((x^e - p = 0) \land \Phi_{p'}(y) = 0)\}$,

where $\Phi_{p'}(X) \in \mathbb{Z}[X]$ is the $p'$-th cyclotomic polynomial. By the proof of Lemma 3.12, we have

- each pairwise union of $T_1$, $T_2$, and $T_3$ is inconsistent;
- $T_1$ and $T_2$ are consistent;

and for a finitely tamely ramified henselian valued field $(K', \nu', \Gamma', k')$ of mixed characteristic $(0, p)$ having absolute ramification index $e$, if $k' \equiv k$, $\Gamma' \equiv \Gamma$, and $R_{\nu'} \equiv R_\nu$ for the $e$-th residue ring $R_{\nu'}$ of $(K', \nu')$, then there is $i \in \{1, 2, 3\}$

$(K', \nu') \models T_i$.

Since $(K, \nu) \models T_0$ and there are at least two different complete theories containing $T_0$, we have $\lambda(T) \geq e + 1$. By Corollary 4.13, we conclude that $\lambda(T) = e + 1$.

For some wild cases, we have a lower bound for $\lambda(T)$.

**Proposition 4.16.** Let $p$ be a prime number and $e$ be a positive integer divided by $p$. Let $(K, \nu, \Gamma, k)$ be a finitely ramified henselian valued field of mixed characteristic $(0, p)$ having absolute ramification index $e \geq 2$. Suppose $k$ is perfect and $\Gamma$ is $\mathbb{Z}$-group. Then $\lambda(T) \geq e + 1$ for $T = \text{Th}(K, \nu)$.

**Proof.** The proof is similar to the proof of Proposition 4.15. Let $T_{p,e}$ and $T_0$ be the theory introduced in the proof of Proposition 4.15. We write $e = sp^r$ for positive integers $s$ and $r$ where $s$ is prime to $p$. Let $\alpha \in \mathbb{Q}^{alg}$ be in the proof of Lemma 3.15 such that $\alpha$ is a uniformizer of $M_\nu$ corresponding to the place above $p$ where $M_\nu = \mathbb{Q}(\alpha)$ is the $r$-th subfield of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of degree $p^r$ over $\mathbb{Q}$. Let $f(X)$ be the minimal polynomial of $\alpha$ over $\mathbb{Q}$. Consider the following theories:

- $T_1 = T_0 \cup \{\exists x(x^e - p = 0)\}$;
- $T_2 = T_0 \cup \{\exists x(x^e - p = 0), \exists x(f(x) = 0)\}$;
- $T_3 = T_0 \cup \{\neg\exists x((x^e - p = 0) \land \exists x(f(x) = 0))\}$

By the proof of Lemma 3.15, we have

- each pairwise union of $T_1$, $T_2$, and $T_3$ is not consistent;
- $T_1$ and $T_2$ are consistent;
and for a ramified henselian valued field \((K', \nu', \Gamma', k')\) of mixed characteristic \((0, p)\) having absolute ramification index \(e\), if \(k' \equiv k\), \(\Gamma' \equiv \Gamma\), and \(R'_e \equiv R_e\) for the \(e\)-th residue ring \(R'_e\) of \((K', \nu')\), then there is \(i \in \{1, 2, 3\}\) such that 
\[
(K', \nu') \models T_i.
\]
Since \((K, \nu) \models T_0\) and there are at least two different complete theories containing \(T_0\), we have \(\lambda(T) \geq e + 1\).

We list some special cases of Proposition 4.15 and Proposition 4.16. For a positive integer \(s\), we say that \(s^\infty\) divides \([k : F_p]\) if there is a subfield \(k_n\) of \(k\) such that \([k_n : F_p]\) is finite and \(s^n\) divides \([k_n : F_p]\) for each \(n \geq 1\).

**Corollary 4.17.** Suppose \((K, \nu, \Gamma, k)\) is a finitely ramified henselian valued field of mixed characteristic \((0, p)\) having absolute ramification index \(e \geq 2\). Let \(T\) be the theory of \(K\). Let \(s\) be the order of the group \(\mu_e \cap k^\times\) where \(\mu_e\) is the group generated by \(\zeta_e\).

Case \(p \nmid e\).

- \(\lambda(T) = 1\) when \(k = k^{alg}\);
- \(\lambda(T) = 1\) when \(K\) is a subfield of \(\mathbb{C}_p\) and \(s^\infty\) divides \([k : F_p]\);
- \(\lambda(T) = e + 1\) when \(K\) is a subfield of \(\mathbb{C}_p\) and \(s^\infty\) does not divide \([k : F_p]\).

Case \(p | e\).

- \(\lambda(T) \geq e + 1\) when \(K\) is a subfield of \(\mathbb{C}_p\).

**References**


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