Reversely Well-Ordered Valuations on Polynomial Rings in Two Variables

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Preliminaries

- The set of nonzero elements of R is denoted R^*.
- The power product $x_1 \cdots x_n$ is abbreviated as $\mathbf{x}$.
- The polynomial ring $K[x_1, \dots, x_n]$ is abbreviated as $K[\mathbf{x}]$.
- The rational function field $K(x_1, \dots, x_n)$ is abbreviated as $K(\mathbf{x})$.
- A monomial is a product of powers of variables.

Ideal Membership in One Variable

Problem: Given $f, f_1, \dots, f_s \in K[x]$, determine whether

$$f \in I = \langle f_1, \dots, f_s \rangle.$$

Easy solution:

1. Write $I = \langle g \rangle$ where $g = \gcd(f_1, \dots, f_s)$.
2. Divide f by g to get quotient q and remainder r.
3. $f \in I$ if and only if $r = 0$.

Monomial Orders

Definition

A monomial order is a total order on the set of monomials such that

$$\text{(i) } \mathbf{x}^\alpha \geq 1;$$
$$\text{(ii) } \mathbf{x}^\alpha > \mathbf{x}^\beta \Rightarrow \mathbf{x}^\alpha \mathbf{x}^\gamma > \mathbf{x}^\beta \mathbf{x}^\gamma.$$

Example (Lexicographical Order)

$\mathbf{x}^\alpha > \mathbf{x}^\beta \Leftrightarrow$ the first nonzero coordinate of $\alpha - \beta$ is positive.

Example (Graded Lex Order)

Compare total degrees and break ties using lexicographic order.

Example: $x^2y >_{\text{lex}} y^5$, but $x^2y <_{\text{grlex}} y^5$.

Definition

Given a polynomial $f \in K[\mathbf{x}]$, the monomial of f that is larger than all of the other monomials appearing in f is called the **leading monomial**, which we denote $\text{lm}(f)$.

Reduction

Consider the following polynomials:

$$f = x^2y + 4xy - 3y^2, \quad g = 2x + y + 1.$$

We perform one step of dividing f by g via long division (with graded lex):

$$\begin{array}{r|l} & \frac{1}{2}xy \\ \hline 2x + y + 1 & x^2y + 4xy - 3y^2 \\ & x^2y + \frac{1}{2}xy^2 + \frac{1}{2}xy \\ \hline & -\frac{1}{2}xy^2 + \frac{7}{2}xy - 3y^2 \end{array}$$

We denote this by

$$f \xrightarrow{g} -\frac{1}{2}xy^2 + \frac{7}{2}xy - 3y^2.$$

Definition

Given $F = \{f_1, \ldots, f_s\}$, we write $f \xrightarrow{F} h$ whenever there exists a sequence of reductions of the form

$$f \xrightarrow{f_{i_1}} h_1 \xrightarrow{f_{i_2}} h_2 \xrightarrow{f_{i_3}} \cdots \xrightarrow{f_{i_j}} h.$$

Gröbner Bases

Question

Does the ideal

$$\langle f_1, f_2 \rangle = \langle 2x^2y - 5xy^2 + y,\ 2xy^2 - 5y^3 \rangle$$

contain y^2?

Yes!

$$y^2 = y \cdot (2x^2y - 5xy^2 + y) + (-x) \cdot (2xy^2 - 5y^3).$$

Unfortunately,

$$y^2 \overset{\{f_1,f_2\}}{\not\longrightarrow} 0$$

regardless of the choice of monomial order.

Definition (Gröbner Bases)

Let I be a nonzero ideal in $K[\mathbf{x}]$ and let $G \subseteq I$ be a set of nonzero polynomials. Then G is a *Gröbner basis* with respect to the monomial order '$<$' if for all $f \in K[\mathbf{x}]$,

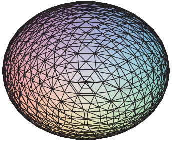
$$f \in I \Leftrightarrow f \xrightarrow{G} 0.$$

Theorem (Buchberger)

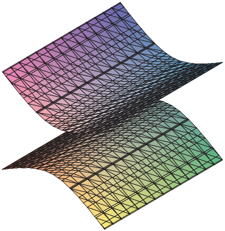
For any monomial order, every nonzero ideal in $K[\mathbf{x}]$ has a finite Gröbner basis with respect to that monomial order.

Intersections of Surfaces

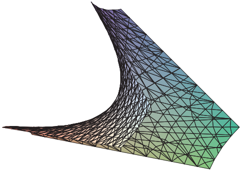
Where do the following three surfaces intersect?



$x^2 + y^2 + z^2 = 4$



$y^2 = z^2 - 1$



$y = xz$

A lexicographic Gröbner basis of $\langle x^2 + y^2 + z^2 - 4, xz - y, y^2 - z^2 + 1 \rangle$ is

$$\{2z^4 - 4z^2 - 1, y^2 - z^2 + 1, x - 2yz^3 + 4yz\}.$$

Valuations

Definition

Given a field extension $L \mid K$ and a totally ordered abelian group $(\Gamma, +, <)$, we say that

$$v : L \to \Gamma \cup \{\infty\}$$

is a K-**valuation on** L if for all $f, g \in L$, the following hold:

(i) $v(f) = \infty$ if and only if $f = 0$;

(ii) $v(fg) = v(f) + v(g)$;

(iii) $v(f + g) \geq \min\{v(f), v(g)\}$;

(iv) If $v(f) = v(g) \neq \infty$, then $\exists! \lambda \in K$ such that $v(f + \lambda g) > v(f)$.

Note that condition (iv) means that K is a field of representatives for the residue field $\mathcal{O}_v / \mathcal{M}_v$.

Valuations from Monomial Orders

Definition

Given a monomial order '$<$' on $K[\mathbf{x}]$, define v to be the K-valuation on $K(\mathbf{x})$ such that for all $f \in K[\mathbf{x}]$,

$$v(f) = -\text{exponent}(\text{lm}(f)).$$

We say v **comes from a monomial order** on $K[\mathbf{x}]$.

Example

Using the lexicographic order,

$$v(x^3y + x^2y^2) = (-3, -1) \in (\mathbb{Z}_{\leq 0})^2.$$

Lemma

Let v be a K-valuation on $K(\mathbf{x})$. If v comes from a monomial order on $K[\mathbf{x}]$, then

$$v(K[\mathbf{x}]^*) \cong (\mathbb{Z}_{\leq 0})^n.$$

What about the converse?

Monomial Orders in Suitable Variables

Definition

Given a K-algebra automorphism $\varphi : K[\mathbf{x}] \to K[\mathbf{x}]$, define v to be the K-valuation on $K(\mathbf{x})$ such that for all $f \in K[\mathbf{x}]$,

$$v(f) = -\text{exponent}(\text{lm}(\varphi(f))).$$

We say v **comes from a monomial order in suitable variables**.

Define $v(f) = -\text{exponent}(\text{lm}(\varphi(f)))$ with lexicographic order, where

$$\begin{array}{rcl} K[x, y] & \xrightarrow{\varphi} & K[u, v] \\ x & \mapsto & u + v \\ y & \mapsto & u - v. \end{array}$$

Then $v(K[x, y]^*) \cong (\mathbb{Z}_{\leq 0})^2$ since

$$\begin{array}{rcccl} v(x - y) & = & -\text{exponent}(\text{lm}(2v)) & = & (0, -1), \\ v(x) & = & -\text{exponent}(\text{lm}(u + v)) & = & (-1, 0). \end{array}$$

Theorem (M_, Sweedler)

Let v be a K-valuation on $K(\mathbf{x})$. Then v comes from a monomial order in suitable variables iff $v(K[\mathbf{x}]^*) \cong (\mathbb{Z}_{\leq 0})^n$.

Value Monoids and Suitable Valuations

Definition

Recall that $v : L \to \Gamma \cup \{\infty\}$ is a K-**valuation on** L if for all $f, g \in L$, the following hold

(i) $v(f) = \infty$ if and only if $f = 0$;

(ii) $v(fg) = v(f) + v(g)$;

(iii) $v(f+g) \geq \min\{v(f), v(g)\}$;

(iv) If $v(f) = v(g) \neq \infty$, then $\exists! \lambda \in K$ such that $v(f + \lambda g) > v(f)$.

Definition

We call $v(K[\mathbf{x}]^*)$ the **value monoid** of v with respect to $K[\mathbf{x}]$. We say that the K-valuation v on $K(\mathbf{x})$ is **suitable relative to** $K[\mathbf{x}]$ if

(v) $\forall f \in K[\mathbf{x}]^* : \; v(f) = 0$ iff $f \in K^*$;

(vi) $v(K[\mathbf{x}]^*)$ is a reversely well-ordered set.

Open Question

Which monoids are of the form $v(K[\mathbf{x}]^*)$ for a K-valuation on $K(\mathbf{x})$ that is suitable relative to $K[\mathbf{x}]$?

Generalized Gröbner Bases

Replacing monomial orders with K-valuations on $K(\mathbf{x})$ that are suitable relative to $K[\mathbf{x}]$, one can still do the following:

- Divide one polynomial by another with respect to the valuation;
- Reduce one polynomial by a finite collection of polynomials with respect to the valuation.

Definition

Let v be a K-valuation on $K(\mathbf{x})$ that is suitable relative to $K[\mathbf{x}]$, and let I be a nonzero ideal of $K[\mathbf{x}]$. Let $G \subseteq I$ be a nonempty set of nonzero polynomials. Then G is a **Gröbner basis** with respect to v if for all $f \in K[\mathbf{x}]^*$,

$$f \in I \Leftrightarrow f \xrightarrow{G} 0.$$

Proposition (Sweedler)

Let v be a K-valuation on $K(\mathbf{x})$ that is suitable relative to $K[\mathbf{x}]$. There is a natural algorithm that will produce a (potentially infinite) Gröbner basis with respect to v.

Generalized Power Series

Definition

Given a function $f : S \to M$, where M is an additive monoid, the support of f, denoted $\mathrm{Supp}(f)$, is the subset of the domain consisting of the elements that are not sent to 0.

Definition

The field of **Hahn power series**, $K((t^{\mathbb{Q}}))$, is the set of all functions from $\mathbb{Q}$ to K with well-ordered support. Addition is defined pointwise and multiplication is defined via convolution. It is often useful to think of such functions $z : \mathbb{Q} \to K$ as series (in the variable t, say) of the form

$$z = \sum_{e \in \mathbb{Q}} z(e)t^e.$$

Example

$$z = t^{-\frac{1}{2}} + t^{-\frac{1}{4}} + t^{-\frac{1}{8}} + t^{-\frac{1}{16}} + \cdots .$$

Squaring z, the support of the resulting series no longer has ordinal type ω:

$$z^2 = t^{-1} + 2t^{-\frac{3}{4}} + 2t^{-\frac{5}{8}} + 2t^{-\frac{9}{16}} + 2t^{-\frac{17}{32}} + \cdots + t^{-\frac{1}{2}} + \cdots .$$

Transcendental Series – Characteristic Zero

Theorem (Puiseux's Theorem)

The series $z \in K((t^{\mathbb{Q}}))$ is transcendental over the field of Laurent series $K((t))$ if and only if the set of denominators that appear in the reduced-form exponents of z is infinite.

Corollary

If the set of denominators that appear in the reduced-form exponents of z is infinite, then z is transcendental over $K(t)$.

Open Question

How can we decide whether a given series $z \in K((t^{\mathbb{Q}}))$ is algebraic or transcendental over $K(t)$?

Example

Consider

$$z = \sum_{i=1}^{\infty} t^{-p^{-i}} = t^{-1/p} + t^{-1/p^2} + t^{-1/p^3} + t^{-1/p^4} + \cdots .$$

By Puiseux's Theorem, z is transcendental over $K(t)$ in case char $K = 0$. If char $k = p$, then

$$z^p = \Big(\sum_{i=1}^{\infty} t^{-p^{-i}}\Big)^p = \sum_{i=1}^{\infty} t^{-p^{-i+1}} = \sum_{i=0}^{\infty} t^{-p^{-i}} = t + z,$$

and so z is algebraic over $K(t)$.

Theorem (Kedlaya)

When K has positive characteristic, the algebraic closure of the field of Laurent series depends on both the support and the coefficients of the series.

Value Group

Definition

Given a series $z \in K((t^{\mathbb{Q}}))$ that is transcendental over $K(t)$, define $\varphi : K(x,y) \to K((t^{\mathbb{Q}}))$ by

$$\begin{aligned} x &\mapsto t^{-1} \\ y &\mapsto z = \sum_{i=1}^{\infty} z(e_i) t^{e_i}. \end{aligned}$$

Using this, we define a valuation $v_z : K(x,y) \to \mathbb{Q}$ given by

$$v_z(f) = \min(\mathrm{Supp}(\varphi(f))).$$

Example ($z = t^{-\frac{1}{2}} + t^{-\frac{1}{4}} + t^{-\frac{1}{8}} + \cdots$)

We have

$$\varphi(y^2 - x) = z^2 - t^{-1} = 2t^{-3/4} + 2t^{-5/8} + 2t^{-9/16} + \cdots$$

and so $v_z(y^2 - x) = -3/4$.

Proposition (MacLane, Schilling)

The value group $v_z(K(x,y)^*)$ is the additive subgroup of $\mathbb{Q}$ generated by $-1, e_1, e_2, e_3, \ldots$.

Bounded Growth

For $n \geq 0$,

$$V_n = \{f \in k[x,y]^* \mid \deg_y(f) \leq n\}.$$

Given $z = t^{-1} + t^{-1/2} + t^{-1/4} + t^{-1/8} + \cdots$, with some work one can show

$$\begin{aligned}
v_z(V_0) &= \mathbb{Z}_{\leq 0}; \\
v_z(V_1) &= \mathbb{Z}_{\leq 0} \cup \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right); \\
v_z(V_2) &= \mathbb{Z}_{\leq 0} \cup \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \cup \left(\mathbb{Z}_{\leq 0} - \frac{3}{4}\right); \\
v_z(V_3) &= \mathbb{Z}_{\leq 0} \cup \left(\mathbb{Z}_{\leq 0} - \frac{1}{2}\right) \cup \left(\mathbb{Z}_{\leq 0} - \frac{3}{4}\right) \cup \left(\mathbb{Z}_{\leq 0} - \frac{5}{4}\right).
\end{aligned}$$

Proposition (M_, Sweedler)

The quotient of monoids $v_z(V_n)/v_z(V_0)$ has cardinality exactly $n+1$.

Exponents and Ramification

Given

$$z = \sum_{i=1}^{\infty} z(e_i)t^{e_i},$$

where

$$e_i = \frac{n_i}{d_i}, \ \gcd(n_i, d_i) = 1, n_i \geq 0,$$

the **exponent sequence** and **ramification sequence** of z are

$$\begin{aligned} \mathbf{e} &= (e_1, e_2, e_3, \ldots), \\ \mathbf{r} &= (r_0, r_1, r_2, \ldots), \end{aligned}$$

where $r_0 = 1$ and $r_i = \mathrm{lcm}(d_1, \ldots, d_i)$.

Example ($z = t^{-9/2} + t^{-9/5} + t^{-17/10} + t^{-17/15} + \cdots$)

We have exponent and ramification sequences given by

$$\begin{aligned} \mathbf{e} &= (-9/2, -9/5, -17/10, -17/15, \ldots), \\ \mathbf{r} &= (1, 2, 10, 10, 30, \ldots). \end{aligned}$$

Reduced Ramification Series

Consider the following series

$$z = t^{-5/2} + t^{-2} + t^{-5/3} + t^{-3/4} + t^{-1/3} + t^{-1/4} + t^{-1/5} + \cdots ,$$

which has ramification sequence $1, 2, 2, 6, 12, 12, 12, 60, \ldots$. If we remove repetitions from the ramification sequence and then extract the corresponding first terms from the original series, we obtain

$$z_{\text{red}} = t^{-5/2} + t^{-5/3} + t^{-3/4} + t^{-1/4} + t^{-1/5} + \cdots .$$

Proposition (M_)

The value monoids $v_z(K[x, y]^*)$ and $v_{z_{red}}(K[x, y]^*)$ are identical.

Open Question

Although the value monoids $v_z(K[x, y]^*)$ and $v_{z_{red}}(K[x, y]^*)$ are identical, the valuations themselves are different. How can we classify equivalence classes of such valuations that share the same value monoid?

Explicit Value Semigroups

We now assume that z has negative support. Define

$$\begin{aligned} \lambda_1 &= e_1, \\ \lambda_{i+1} &= (r_i/r_{i-1})\lambda_i - e_i + e_{i+1}. \end{aligned}$$

Theorem (M_)

Suppose that $z \in K((t^{\mathbb{Q}}))$ has negative support such that no term of the ramification sequence is divisible by char K and that there are no repetitions in the ramification sequence. Then the value monoid $v_z(K[\mathbf{x}]^*)$ is the submonoid of $\mathbb{Q}$ generated by $\{-1, \lambda_1, \lambda_2, \lambda_3, \ldots\}$.

Open Question

For a given valuation, there are efficient algorithms to generate polynomials with a prescribed image in the value monoid $v_z(K[\mathbf{x}]^*)$. Surprisingly, the inverse problem is more challenging. Given a polynomial, how does one efficiently compute its image in the value monoid $v_z(K[\mathbf{x}]^*)$?

A Pathological Example

Example

Given

$$z = t^{-\frac{3}{2}} + \sum_{i=2}^{\infty} t^{1-2^{-i}} = t^{-\frac{3}{2}} + t^{\frac{3}{4}} + t^{\frac{7}{8}} + t^{\frac{15}{16}} + \cdots,$$

the corresponding value monoid $v_z(K[x, y]^*)$ is nonpositive and reversely well ordered when working over characteristic zero!

Open Question

- *Can the value monoid $v_z(K[\mathbf{x}]^*)$ be easily described for examples like those above where some of the support is positive?*
- *When working over positive characteristic, do there exist examples of valuations induced by series with partially positive support such that the corresponding value monoid $v_z(K[\mathbf{x}]^*)$ is reversely well ordered?*
- *For which series does the induced valuation produce a reversely well-ordered value monoid $v_z(K[\mathbf{x}]^*)$?*

Infinite Gröbner Bases

Example

Suppose char $K \neq 2$. The ideal $\langle x, y \rangle$ does not have a finite Gröbner basis with respect to the valuation induced by

$$z = t^{-\frac{1}{2}} + t^{-\frac{1}{4}} + t^{-\frac{1}{8}} + t^{-\frac{1}{16}} + \ldots.$$

Open Question

For any given nonzero ideal in a polynomial ring, there exists a finite set of generators that form a Gröbner basis with respect to all possible monomial orders. Do there exist **any** *nonzero ideals such that there exists a finite set of generators that form a Gröbner basis with respect to all valuations?*

An Interesting Gröbner Basis with Respect to a Valuation

Question (Bernd Sturmfels)

Given a Gröbner basis G with respect to a valuation, does it necessarily follow that there exists a monomial order such that G is a Gröbner basis with respect to the monomial order?

Example

Let K be a field that is not of characteristic 2. Define $f_1 = y^2 - x$ and $f_2 = xy$. Then

$$G = \{f_1, f_2\}$$

is a Gröbner basis for the ideal $\langle f_1, f_2 \rangle$ with respect to the valuation induced by

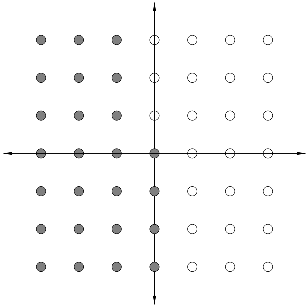
$$z = t^{-\frac{1}{2}} + t^{-\frac{1}{4}} + t^{-\frac{1}{8}} + t^{-\frac{1}{16}} + \dots.$$

However, G is not a Gröbner basis with respect to any monomial order.

Examples of Discrete Valuations of Rank 2

Endow $\mathbb{Z}^2$ with the lexicographic ordering so that its positive elements are ordered pairs (a, b) such that either (i) $a > 0$ or (ii) $a = 0$ and $b > 0$.

Let $v : K[x, y]^* \to \mathbb{Z}^2$ be a valuation. The shaded points below are nonpositive.



Question

Can $v(K[x, y]^*)$ be a reversely well-ordered subset of $\mathbb{Z}^2$ without being isomorphic to $(\mathbb{Z}_{\leq 0})^2$?

Generating Sets for Value Monoids

Example

There is a unique valuation $v : K(x, y) \to \mathbb{Z}^2$ such that

- $\nu(x - y) = (0, -1)$,
- $\nu(x) = (-1, 0)$.

As we have seen before, the value monoid is simply $(\mathbb{Z}_{\leq 0})^2$. It comes from a monomial order in suitable variables.

Example

There is a unique valuation $v : K(x, y) \to \mathbb{Z}^2$ such that

- $\nu(x) = (-2, -2)$,
- $\nu(y) = (-3, -3)$,
- $\nu(y^2 + x^3) = (-2, -1)$.

The value monoid $v_z(K[x, y]^*)$ is nonpositive and reversely well ordered. It is minimally generated by $\{(-2, -2), (-3, -3), (-2, -1)\}$.

Open Question

How can we classify value monoids $v(K[x, y]^*)$ based on the minimum number of generators?

A Most Surprising Example

Example

There is a unique valuation $v : K(x, y) \to \mathbb{Z}^2$ such that

- $\nu(x) = (-2, -2)$,
- $\nu(y) = (-3, -3)$,
- $\nu(xy^2 + y + x^4) = (-2, -1)$.

This valuation exhibits interesting behaviors:

- The value monoid is $v(K[x, y]^*)$ is contained in $(\mathbb{Z}_{<0}) \times \mathbb{Z}$, and hence, is nonpositive.
- However, $v(K[x, y]^*)$ is not reversely well ordered! In particular, it is not finitely generated.

Open Question

Do there exist value monoids (contained in $\mathbb{Z}^2$) that are not reversely well ordered and are not finitely-generated?

Gratitude

Thank you, Franz-Viktor Kuhlmann and Kasia Kuhlmann, for organizing this worskhop!