Weierstrass representation for minimal surfaces

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on
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Consider the soap film created by dipping a cube frame into soap solution. The soap film creates the minimum surface area for a surface with a cube as its boundary.
Water molecules exert a force on each other. Near the surface of the water there is a greater force pulling the molecules toward the center of the water, creating surface tension that tends to minimize the surface area of the shape. Soap solution has a lower surface tension than water and this permits the formation of soap films that also tend to minimize geometric properties such as length and area.
By dipping into soap solution a wire frame of a slinky (or helix) with a straw connecting the ends of the slinky, we can create part of the minimal surface known as the **helicoid**.
Other examples of soap films
By dipping a 3-dimensional version of the wire frame (a box frame missing two parallel edges on the top and two parallel edges on the bottom) into soap solution we can create part of the minimal surface known as **Scherk’s doubly periodic surface**.
Other examples of soap films
Plateau’s Problem

Physical interpretation of minimal surfaces is related to J. Plateau work and is called Plateau’s Problem.

J.A.F. Plateau, 1801–1883
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Plateau described results and experiments on surface tension in 1873 in *Statique expérimentale et théorétique des liquides soumis aux seules forces moléculaires.*

However Plateau’s Problem was raised by Joseph-Louis Lagrange in 1760.

J.A.F. Plateau, 1801–1883
Plateau’s Problem

For any closed curve $\gamma \in \mathbb{R}^3$ of the finite length there is a surface $S$ of the lowest area spanned on the curve $\gamma$. 
Plateau’s Problem was solved independently in years 1960-1961 by Jesse Douglas and Tibor Radó. J. Douglas got Field’s medal in 1936 for this result!
In mathematics the minimal surface is a surface, such that the main curvature is equal to zero on every point on this surface. Let us explain what does it mean:

- consider the surface $S \in \mathbb{R}^3$ and any point $p \in S$. 

**Minimal surfaces**
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- consider all planes \( P \) including \( \vec{n} \),
- get the curve \( \alpha \) as an intersection of plane \( P \) and surface \( S \),
- determine curvature of \( \alpha \) (i.e. score \( |\alpha''| \)).
Minimal surfaces

In mathematics the minimal surface is a surface, such that the main curvature is equal to zero on every point on this surface. Let us explain what does it mean:

1. Consider the surface $S \in \mathbb{R}^3$ and any point $p \in S$,
2. Determine normal vector $\vec{n}$ to the surface $S$ in $p$ (i.e. normalized vector perpendicular to $S$ in $p$),
3. Consider all planes $P$ including $\vec{n}$,
4. Get the curve $\alpha$ as an intersection of plane $P$ and surface $S$,
5. Determine curvature of $\alpha$ (i.e. score $|\alpha''|$),
6. Determine normal curvature in the fixed direction $w$, i.e. $k(w) = \alpha'' \cdot n$
Minimal surfaces

The normal curvature measures how much the surface bends towards $\mathbf{n}$ as you travel in the direction of the tangent vector $\mathbf{w}$ starting at $p$. As we rotate the plane about $\mathbf{n}$, we get a set of curves on the surface each of which has a value for its curvature. Let $k_1$ and $k_2$ be the maximum and minimum curvature values at $p$. The directions in which the normal curvature attains its absolute maximum and absolute minimum values are known as the principal directions.
The **mean curvature** (i.e., average curvature) of a surface $S$ at $p$ is

$$H := \frac{k_1 + k_2}{2}$$

**Definition**

A **minimal surface** is a surface $S$ with mean curvature $H = 0$ at all points $p \in S$. 

**Definition**

**Weierstrass representation**
Definition

Let us consider the surface $S \subset \mathbb{R}^3$. Let $D \subset \mathbb{R}^2$ be an open set. Then $S$ can be presented by mapping $x: D \to \mathbb{R}^3$, where

$$x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)),$$

, i.e. $S$ is an image of the set $x(D)$ by map $x$. We demand that $x \in C^2(D)$. The mapping above is called a parametrization of the surface $S$. 

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Weierstrass representation
Minimal surfaces

Theorem

Let $S$ be the surface with parametrization

$$x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$$

and $p$ be any point on this surface. Let us denote:

$$E = \sum_{j=1}^{3} \left( \frac{\partial x_j}{\partial u} \right)^2, \quad F = \sum_{j=1}^{3} \frac{\partial x_j}{\partial u} \frac{\partial x_j}{\partial v}, \quad G = \sum_{j=1}^{3} \left( \frac{\partial x_j}{\partial v} \right)^2,$$

and for vector $\vec{n} = [n_1, n_2, n_3]$ normal to $S$ in the point $p$:

$$e = \sum_{j=1}^{3} \frac{\partial^2 x_j}{\partial u^2} \cdot n_j, \quad f = \sum_{j=1}^{3} \frac{\partial^2 x_j}{\partial u \partial v} \cdot n_j, \quad g = \sum_{j=1}^{3} \frac{\partial^2 x_j}{\partial v^2} \cdot n_j$$

Then

$$H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}.$$
Corollary

The surface $S$ is minimal if

$$\frac{Eg + Ge - 2Ff}{2(EG - F^2)} = 0.$$
Examples

Enneper’s surface (Alfred Enneper, 1864)

\[ x(u, v) = \left( u - \frac{1}{3}u^3 + uv^2, v - \frac{1}{3}v^3 + u^2v, u^2 - v^2 \right), \]

where \((u, v) \in \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}\).
Catenoid (Leonhard Euler, 1744)

\[
x(u, v) = (a \cosh(v) \cos(u), a \cosh(v) \sin(u), av),
\]

where \((u, v) \in \{(x, y) \in \mathbb{R}^2 : 0 \leq u \leq 2\pi, -\pi \leq v \leq \pi\}\).
Examples

Helikoid (Leonhard Euler in 1774 and Jean Baptiste Meusnier in 1776)

\[ x(u, v) = (a \sinh(v) \cos(u), a \sinh(v) \sin(u), au), \]

where \((u, v) \in \{(x, y) \in \mathbb{R}^2 : -\pi \leq u \leq \pi, -\frac{2}{3}\pi \leq v \leq \frac{2}{3}\pi\}\).
Scherk’s doubly periodic surface: (Heinrich Scherk, 1834)

\[ x(u, v) = \left( u, v, \ln \left( \frac{\cos(u)}{\cos(v)} \right) \right), \]

where \((u, v) \in \{(x, y) \in \mathbb{R}^2 : -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}\}.\]
Examples

Pieces of Scherk’s doubly periodic surface can be put together in the $xy$-plane in a checkerboard fashion. They repeat (or are periodic) in two directions, $x$ and $y$.  

Scherk’s singly periodic surface (Heinrich Scherk, 1834)

\[ x(u, v) = (\text{arcsinh}(u), \text{arcsinh}(v), \text{arcsin}(uv)), \]

where \((u, v) \in \{(x, y) \in \mathbb{R}^2 : -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}\}\).
Individual pieces of Scherk’s singly periodic surface can fit together creating a tower in the $z$ direction. You can visualize adding two pieces together by taking one piece of Scherk’s singly periodic surface and adding it to another piece that has been reflected across the $xy$-plane and shifted up in the $z$ direction.
Hennenberg’s surface (Ernst Lebrecht Henneberg, 1875)

\[ x(u, v) = (-1 + \cosh(2u) \cos(2v), -\sinh(u) \sin(v) \]
\[ - \frac{1}{3} \sinh(3u) \sin(3v), -\sinh(u) \cos(v) \]
\[ + \frac{1}{3} \sinh(3u) \cos(3v)), \]

where \((u, v) \in \{(x, y) \in \mathbb{R}^2 : 0 \leq u \leq 2\pi, -\pi \leq v \leq \pi\}\).
Catalan’s surface (Eugene Charles Catalan, 1843)

\[ x(u, v) = (1 - \cos(u) \cosh(v), 4 \sin\left(\frac{u}{2}\right) \sinh\left(\frac{v}{2}\right), u - \sin(u) \cosh(v)), \]

where \((u, v) \in \{(x, y) \in \mathbb{R}^2 : 0 \leq u \leq 2\pi, -\pi \leq v \leq \pi\}.\]
There is an extensive list of minimal surfaces, but we have no way of listing all of them.
So, we often focus on trying to classify minimal surfaces. This means, we try to find results that include all possibilities for minimal surfaces with specific properties. The simplest example of this is:
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There is an extensive list of minimal surfaces, but we have no way of listing all of them. So, we often focus on trying to classify minimal surfaces. This means, we try to find results that include all possibilities for minimal surfaces with specific properties. The simplest example of this is:

A nonplanar minimal surface in $\mathbb{R}^3$ that is also a surface of revolution is contained in a catenoid.
Isothermal parametrization

**Definition**

A parametrization $x$ is isothermal if $E = x_u \cdot x_u = x_v \cdot x_v = G$ and $F = x_u \cdot x_y = 0$. 
Definition

A parametrization $x$ is isothermal if $E = x_u \cdot x_u = x_v \cdot x_v = G$ and $F = x_u \cdot x_y = 0$.

Because $E$, $F$, and $G$ describe how lengths on a surface are distorted as compared to their usual measurements in $\mathbb{R}^3$, if $F = x_u \cdot x_y = 0$ then $x_u$ and $x_v$ are orthogonal and if $E = G$ then the amount of distortion is the same in the orthogonal directions. Thus, we can think of an isothermal parametrization as mapping a small square in the domain to a small square on the surface.
Sometimes an isothermal parametrization is called a conformal parametrization, because the angle between a pair of curves in the domain is equal to the angle between the corresponding pair of curves on the surface.
Theorem

Every minimal surface in $\mathbb{R}^3$ has an isothermal parametrization.

Theorem

If the parametrization $x$ is isothermal, then

$$x_{uu} + x_{vv} = 2EH\overrightarrow{n}, \quad (1)$$

where $E$ is a coefficient of the first fundamental form and $H$ is the mean curvature.
Corollary 1

A surface $S$ with an isothermal parametrization $x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$ is minimal if and only if $x_1$, $x_2$, and $x_3$ are harmonic.

Proof:
$(\Rightarrow)$
If $S$ is minimal, then $H = 0$ and so by (1) we have $x_{uu} + x_{vv} = 0$, and hence the coordinate functions are harmonic.

$(\Leftarrow)$
Suppose $x_1$, $x_2$, and $x_3$ are harmonic. Then $x_{uu} + x_{vv} = 0$. So by (1) we have $2(x_u \cdot x_u)H \vec{n} = 0$. But $\vec{n} \neq 0$ and $E = x_u \cdot x_u \neq 0$. Hence $H = 0$ and $S$ is minimal.

□
Suppose \( S \) is a minimal surface with an isothermal parametrization \( x(u, v) \). Let \( z = u + iv \) be a point in the complex plane, so \( \bar{z} = u - iv \). Solving for \( u, v \) in terms of \( z, \bar{z} \) we get

\[
    u = \frac{z + \bar{z}}{2} \quad v = \frac{z - \bar{z}}{2i}
\]

The parametrization of the minimal surface \( S \) can be written in terms of the complex variables \( z \) and \( \bar{z} \) as

\[
    x(z, \bar{z}) = (x_1(z, \bar{z}), x_2(z, \bar{z}), x_3(z, \bar{z})).
\]
Lemma 1

Let \( f(u, v) = x(u, v) + iy(u, v) \) be a complex function. Using 
\[ u = \frac{z + \overline{z}}{2} \quad \text{and} \quad v = \frac{z - \overline{z}}{2i}, \]
we have

\( \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} \right) + \frac{i}{2} \left( \frac{\partial y}{\partial u} - \frac{\partial x}{\partial v} \right), \)

\( \frac{\partial f}{\partial \overline{z}} = \frac{1}{2} \left( \frac{\partial x}{\partial u} - \frac{\partial y}{\partial v} \right) + \frac{i}{2} \left( \frac{\partial y}{\partial u} + \frac{\partial x}{\partial v} \right). \)

ii. \( f \) is analytic \( \iff \frac{\partial f}{\partial \overline{z}} = 0. \)

iii. \( 4 \left( \frac{\partial}{\partial \overline{z}} \left( \frac{\partial f}{\partial z} \right) \right) = f_{uu} + f_{vv}. \)
Theorem I

Let \( S \) be a surface with parametrization \( \mathbf{x} = (x_1, x_2, x_3) \) and let \( \phi = (\phi_1, \phi_2, \phi_3) \), where
\[
\phi_k = \frac{\partial x_k}{\partial z}.
\]

Let \( \phi^2 \) denote \( (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2 \).

Then \( \mathbf{x} \) is isothermal \( \iff \phi^2 \equiv 0 \).

If \( \mathbf{x} \) is isothermal then \( S \) is minimal \( \iff \) each \( \phi_k \) is analytic.
Proof of theorem I:
Applying the complex differential operator $\frac{\partial f}{\partial z}$ from lemma I and squaring the terms, we have

$$(\phi_k)^2 = \left( \frac{\partial x_k}{\partial z} \right)^2$$

$$= \left[ \frac{1}{2} \left( \frac{\partial x_k}{\partial u} - i \frac{\partial x_k}{\partial v} \right) \right]^2$$

$$= \frac{1}{4} \left[ \left( \frac{\partial x_k}{\partial u} \right)^2 - \left( \frac{\partial x_k}{\partial v} \right)^2 - 2i \frac{\partial x_k}{\partial u} \frac{\partial x_k}{\partial v} \right]$$

Also,

$$x_u \cdot x_u = \left( \frac{\partial x_1}{\partial u} \right)^2 + \left( \frac{\partial x_2}{\partial u} \right)^2 + \left( \frac{\partial x_3}{\partial u} \right)^2 = \sum_{k=1}^{3} \left( \frac{\partial x_k}{\partial u} \right)^2$$
Similarly we have $x_v \cdot x_v = \sum_{k=1}^{3} \left( \frac{\partial x_k}{\partial v} \right)^2$. Hence,

$$\phi^2 = (\phi_1)^2 + (\phi_2)^2 + (\phi_3)^2$$

$$= \frac{1}{4} \left[ \sum_{k=1}^{3} \left( \frac{\partial x_k}{\partial u} \right)^2 - \sum_{k=1}^{3} \left( \frac{\partial x_k}{\partial v} \right)^2 - \sum_{k=1}^{3} \frac{\partial x_k}{\partial u} \frac{\partial x_k}{\partial v} \right]$$

$$= \frac{1}{4} (x_u \cdot x_u - x_v \cdot x_v - 2i(x_u \cdot x_v))$$

$$= \frac{1}{4} (E - G - 2iF)$$
Thus $x$ is isothermal $\iff E = G$, and $F = 0 \iff \phi^2 \equiv 0$.

Suppose that $x$ is isothermal. By corollary I, it it suffices to show that for each $k$, $x_k$ is harmonic $\iff \phi_k$ is analytic. By lemma I this follows because

$$\frac{\partial^2 x_k}{\partial u \partial u} + \frac{\partial^2 x_k}{\partial v \partial v} = 4 \left( \frac{\partial}{\partial z} \left( \frac{\partial f}{\partial z} \right) \right) = 4 \left( \frac{\partial}{\partial z} (\phi_k) \right) = 0.$$
Let’s apply this theorem. Suppose we have analytic functions $\phi_k$ and we want to find the functions $x_k$. If $\mathbf{x}$ is isothermal, then

$$|\phi^2| = \left| \frac{\partial x_1}{\partial z} \right|^2 + \left| \frac{\partial x_2}{\partial z} \right|^2 + \left| \frac{\partial x_3}{\partial z} \right|^2 = \frac{1}{4} \left( \sum_{k=1}^{3} \left( \frac{\partial x_k}{\partial u} \right)^2 + \sum_{k=1}^{3} \left( \frac{\partial x_k}{\partial v} \right)^2 \right)$$

$$= \frac{1}{4} (\mathbf{x}_u \cdot \mathbf{x}_u + \mathbf{x}_v \cdot \mathbf{x}_v) = \frac{1}{4} (E + G) = \frac{E}{2}.$$
Let's apply this theorem. Suppose we have analytic functions $\phi_k$ and we want to find the functions $x_k$. If $x$ is isothermal, then

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$$= \frac{1}{4} (x_u \cdot x_u + x_v \cdot x_v) = \frac{1}{4} (E + G) = \frac{E}{2}.$$

So if $|\phi^2| = 0$, then the coefficients of the first fundamental form are zero and $S$ is a point.

We need to solve $\phi_k = \frac{\partial x_k}{\partial z}$ for $x_k$ since the parametrization of the surface is given as $x = (x_1, x_2, x_3)$. 
Since $x_k$ is a function of the two variables $u$ and $v$, we can write

$$dx_k = \frac{\partial x_k}{\partial u} du + \frac{\partial x_k}{\partial v} dv. \quad (2)$$

Also $dz = du + idv$ so by lemma 1 we have

$$\phi_k dz = \frac{\partial x_k}{\partial z} dz = \frac{1}{2} \left( \frac{\partial x_k}{\partial u} - i \frac{\partial x_k}{\partial v} \right) (du + idv)$$

$$= \frac{1}{2} \left[ \frac{\partial x_k}{\partial u} du + \frac{\partial x_k}{\partial v} dv + i \left( \frac{\partial x_k}{\partial u} dv - \frac{\partial x_k}{\partial v} du \right) \right],$$

$$\overline{\phi_k dz} = \frac{\partial \overline{x_k}}{\partial z} dz = \frac{1}{2} \left( \frac{\partial x_k}{\partial u} + i \frac{\partial x_k}{\partial v} \right) (du - idv)$$

$$= \frac{1}{2} \left[ \frac{\partial x_k}{\partial u} du + \frac{\partial x_k}{\partial v} dv - i \left( \frac{\partial x_k}{\partial u} dv - \frac{\partial x_k}{\partial v} du \right) \right].$$
Weierstrass representation

Adding we get

\[
\frac{\partial x_k}{\partial u} du + \frac{\partial x_k}{\partial v} dv = \phi + k dz + \overline{\phi_k} dz = 2 \text{Re}\{\phi_k dz}\]  

(3)

Combining (2) and (3) we have

\[dx_k = 2 \text{Re}\{\phi_k dz}\]

Therefore \(x_k = 2 \text{Re} \int \phi_k dz + c_k\).

Since adding \(c_k\) translates the image by a constant amount and multiplying a coordinate function by 2 scales the surface, the constants do not affect the geometric shape of the surface. Hence, we do not need them and we will let our coordinate function be:

\[x_k = \text{Re} \int \phi_k dz.\]
Weierstrass representation

Corollary

If we have analytic functions \( \phi_k \) \( (k = 1, 2, 3) \) such that \( \phi^2 \equiv 0 \) and \( |\phi^2| \neq 0 \) and is finite, then the parametrization

\[
x = \left( \text{Re} \int \phi_1(z)dz, \text{Re} \int \phi_2(z)dz, \text{Re} \int \phi_3(z)dz \right)
\]

(4)

defines a minimal surface.
For example, consider the functions $p(z)$ and $q(z)$ such that

$$
\phi_1 = p(1 + q^2)
$$
$$
\phi_2 = -ip(1 - q^2)
$$
$$
\phi_3 = -2ipq.
$$

Then

$$
\phi^2 = [p(1 + q^2)]^2 + [-ip(1 - q^2)]^2 + [-2ipq]^2
$$
$$
= [p^2 + 2p^2q^2 + p^2q^4] - [p^2 - 2p^2q^2 + p^2q^4] - [4p^2q^2]
$$
$$
= 0,
$$
Weierstrass representation

and

\[ |\phi|^2 = |p(1 + q^2)|^2 + |-ip(1 - q^2)|^2 + |-2ipq|^2 \]
\[ = |p|^2[(1 + q^2)(1 + \overline{q}^2) + (1 - q^2)(1 - \overline{q}^2) + 4q\overline{q}] \]
\[ = |p|^2[2(1 + 2q\overline{q} + q^2\overline{q}^2)] , \]
\[ = 4|p|^2(1 + |q|^2)^2 \neq 0. \]

For \( \phi_k \) to be analytic, \( p, pq^2, \) and \( pq \) have to be analytic. If \( p \) is analytic with a zero of order \( 2m \) at \( z_0 \), then \( q \) can have a pole of order no larger than \( m \) at \( z_0 \). A function that is analytic in a domain \( D \) except possibly at poles is a meromorphic function in \( D \). This leads to the following result.
Every regular minimal surface has a local isothermal parametric representation \( x = (x_1, x_2, x_3) \) of the form

\[
\begin{align*}
x_1(z) &= \text{Re} \left\{ \int_a^z p(1 + q^2) \, dz \right\}, \\
x_2(z) &= \text{Re} \left\{ \int_a^z -ip(1 - q^2) \, dz \right\}, \\
x_3(z) &= \text{Re} \left\{ \int_a^z -2ipq \, dz \right\},
\end{align*}
\]

where \( p \) is an analytic function and \( q \) is a meromorphic function in a domain \( \Omega \in \mathbb{C} \), having the property that where \( q \) has a pole of order \( m \), \( p \) has a zero of order at least \( 2m \), and \( a \in \Omega \) is a constant.
Example

For $p(z) = 1$ and $q(z) = iz$ we get:

$$x(z) = \left( \operatorname{Re} \left\{ \int_0^z (1 - z^2) \, dz \right\}, \operatorname{Re} \left\{ \int_0^z -i(1 + z^2) \, dz \right\}, \operatorname{Re} \left\{ \int_a^z 2zpq \, dz \right\} \right)$$

$$= \left( \operatorname{Re} \left\{ z - \frac{1}{3} z^3 \right\}, \operatorname{Re} \left\{ -i \left( z + \frac{1}{3} z^3 \right) \right\}, \operatorname{Re} \left\{ z^2 \right\} \right).$$

Letting $z = u + iv$, this yields

$$x(z) = \left( u - \frac{1}{3} u^3 + u v^2, v - \frac{1}{3} v^3 + u^2 v, u^2 - v^2 \right),$$

which gives Enneper’s surface.
References


Thank you for patience and listening.