Pairs of Definition and Minimal Pairs

An overview of results by S.K. Khanduja, V. Alexandru, N. Popescu and A. Zaharescu

Hanna Ćmiel and Piotr Szewczyk

Institute of Mathematics, University of Szczecin

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Definition (minimal pair)

A pair $(\alpha, \delta) \in \widetilde{K} \times \widetilde{v}\widetilde{K}$ is said to be a minimal pair (more precisely, a (K, v)-minimal pair) if for every $\beta \in \widetilde{K}$ we have

$$\widetilde{v}(\alpha - \beta) \geq \delta \Rightarrow [K(\alpha) : K] \leq [K(\beta) : K],$$

i.e. α has least degree over K in the closed ball

$$B(\alpha, \delta) = \{\beta \in \widetilde{K} \mid \widetilde{v}(\alpha - \beta) \geq \delta\}.$$

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Example (minimal pair)

Let $f(x) \in \mathcal{O}[x]$ be a monic polynomial of degree $m \geq 1$ with $\big(fv\big)(x)$ irreducible over Kv and let α be the root of $f(x)$ in $\widetilde{K}$. Then (α, δ) is a minimal pair for every positive $\delta \in v\widetilde{K}$.

Valuation given by a minimal pair

Let $(\alpha, \delta) \in \widetilde{K} \times \widetilde{v}\widetilde{K}$ be a (K, v)-minimal pair. The mapping $\widetilde{w}_{\alpha\delta}$ defined on $\widetilde{K}(x)$ associated with this minimal pair is given by

$$\widetilde{w}_{\alpha\delta}\left(\sum_{i=0}^{n} c_i(x-\alpha)^i\right) = \min_i \left\{\widetilde{v}(c_i) + i\delta\right\}, \; c_i \in \widetilde{K}. \tag{1}$$

It is shown in [1] that $\widetilde{w}_{\alpha\delta}$ is indeed a valuation on $\widetilde{K}$. By $w_{\alpha\delta}$ we will denote the restriction of $\widetilde{w}_{\alpha\delta}$ to K.

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Example

For the (K, v)-minimal pair $(0, 0)$ we acquire the well known Gauss valuation:

$$\widetilde{w}_{\alpha\delta}\left(\sum_{i=0}^{n} c_i x^i\right) = \min_i \left\{\widetilde{v}(c_i)\right\}.$$

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Theorem A, [2]

The valuation $w_{\alpha\delta}$ defined by (1) is a residue transcendental extension of v to $K(x)$. Conversely, for any residue transcendental extension of v to $K(x)$ there exists a minimal pair (α, δ) such that $w = w_{\alpha\delta}$.

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Theorem B, [2]

If (α, δ), (β, η) are two (K, v)-minimal pairs then $w_{\alpha\delta} = w_{\beta\eta}$ if and only if $\delta = \eta$ and $\widetilde{v}(\alpha' - \beta) \geq \delta$ for some K-conjugate α' of α.

Minimal pairs – different approach

Let w be a given extension of v to $K(x)$ and $\widetilde{w}$ an extension of w to $\widetilde{K}(x)$. Consider the set

$$\widetilde{w}(x - \widetilde{K}) := \{\widetilde{w}(x - a) \mid a \in \widetilde{K}\}.$$

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Theorem 1, [3]

w is a residue transcendental extension if and only if:

1. $\widetilde{v}\widetilde{K} = \widetilde{w}\widetilde{K}(x)$,
2. the set $\widetilde{w}(x - \widetilde{K})$ is upper bounded in $\widetilde{w}\widetilde{K}(x)$,
3. $\widetilde{w}\widetilde{K}(x)$ contains its upper bound.

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Let δ be the upper bound of $\widetilde{w}(x - \widetilde{K})$. Then there exists $\alpha \in \widetilde{K}$ such that $\delta = \widetilde{w}(x - \alpha)$ and thus ([3]) $\widetilde{w}$ is a residue transcendental extension of $\widetilde{v}$ defined by (1). The pair (α, δ) is called a *pair of definition*.

Definition

A pair of definition (α, δ) is called *minimal* (or *minimal relative to K*) if it is a minimal pair in the sense of the previous definition.

Let w_1, w_2 be two residue transcendental extensions of v to $K(x)$. We say that w_2 *dominates* w_1 (written $w_1 \leq w_2$) if $w_1\big(f(x)\big) \leq w_2\big(f(x)\big)$ for all polynomials $f \in K[x]$. If $w_2 \geq w_1$ and there exists $f \in K[x]$ such that $w_1(f) < w_2(f)$, we say that w_2 *well dominates* w_1, which we will denote as $w_1 < w_2$.

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Proposition 1, [4]

Let K be algebraically closed and let w_1, w_2 be two residue transcendental extensions of v to $K(x)$. Let (α_i, δ_i) be a pair of definition of w_i, $i = 1, 2$. The following statements are equivalent:

1. $w_1 \leq w_2$
2. $\delta_1 \leq \delta_2$ and $v(\alpha_1 - \alpha_2) \geq \delta_1$.

Moreover, $w_1 < w_2$ if and only if $\delta_1 < \delta_2$ and $v(\alpha_1 - \alpha_2) \geq \delta_1$.

By an *ordered system of residue transcendental extensions of v to $K(x)$* (for brevity call it an *ordered system*) we mean a family $(w_i)_{i \in I}$ of residue transcendental extensions of v to $K(x)$, where I is a well ordered set without a last element and such that w_j dominates w_i when $i < j$.

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For an ordered system $(w_i)_{i\in I}$ and any given $f \in K[x]$ let us define the mapping

$$w(f) := \sup_{i\in I} w_i(f).$$

As stated in [4], w is a valuation on $K[x]$. It will be called *the limit of the given system* $(w_i)_{i\in I}$ and denoted by $w = \sup_i w_i$.

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As stated in [4], w is a valuation on $K[x]$. It will be called *the limit of the given system* $(w_i)_{i\in I}$ and denoted by $w = \sup_i w_i$.
For each $i \in I$ we denote by (α_i, δ_i) a pair of definition of w_i. Then by Proposition 1, the set $(\delta_i)_{i\in I}$ is a well ordered subset of vK. Moreover, if for every $i, j \in I$, $i < j$, w_j well dominates w_i, then $(\alpha_i)_{i\in I}$ is a pseudo-convergent sequence on K.

Theorem 2, [4]

Let K be a field, and let $(\widetilde{w}_i)_{i\in I}$ be an ordered system of residue transcendental extensions of $\widetilde{v}$ to $\widetilde{K}(x)$. For every $i \in I$ we denote by (α_i, δ_i) a fixed minimal pair of definition of $\widetilde{w}_i$ with respect to K. Denote by w_i the restriction of $\widetilde{w}_i$ to $K(x)$ and by v_i the restriction of $\widetilde{v}$ to $K(\alpha_i)$, $i \in I$. Then

a) For all $i, j \in I$, $j < j$ one has $w_i < w_j$, i.e. $(w_i)_{i\in I}$ is an ordered system of residue transcendental extensions of v to $K(x)$.

b) For all $i, j \in I$, $i < j$ one has $Kv_i \subseteq Kv_j$ and $v_iK \subseteq v_jK$.

c) Assume that $\widetilde{w} = \sup \widetilde{w}_i$ and $\widetilde{w}$ is not a residue transcendental extension of $\widetilde{v}$ to $\widetilde{K}(x)$. Let w be the restriction of $\widetilde{w}$ to $K(x)$. Then $w = \sup_i w_i$. Moreover, one has

$$Kw = \bigcup_{i\in I} Kv_i \quad \text{and} \quad wK = \bigcup_{i\in I} v_iK.$$

Theorem 3, [4]

Let w be a given value transcendental extension of v to $K(x)$. Consider a cofinal well ordered set $\{\delta_i \mid i \in I\} \subseteq vK$ and some α_i such that

$$w(x - \alpha_i) = \delta_i, \ i \in I.$$

Let $w_i = w_{\alpha_i \delta_i}$. Then

a) $w_i < w_j$ if $i < j$, i.e. $\{w_i\}_{i \in I}$ is an ordered system of residue transcendental extensions of v to $K(x)$. Moreover, for every $i < j$ w_j well dominates w_i.

b) $w_i \leq w$ for all $i \in I$ and $w = \sup_{i \in I} w_i$.

Theorem 4, [4]

Let w be a value transcendental extension of v to $K(x)$. Then there exists a pair $(\alpha, \delta) \in K \times wK(x)$ such that $w(x-\alpha) = \delta$. Moreover, $wK(x) = vK \oplus \mathbb{Z}\delta$ and w is defined by

$$w\left(\sum_{i=0}^{n} a_i (x-\alpha)^i\right) = \inf_i \left(v(a_i) + i\delta\right), \; a_i \in K. \tag{2}$$

Conversely, let Γ be an ordered group which contains vK as a subgroup, and $\delta \in \Gamma$ be such that $\mathbb{Z}\delta \cap vK = 0$. Let $\alpha \in K$ and let $w : K(x) \to \Gamma$ be defined by the equality (2). Then w is a value transcendental extension of v to $K(x)$. Moreover, $wK(x) = vK \oplus \mathbb{Z}\delta$ and $Kw = Kv$.

Theorem 5, [4]

Let w be a value transcendental extension of v to $K(x)$, let $\{\delta_i \mid i \in I\}$ be a set cofinal in $\widetilde{w}\widetilde{K}(x)$. Choose $\alpha_i \in \widetilde{K}$, $i \in I$, such that (α_i, δ_i) are minimal pairs. Take w_i to be the restriction of $w_{\alpha_i \delta_i}$ to $K(x)$ and v_i to be the restriction of $\widetilde{v}$ to $K(\alpha_i)$. Then

- $w_i < w_j$, $Kv_i \subseteq Kv_j$ and $v_i K \subseteq v_j K$ whenever $i < j$.
- $(w_i)_{i \in I}$ is an ordered system of residue transcendental extensions of v to $K(x)$ and $w = \sup_i w_i$. Moreover, we have

$$K(x)w = \bigcup_{i \in I} Kv_i, \quad wK(x) = \bigcup_{i \in I} v_i K.$$

Theorem 6, [4]

Let w be a value transcendental extension of v to $K(x)$ and (α, δ) a minimal pair of definition of w with respect to K. Denote by f the monic minimal polynomial of α over K and let $\gamma = w(f)$. If $g \in K[x]$ is a polynomial with f-expansion of the form

$$g = \sum_{i=0}^{n} g_i f^i, \; g_i \in K[x], \; \deg g_i < \deg f,$$

then

$$w(g) = \inf \left(v(g_i(\alpha)) + i\gamma \right).$$

Moreover, if v_1 is the restriction of $\widetilde{v}$ to $K(\alpha)$, then

$$K(x)w = K(\alpha)v_1 \quad \text{and} \quad wK(x) = v_1 K(\alpha) \oplus \mathbb{Z}\gamma.$$

Results on minimal pairs

Given an element $\alpha \in \widetilde{K}$, are we able to find $\delta \in \widetilde{v}\widetilde{K}$ such that (α, δ) is a minimal pair?

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Theorem, [5]

Let (K, v) be henselian.

- If $\alpha \in \widetilde{K}$ is separable over K, then there exists an element $\delta \in \widetilde{v}\widetilde{K}$ such that (α, δ) is a minimal pair.
- If K is complete with respect to v, then there exists an element $\delta \in \widetilde{v}\widetilde{K}$ such that (α, δ) is a minimal pair.

Results on minimal pairs

Given some extension Γ of vK and some extension k of Kv, can we construct an extension w of v to $K(x)$ such that $wK(x) = \Gamma$ and $K(x)w = k$?

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Assume first that $(\Gamma : vK) < \infty$ and $[k : Kv] < \infty$.

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a) there exists a value transcendental extension w such that

$$K(x)w = k \quad \text{and} \quad wK(x) = \Gamma \oplus \mathbb{Z}\lambda \tag{3}$$

for λ in some group extension for any given ordering;

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Conversely,

a) if w is a value transcendental extension then 3 holds;

b) if w is a residue transcendental extension then 4 holds. In particular, $K(x)w$ is a rational function field over a finite extension of Kv (Ruled Residue Theorem, [7]).

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Assume now that $\Gamma \supseteq vK$ and $k \supseteq Kv$ are countably generated and at least one of them is infinite.

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Results on minimal pairs

Assume now that $\Gamma \supseteq vK$ and $k \supseteq Kv$ are countably generated and at least one of them is infinite. Then there exists an extension w such that

$$wK(x) = \Gamma \quad \text{and} \quad K(x)w = k. \tag{5}$$

Conversely, if (5) holds, then both extensions are countably generated.

Bibliography

1 S.K. Khanduja. *On valuations of $K(x)$*.

2 S.K. Khanduja, N. Popescu and K.W. Roggenkamp. *On minimal pairs and residually transcendental extensions of valuations*.

3 V. Alexandru, N. Popescu and A. Zaharescu. *A theorem of characterization of residual transcendental extensions of a valuation*.

4 V. Alexandru, N. Popescu and A. Zaharescu. *All valuations on $K(x)$*.

5 V. Alexandru, N. Popescu and A. Zaharescu. *Minimal pairs of definition of a residual transcendental extension of a valuation*.

6 F.-V. Kuhlmann. *Value groups, residue fields and bad places of rational function fields*.

7 J. Ohm. *The Ruled Residue Theorem for simple transcendental extensions of valued fields*.