Model theory of separably closed valued fields

Florian Felix

WWU Muenster

May 31, 2019
Overview

1. Setting and Introduction

2. Quantifier Elimination

3. NIP
Facts and questions

Fact

The theory of algebraically closed nontrivially valued fields $ACVF_{p,q}$ in the language $\mathcal{L}_{div} = \mathcal{L}_{Ring} \cup \{|\}$ admits quantifier elimination.
Facts and questions

Fact
The theory of algebraically closed nontrivially valued fields $ACVF_{p,q}$ in the language $\mathcal{L}_{div} = \mathcal{L}_{Ring} \cup \{|\}$ admits quantifier elimination.

• Without additions to the language that is basically everything we can hope for.
Facts and questions

Fact

The theory of algebraically closed nontrivially valued fields $ACVF_{p,q}$ in the language $\mathcal{L}_{div} = \mathcal{L}_{\text{Ring}} \cup \{||\}$ admits quantifier elimination.

- Without additions to the language that is basically everything we can hope for.
- But what do we have to add in a more generalized setting, say separably closed valued fields?
Lambda-Functions

Definition

\( \mathcal{L}_p \) denotes the following first-order language

\[
\mathcal{L}_p := \mathcal{L}_{\text{Ring}} \cup \{\lambda_{n,j}(\cdot)\}_{0 \leq n \leq \omega, 0 \leq j < p^n}
\]

where each \( \lambda_{n,j}(\cdot) \) is an \((n + 1)\)-ary function symbol, which is usually going to be written as \( \lambda_{n,j}(x; y_1, \ldots, y_n) \) if the input is the tuple \((x, y_1, \ldots, y_n)\); we call the entries \( y_1, \ldots, y_n \) basis entries. Let \( \mathcal{L}_{p,\text{div}} \) be \( \mathcal{L}_p \cup \mathcal{L}_{\text{div}} \).
Definition

\( \mathcal{L}_p \) denotes the following first-order language

\[
\mathcal{L}_p := \mathcal{L}_{\text{Ring}} \cup \{ \lambda_{n,j}(\cdot) \}_{0 \leq n \leq \omega, 0 \leq j < p^n}
\]

where each \( \lambda_{n,j}(\cdot) \) is an \((n + 1)\)-ary function symbol, which is usually going to be written as \( \lambda_{n,j}(x; y_1, \ldots, y_n) \) if the input is the tuple \((x, y_1, \ldots, y_n)\); we call the entries \( y_1, \ldots, y_n \) basis entries. Let \( \mathcal{L}_{p,\text{div}} \) be \( \mathcal{L}_p \cup \mathcal{L}_{\text{div}} \).

- The added \( \lambda \)-functions will allow for quantifier elimination, but we first have to define what they even do.
Definition

Let $SCVF_{p,e}$ the $\mathcal{L}_{p,div}$ theory of separably closed nontrivially valued fields of characteristic $p$ and imperfection degree $e$, where $\lambda_{n,j}(x, y_1, \ldots, y_n) = 0$, if $y_1, \ldots, y_n$ are not $p$-independent or don’t generate $x$. Otherwise it stands for one fixed coordinate if you represent $x$ as a linear combination of $\prod_{i=1}^{n} y_{f(i)}^{i}$ over $L^p$, for any $f(i) : \{0, \ldots n\} \rightarrow \{0, \ldots, p - 1\}$, where $L$ is our underlying field. Note that $e = \infty$ is allowed.
Lambda-Functions

Definition

Let $SCVF_{p,e}$ the $L_{p,\text{div}}$ theory of separably closed nontrivially valued fields of characteristic $p$ and imperfection degree $e$, where $\lambda_{n,j}(x, y_1, \ldots, y_n) = 0$, if $y_1, \ldots, y_n$ are not $p$-independent or don’t generate $x$. Otherwise it stands for one fixed coordinate if you represent $x$ as a linear combination of $\prod_{i=1}^{n} y_i^{f(i)}$ over $L^p$, for any $f(i) : \{0, \ldots n\} \to \{0, \ldots, p-1\}$, where $L$ is our underlying field. Note that $e = \infty$ is allowed.

- This allows for enough control to gain quantifier elimination
Quantifier Elimination

It should come to no great suprise that we need some way of extending embeddings if we want to proof quantifier elimination. With the $\lambda$-functions it is easy to define predicates of $p$-independence $Q_n(x_1, \ldots, x_n)$ that are true iff $x_1, \ldots, x_n$ are $p$-independent. The following lemma is basically the crux for quantifier elimination.

**Theorem (Hong, 2016)**

Suppose that $\mathcal{K}_1 := (K_1, \nu_1)$ and $\mathcal{K}_2 := (K_2, \nu_2)$ are two models of $SCVF_{p,e}$ and $\mathcal{K}_2$ is $|K_1|^+$-saturated. Suppose that $f : F \rightarrow K_2$ is an $\mathcal{L}_{\text{div}}$-embedding, and $F$ is a separably closed subfield of $K_1$. Suppose that for some $n \geq 0, a_1, \ldots, a_n \in F$ satisfy that $\mathcal{K}_1 \models Q_n(a_1, \ldots, a_n)$ and $\mathcal{K}_2 \models Q_n(f(a_1), \ldots, f(a_n))$ Then for any $a \in K_1$ transcendental over $F$ and $\mathcal{K}_1 \models Q_{n+1}(a_1, \ldots, a_n, a)$, there is an $\mathcal{L}_{\text{div}}$-embedding $\bar{f} : F(a)^{\text{sep}} \rightarrow \mathcal{K}_2$ extending $f$, such that $\mathcal{K}_2 \models Q_{n+1}(f(a_1), \ldots, f(a_n), f(a))$
Theorem (Hong, 2016)

\( SCVF_{p,e} \) admits quantifier elimination
Quantifier Elimination

Theorem (Hong, 2016)

$SCVF_{p,e}$ admits quantifier elimination

- The idea is to get the right understanding for $p$-basis and see that the $\lambda$-functions don’t really interact with the rest of the language - that is basically the essence of the previous theorem
Theorem (Hong, 2016)

$\text{SCVF}_{p,e}$ admits quantifier elimination

- The idea is to get the right understanding for $p$-basis and see that the $\lambda$-functions don't really interact with the rest of the language - that is basically the essence of the previous theorem
- Then it is just a typical quantifier elimination "extend-the-embedding-into-a-highly-saturated-model"-proof.
Theorem (Hong, 2016)

$SCVF_{p,e}$ admits quantifier elimination

- The idea is to get the right understanding for $p$-basis and see that the $\lambda$-functions don’t really interact with the rest of the language - that is basically the essence of the previous theorem.
- Then it is just a typical quantifier elimination "extend-the-embedding-into-a-highly-saturated-model"-proof.

Corollary

$SCVF_{p,e}$ is complete.
Theorem (Hong, 2016)

$SCVF_{p,e}$ admits quantifier elimination

- The idea is to get the right understanding for $p$-basis and see that the $\lambda$-functions don’t really interact with the rest of the language - that is basically the essence of the previous theorem
- Then it is just a typical quantifier elimination "extend-the-embedding-into-a-highly-saturated-model"-proof.

Corollary

$SCVF_{p,e}$ is complete.

- This follows easily, because $(F_p, v_{triv})$ is a prime structure for $SCVF_{p,e}$
Definition

Let $\phi(x; y)$ be a partitioned formula. We say that a set $A$ of $|x|$-tuples is \textit{shattered} if we can find a family $(b_I)_{I \subseteq A}$ of $|y|$-tuples such that

$$\mathcal{U} \models \phi(a; b_I) \iff a \in I \quad \text{for all } a \in A.$$  

We say that $\phi(x; y)$ is NIP, if there is not infinite $A$ that is shattered by $\phi(x; y)$, otherwise we say it is IP. And a theory $T$ is NIP if every formula modulo $T$ is NIP.
Definition

Let $\phi(x; y)$ be a partitioned formula. We say that a set $A$ of $|x|$-tuples is \textit{shattered} if we can find a family $(b_I)_{I \subseteq A}$ of $|y|$-tuples such that

$$\mathcal{U} \models \phi(a; b_I) \iff a \in I \quad \text{for all } a \in A.$$ 

We say that $\phi(x; y)$ is NIP, if there is not infinite $A$ that is shattered by $\phi(x; y)$, otherwise we say it is IP. And a theory $T$ is NIP if every formula modulo $T$ is NIP.

- In some sense NIP means that a theory is not too wild in its combinatorical properties.
Definition

Let $\phi(x; y)$ be a partitioned formula. We say that a set $A$ of $|x|$-tuples is **shattered** if we can find a family $(b_I)_{I \subseteq A}$ of $|y|$-tuples such that

$$U \models \phi(a; b_I) \iff a \in I \text{ for all } a \in A.$$  

We say that $\phi(x; y)$ is NIP, if there is not infinite $A$ that is shattered by $\phi(x; y)$, otherwise we say it is IP. And a theory $T$ is NIP if every formula modulo $T$ is NIP.

- In some sense NIP means that a theory is not too wild in its combinatorical properties
- The notion of NIP was introduced by Shelah in his famous classification theory. It is one way to generalize stable theories.
Example

The theory $T_{RG}$ of the Random Graph in the language $\{R\}$ is IP.
Example

The theory $T_{RG}$ of the Random Graph in the language $\{R\}$ is IP.

- Consider the formula $R$
Example

The theory $T_{RG}$ of the Random Graph in the language \{R\} is IP.

- Consider the formula $R$
- For any $n \in \mathbb{N}$ just take any elements $a_1, \ldots, a_n$ in a model of $T_{RG}$
Example

The theory $T_{RG}$ of the Random Graph in the language $\{R\}$ is IP.

• Consider the formula $R$
• For any $n \in \mathbb{N}$ just take any elements $a_1, \ldots, a_n$ in a model of $T_{RG}$
• Then for any subset $I \subset \{a_1, \ldots, a_n\}$ there is with $T_{RG}$ some $b_I$ such that $a_i \leq b_I$ iff $i \in I$
Example

The theory $T_{RG}$ of the Random Graph in the language \{R\} is IP.

- Consider the formula $R$
- For any $n \in \mathbb{N}$ just take any elements $a_1, \ldots, a_n$ in a model of $T_{RG}$
- Then for any subset $I \subset \{a_1, \ldots, a_n\}$ there is with $T_{RG}$ some $b_I$ such that $a_i \leq b_I$ iff $i \in I$
- By compactness we are done
**Definition**

Let $\mathcal{M}$ be a model of $T$ and let $\mathcal{M} \leq \mathcal{B}$ be an extension. Let $p \in S(M)$ some type, we call $q \in S(B)$ coheir of $p$ to $B$, if $p \subset q$ and $q$ is finitely satisfiable in $M$ i.e. for every $\phi(x, \overline{u}) \in q$ there is some $m \in M$ such that $M \models \phi(m, \overline{u})$. A *global coheir* of $p$ is a coheir of $B$ to a monster model $\mathcal{U}$.
Fun with types

**Definition**

Let $\mathcal{M}$ be a model of $T$ and let $\mathcal{M} \leq \mathcal{B}$ be an extension. Let $p \in S(\mathcal{M})$ some type, we call $q \in S(\mathcal{B})$ coheir of $p$ to $\mathcal{B}$, if $p \subset q$ and $q$ is finitely satisfiable in $\mathcal{M}$ i.e. for every $\phi(x, \bar{u}) \in q$ there is some $m \in \mathcal{M}$ such that $\mathcal{M} \models \phi(m, \bar{u})$. A global coheir of $p$ is a coheir of $\mathcal{B}$ to a monster model $\mathcal{U}$.

This notion allows us to formulate a very practical Lemma.

**Fact**

A complete theory $T$ is $\text{NIP}$ iff for all $\mathcal{M} \models T$ and for all $p \in S(\mathcal{M})$, $p$ has at most $2^{|\mathcal{M}| + |T|}$ global coheirs.
Theorem

$SCVF_{p,e}$ is NIP.
Theorem

\( SCVF_{p,e} \) is NIP.

- We will make use of the fact that \( ACVF_{p,p} \) is NIP.
Theorem

$SCVF_{p,e}$ is NIP.

- We will make use of the fact that $ACVF_{p,p}$ is NIP.
- So let $M \models SCVF_{p,e}$ and $U$ a monster model.
Theorem

$SCVF_{p,e}$ is NIP.

- We will make use of the fact that $ACVF_{p,p}$ is NIP.
- So let $M \models SCVF_{p,e}$ and $U$ a monster model.
- Let $p \in S(M)$ and $a$ a realization of $p$. With Löwenheim-Skolem we can realize it in $M \preceq N$ with $|M| = |N|$
Theorem

$SCVF_{p,e}$ is NIP.

- We will make use of the fact that $ACVF_{p,p}$ is NIP.
- So let $M \models SCVF_{p,e}$ and $U$ a monster model.
- Let $p \in S(M)$ and $a$ a realization of $p$. With Löwenheim-Skolem we can realize it in $M \preceq N$ with $|M| = |N|$
- Due to quantifier elimination it is enough to consider quantifier-free types
Theorem

$SCVF_{p,e}$ is NIP.

- We will make use of the fact that $ACVF_{p,p}$ is NIP.
- So let $M \models SCVF_{p,e}$ and $U$ a monster model.
- Let $p \in S(M)$ and $a$ a realization of $p$. With Löwenheim-Skolem we can realize it in $M \preceq N$ with $|M| = |N|$
- Due to quantifier elimination it is enough to consider quantifier-free types
- $ACVF_{p,p}$ is NIP, so $qftp_{div}(N/M^\text{alg})$ has at most $|N|2^{|N|+\aleph_0} = 2^{|N|+\aleph_0}$ coheirs to $U^\text{alg}$
Assertion

Let $q \in S(U)$ be a coheir of $tp(N/M)$ in $\mathcal{L}_{p, \text{div}}$ and $N' \models q$, then for every $q$ we get an unique coheir $q' \in qtp(N/M^\text{alg})$. 
Assertion

Let $q \in S(U)$ be a coheir of $tp(N/M)$ in $\mathcal{L}_{p,div}$ and $N' \models q$, then for every $q$ we get an unique coheir $q' \in qftp(N/M^{alg})$

- We see this by restricting us to formulas that do not involve $\lambda$-functions and note that the $\mathcal{L}_{p,div}$ structure on $U(N')$ is uniquely determined by $U(N')^{sep}$, since $N'$ and $U$ are $p$-independent over $M$. 
Assertion

Let $q \in S(U)$ be a coheir of $tp(N/M)$ in $\mathcal{L}_{p,\text{div}}$ and $N' \models q$, then for every $q$ we get an unique coheir $q' \in qftp(N/M^\text{alg})$

- We see this by restricting us to formulas that do not involve $\lambda$-functions and note that the $\mathcal{L}_{p,\text{div}}$ structure on $U(N')$ is uniquely determined by $U(N')^{\text{sep}}$, since $N'$ and $U$ are $p$-independent over $M$.

- We show this by assuming they were not:
Assertion

Let \( q \in S(U) \) be a coheir of \( tp(N/M) \) in \( \mathcal{L}_{p,\text{div}} \) and \( N' \models q \), then for every \( q \) we get an unique coheir \( q' \in qftp(N/M^{alg}) \)

- We see this by restricting us to formulas that do not involve \( \lambda \)-functions and note that the \( \mathcal{L}_{p,\text{div}} \) structure on \( U(N') \) is uniquely determined by \( U(N')^{sep} \), since \( N' \) and \( U \) are \( p \)-independent over \( M \).
- We show this by assuming they were not:
- Then, if \( N' \) and \( U \) would not be \( p \)-independent there would be a \( c \in U \setminus \{M\} \) and a finite \( p \)-independent subset \( \{x_1, \ldots, x_n\} \subset N' \)- we denote the monomials \( \prod_{i=1}^{n} x_i^{f(i)} \) over those subsets by \( P_1 \ldots P_{p^n} \) such that there are some \( m_1 \ldots m_{p^n} \in M \) not all zero with \( c = \sum_{i=1}^{p^n} m_i P_i \).
Assertion

Let $q \in S(U)$ be a coheir of $tp(N/M)$ in $\mathcal{L}_{p,\text{div}}$ and $N' \models q$, then for every $q$ we get an unique coheir $q' \in qftp(N/M^{alg})$

- We see this by restricting us to formulas that do not involve $\lambda$-functions and note that the $\mathcal{L}_{p,\text{div}}$ structure on $U(N')$ is uniquely determined by $U(N')^{\text{sep}}$, since $N'$ and $U$ are $p$-independent over $M$.
- We show this by assuming they were not:
- Then, if $N'$ and $U$ would not be $p$-independent there would be a $c \in U \setminus \{M\}$ and a finite $p$-independent subset $\{x_1, \ldots, x_n\} \subset N'$- we denote the monomials $\prod_{i=1}^{n} x_i^{f(i)}$ over those subsets by $P_1 \ldots P_{p^n}$ such that there are some $m_1 \ldots m_{p^n} \in M$ not all zero with $c = \sum_{i=1}^{p^n} m_i P_i$
- This would mean that $c' \in N' \setminus M$ and this is a contradiction, because our coheir would not be finitely satisfiable in $M$
Assertion

Let \( q \in S(U) \) be a coheir of \( tp(N/M) \) in \( \mathcal{L}_{p,div} \) and \( N' \models q \), then for every \( q \) we get an unique coheir \( q' \in qftp(N/M^{alg}) \)

- We see this by restricting us to formulas that do not involve \( \lambda \)-functions and note that the \( \mathcal{L}_{p,div} \) structure on \( U(N') \) is uniquely determined by \( U(N')^{sep} \), since \( N' \) and \( U \) are \( p \)-independent over \( M \).
- We show this by assuming they were not:
- Then, if \( N' \) and \( U \) would not be \( p \)-independent there would be a \( c \in U \setminus \{M\} \) and a finite \( p \)-independent subset \( \{x_1, \ldots, x_n\} \subset N' \) - we denote the monomials \( \prod_{i=1}^{n} x_i^{f(i)} \) over those subsets by \( P_1 \ldots P_{p^n} \) such that there are some \( m_1 \ldots m_{p^n} \in M \) not all zero with \( c = \sum_{i=1}^{p^n} m_i P_i \)
- This would mean that \( c' \in N' \setminus M \) and this is a contradiction, because our coheir would not be finitely satisfiable in \( M \)
- \( tp(N/M) \) has at most \( 2^{\aleph_0} \) many coheirs, so there at most \( 2^{\aleph_0} \) coheirs of \( tp(a/M) \) and we are done
References


