Intersecting valuation rings in the Zariski-Riemann space of a field

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Motivation from Birational Algebra

Problem: Find a framework for classifying/describing/studying integrally closed domains when viewed **as intersections of valuation rings**.

Outline of talk:

(a) The Zariski-Riemann space as a locally ringed spectral space
(b) Affine subsets of the Zariski-Riemann space and Prüfer domains
(c) Geometric criteria for Prüfer intersections
(d) Patch topology and intersection representations
(e) Quadratic transforms of regular local rings
(f) Overrings of two-dimensional Noetherian domains

The Zariski-Riemann space

Let F be a field and D be a subring of F (e.g., D is prime subring of F).

$\mathfrak{X} =$ projective limit of the projective models of F/D.

$\mathfrak{X}$ is the Zariski-Riemann space of valuation rings of F containing D with the topology inherited from the projective limit

Basis of the topology is given by sets of form

$$\mathcal{U}(x_1, \ldots, x_n) = \{V \in \mathfrak{X} : x_1, \ldots, x_n \in V\}.$$

$\mathfrak{X}$ is quasicompact (Zariski, 1944).

Why Zariski cared: finite resolving system can replace an infinite one.

Theorem (Dobbs-Federer-Fontana, Heubo-Kwegna, Kuhlmann,...)

$\mathfrak{X}$ is a spectral space.

Proof: $\mathfrak{X} \simeq$ prime spectrum of the Kronecker function ring of F/D.

Why the name?

Nagata, 1962:

> *The name of Riemann is added because Zariski called this space 'Riemann manifold' in the case of a projective variety, though this is not a Riemann manifold in the usual sense in differential geometry. The writer believes that the motivation for the terminology came from the case of a curve. Anyway, the notion has nearly nothing to do with Riemann, hence the name 'Zariski space' is seemingly preferable. But, unfortunately, the term 'Zariski space' has been used in a different meaning [e.g., a Noetherian topological space for which every nonempty closed irreducible subset has a unique generic point]. Therefore we are proposing the name 'Zariski-Riemann space'.*

A subset of $\mathfrak{X}$ is qcpt and open **iff** it is a finite union of sets of form

$$\mathcal{U}(x_1, \ldots, x_n) = \{V \in \mathfrak{X} : x_1, \ldots, x_n \in V\}.$$

Inverse topology: closed basis of **qcpt open sets**

...Also called the dual topology.

Patch topology: basis of **qcpt opens and their complements**

...compact, Hausdorff and zero-dimensional.

Every cpct Hausdorff zero-dim'l space having an isolated point arises as the patch space of a Zariski-Riemann space (even an affine scheme).

Sheaf structures

Intersection presheaf: $\mathcal{O}(U) := \bigcap_{V \in U} V$ (U open in $\mathfrak{X}$).

Zariski topology $\Rightarrow$ $\mathcal{O}$ is a sheaf.

Inverse or patch topologies: must sheafify.

Zariski topology: Main virtue: compatible with morphisms into schemes.

Patch topology: Result is a "Pierce" sheaf: ringed Stone space with indecomposable stalks. $\mathcal{O}(\mathfrak{X})$ is a complicated ring with many idempotents. The geometry here is really algebra is disguise: The global sections functor is exact, hence cohomologically trivial.

Inverse topology: Ring of global sections can contain idempotents. Disadvantage: Stalks need not be valuation rings (but are a kind of pullback ring).

Affine subsets of $\mathfrak{X}$

$\mathfrak{X}$ = Zariski-Riemann space of F/D with the Zariski topology.

"Non-degenerate" case: When is $Z \to \text{Spec}(\bigcap_{V \in Z} V)$ an isomorphism?

I.e., which subspaces of $\mathfrak{X}$ are affine schemes?

Proposition.
$Z \subseteq \mathfrak{X}$ is an affine scheme **iff**
Z is inverse closed and $A = \bigcap_{V \in Z} V$ is a Prüfer domain with q.f. F.

An integral domain A is a Prüfer domain if A_M is a valuation domain for each maximal ideal M of A.

So to detect when an intersection of valuation rings is Prüfer is the same as detecting when a subspace of $\mathfrak{X}$ is an affine scheme.

Prüfer domains

Prüfer domains are a fundamental object of study in non-Noetherian commutative ring theory and multiplicative ideal theory.

≥ 100 characterizations (ideal-theoretic, module-theoretic, homological)

Examples:

- Finite intersection of valuation rings
- Ring of entire functions
- The ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$
- Real holomorphy rings

Criterion for affiness

$D =$ subring of the field F.

Recall: $A = \bigcap_{V \in Z} V$ is a Prüfer domain **iff** $Z \subseteq$ affine scheme in $\mathfrak{X}$

Theorem. (O.–, 2014)

$A = \bigcap_{V \in Z} V$ is Prüfer with quotient field F and torsion Picard group	$\Longleftarrow$	image of each D-morphism $Z \to \mathbb{P}^1_D$ is in a distinguished affine open subset of $\mathbb{P}^1_D$

$\mathbb{P}^1_D$ is covered by many affine open subsets.

What conditions guarantee $Z \to \mathbb{P}^1_D$ lands in one of them?

Applications

Three classical independent results about Prüfer intersections can now be reduced to prime avoidance arguments...

Corollary. (Nagata) $A = \bigcap_{V \in Z} V$ is Prüfer when Z is finite.

Proof.

Let $\phi : Z \to \mathbb{P}^1_D$ be a morphism.

Its image is finite.

Prime Avoidance $\Rightarrow \exists f \in D[T_0, T_1]$ not in any prime ideal in Im ϕ.

$\{P \in \mathbb{P}^1_D : f \notin P\}$ is an **affine** open set containing Im ϕ.

So by the theorem, A is Prüfer.

(In fact, f can be chosen to be linear and this implies that A is Bézout.)

Corollary. (Dress, Gilmer, Loper, Roquette, Rush)
$A = \bigcap_{V \in Z} V$ is Prüfer when there exists a nonconstant monic polynomial $f \in A[T]$ that has no root in residue field of any $V \in Z$.

Proof.

Let $\phi : Z \to \mathbb{P}^1_D$ be a morphism.

Let $\overline{f}$ be the homogenization of f.

Then $\{P \in \mathbb{P}^1_D : \overline{f} \notin P\}$ is an **affine** open set containing Im ϕ.

So by the theorem, A is Prüfer with torsion Picard group.

Example: Use $f(X) = X^2 + 1$ to show the real holomorphy ring is Prüfer.

Corollary. (Roitman)

$A = \bigcap_{V \in Z} V$ is Bézout when A contains a field of cardinality $> |Z|$.

Proof.

Let $\phi : Z \to \mathbb{P}^1_D$ be a morphism.

Use the fact that there are more units in A than valuation rings in Z to construct a homogeneous $f \in D[T_0, T_1]$ that is not contained in any prime ideal in the image of ϕ.

Then $\{P \in \mathbb{P}^1_D : f \notin P\}$ is an **affine** open set containing Im ϕ.

So by the theorem, A is Prüfer.

(In fact, f can be chosen to be linear, so A is a Bézout domain.)

Local uniformization

Corollary

$D =$ quasi-excellent local Noetherian domain with quotient field F.

$Z =$ valuation rings dominating D that **don't admit local uniformization**.

If $Z \neq \emptyset$, then $\bigcap_{V \in Z} V$ is a Prüfer domain with torsion Picard group.

So if nonempty, Z lies in an **affine** scheme in $\mathfrak{X}$.

Patch topology

Patch topology: **qcpt opens and their complements** as a basis

Conrad-Temkin, Favre-Jonsson, Finnocchiaro-Fontana-Loper, Huber-Knebusch, Knaf-Kuhlmann, Kuhlmann, O.-, Prestel-Schwartz,...

Patch density: useful for replacing a valuation with a "better" one that behaves the same on a finite set of data

Suppose Z is patch dense in $\mathfrak{X}$ and $V \in \mathfrak{X}$.

$x_1, \ldots, x_n \in V$, $y_1, \ldots, y_m \in \mathfrak{M}_V \implies \exists W \in Z$ with same property

F.-V. Kuhlmann has proved a number of deep theorems for patch density in the space of valuations on a function field.

Patch density is useful for understanding the ideal theory of real holomorphy rings (O-, 2005).

Patch limit points

Theorem. (Finnocchiaro, Fontana, Loper, 2013)
V is a patch limit point of $Z \subseteq \mathfrak{X}$ **iff**

$$V = \{x \in F : \{V \in Z : x \in V\} \in \mathcal{F}\},$$

where $\mathcal{F}$ is a nonprincipal ultrafilter on Z.

Theorem (O-, 2014)
Suppose Spec(D) is a Noetherian space.
$V \in \{\text{patch closure of } Z\}$ **iff** in every projective model X of F/D,

V maps to a generic point for a subset of the image of Z in X.

So patch limit points arise from generic points in the projective models.

Example: Accounting for all valuations

A non-constructive "construction" of all valuations in a function field...

Theorem (Kuhlmann, 2004)

$F/k =$ function field.

The set of DVRs in $\mathfrak{X}$ whose residue fields are finite over k is patch dense.

$\Longrightarrow$ every valuation ring in $\mathfrak{X}$ is an ultrafilter limit of such DVRs.

These DVRs arise from prime ideals in the generic formal fiber of local rings of closed points in projective models of (Heinzer-Rotthaus-Sally, 1993).

"Taking completions" then "taking ultrafilter limits" give all valuations.

Application: Intersection representations

A subset Z of $\mathfrak{X}$ represents a ring R if $R = \bigcap_{V \in Z} V$.

Theorem (O-, 2015)

Every patch closed rep. of a ring contains a minimal patch closed rep.

isolated point $\iff$ irredundant member of the representation

($\Longrightarrow$ consequences for uniqueness and existence of representations).

Theorem

(A, M) = integrally closed local domain with $\text{End}(M) = A$.

$\Longrightarrow$ A is a val'n domain or $\exists$ perfect dominating representation of A.

Corollary

A = completely integrally closed local domain

$\exists$ dominating rep'n with countably many limit points $\Longrightarrow$ valuation ring.

More Prüfer criteria

Theorem

$(D, \mathfrak{m})$ = local subring of F that is not a field.

Z = set of dominating, rank 1 valuations rings.

$$|\{\text{limit points of } Z\}| < \aleph_0 \cdot |D/\mathfrak{m}| \implies \bigcap_{V \in Z} V \text{ is a Prüfer domain.}$$

Corollary.

$$|\{\text{limit points of } Z\}| = \text{finite} \implies \bigcap_{V \in Z} V \text{ is a Prüfer domain.}$$

Application: "Order holomorphy ring"

Let R = RLR, X = blow-up of Spec(R) at the maximal ideal.
V_x = order valuation ring of $\mathcal{O}_{X,x}$ (x = closed point)

$\implies \bigcap_x V_x$ is a Prüfer domain. (What information does it contain?)

Application: Quadratic Transforms

Theorem. (Heinzer, Loper, O-, Schoutens, Toenskoetter)
R = RLR of dimension > 1; $\{R_i\}$ = sequence of local quadratic transforms.

$$\Longrightarrow \quad \bigcup_i R_i = V \cap T,$$

T = smallest Noetherian overring (it's a localization of one of the R_i)
V = unique patch limit point of the order valuation rings of the R_i's.

Theorem. (Heinzer, O.-, Toenskoetter)
$\exists$ an explicit asymptotic description of V (in particular, rank V = 2).

Application of Prüfer criterion:
V is a localization of the intersection of the order valuation rings.

Reason: The intersection of the order valuation rings is a Prüfer domain.

Application: Overrings of two-dim'l Noetherian domains

Suppose D is a two-dimensional Noetherian domain with q.f. F.

Goal: Describe the integrally closed rings $\bigcap_{V \in Z} V$ between D and F.

Special case: $\exists$ morphism $Z \to \mathbb{P}^1_D$ with "small" fibers.

This is in keeping with the philosophy of understanding intersections of valuation rings when there are not "too many" of them.

Theorem. Suppose $\exists$ morphism $Z \to \mathbb{P}^1_D$ with **Noetherian** fibers. Then

(1) $\exists$ unique strongly irredundant representation of $\mathcal{O}(Z) = \bigcap_{V \in Z} V$.

(2) $\exists$ local classification of the ring $\mathcal{O}(Z)$ (somewhat involved).

Note: This includes all integrally closed Noetherian domains.

Application to Rees valuations

Theorem (O-, Tartarone, 2013)
Suppose

$D =$ two-dimensional regular local ring with regular parameter f.

D is equicharacteristic or has mixed characteristic with f a prime integer.

$A =$ integral closure of a finitely generated D-subalgebra of D_f.

Then distinct height one prime ideals of A lying over the maximal ideal of D are comaximal.

Thus A is "essentially one-fibered."

Question: If f is not a regular parameter, is there a bound on the number of height one primes of A lying over the maximal ideal of D and contained in the same maximal ideal of A? I.e., is each A essentially n-fibered?

"Yes" $\Rightarrow$ nice consequences for not-necessarily-Noetherian overrings of D.

Thank you