On the existential theory of equicharacteristic henselian valued fields

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joint work with Sylvy Anscombe

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The full theory

Some axiomatizations of theories of henselian valued fields:

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Theorem (Denef–Schoutens 2003)

resolution of singularities $\Longrightarrow \mathrm{Th}_\exists(\mathbb{F}_p((t)), v_t)$ *decidable*

The $\forall^k\exists$-theory

We work in a language with 3 sorts K, Γ, k:

$$\mathcal{L} = \left\{ +^K, -^K, \cdot^K, 0^K, 1^K, +^\Gamma, 0^\Gamma, <^\Gamma, +^k, -^k, \cdot^k, 0^k, 1^k, v, \mathrm{res} \right\}.$$

An $\forall^k\exists$-sentence is a sentence of the form

$$(\forall x_1, \ldots, x_r \in k)(\exists y_1, \ldots, y_s \in K)\ \varphi(\mathbf{x}, \mathbf{y})$$

with φ a quantifier-free $\mathcal{L}$-formula.

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Theorem

Let (K, v) and (L, w) be equichar. nontriv. hens. valued fields with $Kv \equiv Lw$. Suppose that $(K, v) \models \forall^k \mathbf{x} \exists \mathbf{y} \varphi(\mathbf{x}, \mathbf{y})$.

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1. *If Kv is perfect, then $(L, w) \models \forall^k\mathbf{x}\exists\mathbf{y}\varphi(\mathbf{x}, \mathbf{y})$.*
2. *In general, there exists $n \in \mathbb{N}$ such that, with $p = \mathrm{char}(Kv)$,*

$$(\forall x_1, \dots, x_r \in Lv)(\exists y_1, \dots, y_s \in L^{p^{-n}})\varphi(\mathbf{x}, \mathbf{y})$$

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The existential theory

$$\mathrm{Th}_{\exists}(\mathbb{F}_p((t)), v_t) \text{ ``='' } (K, v) \textit{ equichar.nontriv.hens.} + Kv \equiv \mathbb{F}_p$$

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But note: $\exists$-$\mathcal{L}$ versus $\exists$-$\mathcal{L}(t) = \forall_1^K \exists$-$\mathcal{L}$ (Denef–Schoutens)

A simple proof

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Tame fields with decidable value group and decidable residue field are decidable (Kuhlmann 2015).
This implies that $\mathrm{Th}_\exists(\mathbb{F}_p((t)), v_t)$ is decidable.

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