Extensions of valuations to rational function fields of arbitrary finite transcendence degree

Franz-Viktor Kuhlmann

AMS Sectional Meeting
Hawaii, March 2019
The extension problem

Given:

• a valued field \((K,v)\),
• a rational function field \(F = K(x_1, \ldots, x_n)\),

describe:

• which value groups and residue fields can appear,
• which defects can appear.

It is a big mistake to believe that these questions can be answered simply by induction on the transcendence degree \(n\).

Franz-Viktor Kuhlmann
Extensions of valuations
The extension problem

Given:
- a valued field \((K, \nu)\),
Given:

- a valued field \((K, v)\),
- a rational function field \(F = K(x_1, \ldots, x_n)\),

It is a big mistake to believe that these questions can be answered simply by induction on the transcendence degree \(n\).
The extension problem

Given:
- a valued field \((K, v)\),
- a rational function field \(F = K(x_1, \ldots, x_n)\),
describe:
- which value groups and residue fields can appear,
- which defects can appear.

It is a big mistake to believe that these questions can be answered simply by induction on the transcendence degree \(n\).

Franz-Viktor Kuhlmann
Extensions of valuations
The extension problem

Given:

- a valued field \((K, v)\),
- a rational function field \(F = K(x_1, \ldots, x_n)\),

describe:

- which value groups and residue fields can appear,
The extension problem

Given:
- a valued field \((K, v)\),
- a rational function field \(F = K(x_1, \ldots, x_n)\),

describe:
- which value groups and residue fields can appear,
- which defects can appear.
Given:
• a valued field \((K, v)\),
• a rational function field \(F = K(x_1, \ldots, x_n)\),

describe:
• which value groups and residue fields can appear,
• which defects can appear.

It is a big mistake to believe that these questions can be answered.
The extension problem

Given:
• a valued field \((K, v)\),
• a rational function field \(F = K(x_1, \ldots, x_n)\),

describe:
• which value groups and residue fields can appear,
• which defects can appear.

It is a big mistake to believe that these questions can be answered simply by induction on the transcendence degree \(n\).
In the case of algebraic function fields,
In the case of algebraic function fields, the extension problem breaks down into two steps:
In the case of algebraic function fields, the extension problem breaks down into two steps:

1) first tackle the rational function field generated by a transcendence basis,
In the case of algebraic function fields, the extension problem breaks down into two steps:

1) first tackle the rational function field generated by a transcendence basis,

2) then use ramification theory for the remaining algebraic extension.
In the case of algebraic function fields, the extension problem breaks down into two steps:
1) first tackle the rational function field generated by a transcendence basis,
2) then use ramification theory for the remaining algebraic extension.

In this presentation, we will address part 1).
More advanced problems

Consider the space of all extensions of \( v \) from \( K \) to \( F \) with some natural topology,

• describe the properties of this topological space.
  • Is the space path-connected?
  • Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field \( \mathbb{R}(x,y) | \mathbb{R} \), see [BM].
Consider the space of all extensions of \( v \) from \( K \) to \( F \) with some natural topology, e.g. the Zariski topology.

Problem:

• describe the properties of this topological space. For instance, one can ask:
  • Is the space path-connected?
  • Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field \( \mathbb{R}(x,y) | \mathbb{R} \), see [BM].
Consider the space of all extensions of \( v \) from \( K \) to \( F \) with some natural topology, e.g. the Zariski topology, its associated patch topology,
More advanced problems

Consider the space of all extensions of \( v \) from \( K \) to \( F \) with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings.

Problem:
• describe the properties of this topological space. For instance, one can ask:
• Is the space path-connected?
• Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field \( R(x, y) | R \), see [BM].
Consider the space of all extensions of $v$ from $K$ to $F$ with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.
More advanced problems

Consider the space of all extensions of $v$ from $K$ to $F$ with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
- Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field $R(x, y) | R$, see [BM].
Consider the space of all extensions of $v$ from $K$ to $F$ with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
Consider the space of all extensions of \( v \) from \( K \) to \( F \) with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
- Which subsets,
More advanced problems

Consider the space of all extensions of $v$ from $K$ to $F$ with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

• describe the properties of this topological space.

For instance, one can ask:

• Is the space path-connected?

• Which subsets, characterized e.g. by conditions on value group and residue field,
Consider the space of all extensions of \( v \) from \( K \) to \( F \) with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
- Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field \( \mathbb{R}(x, y) | \mathbb{R} \), see [BM].
More advanced problems

Consider the space of all extensions of \( v \) from \( K \) to \( F \) with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
- Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field \( \mathbb{R}(x, y)|\mathbb{R} \),
More advanced problems

Consider the space of all extensions of $v$ from $K$ to $F$ with some natural topology, e.g. the Zariski topology, its associated patch topology, or some natural topology like the one induced by orderings. Problem:

- describe the properties of this topological space.

For instance, one can ask:

- Is the space path-connected?
- Which subsets, characterized e.g. by conditions on value group and residue field, lie dense in this space?

These questions have been answered by Brown and Merzel in the case of the function field $\mathbb{R}(x,y)|\mathbb{R}$, see [BM].
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that...
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

Theorem

Every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of $\mathbb{Q}$ can be realized as value group and residue field of a place of the rational function field $\mathbb{Q}(x, y) | \mathbb{Q}$ whose restriction to $\mathbb{Q}$ is the identity.
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure),
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure), the same is not necessarily true for the associated extensions of their invariants.
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field.
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

**Theorem**

*Every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of $\mathbb{Q}$*
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

**Theorem**

*Every additive subgroup of \( \mathbb{Q} \) and every countably generated algebraic extension of \( \mathbb{Q} \) can be realized as value group and residue field of a place*
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

**Theorem**

*Every additive subgroup of \( \mathbb{Q} \) and every countably generated algebraic extension of \( \mathbb{Q} \) can be realized as value group and residue field of a place of the rational function field \( \mathbb{Q}(x,y)|\mathbb{Q} \).*
Since the work of MacLane and Schilling in the 1930’s ([MS]) it is known that while a rational function field is a finitely generated extension of its base field (of comparably “simple” algebraic structure), the same is not necessarily true for the associated extensions of their invariants value group and residue field. For example:

**Theorem**

*Every additive subgroup of $\mathbb{Q}$ and every countably generated algebraic extension of $\mathbb{Q}$ can be realized as value group and residue field of a place of the rational function field $\mathbb{Q}(x, y)|\mathbb{Q}$ whose restriction to $\mathbb{Q}$ is the identity.*
The defect

We denote the value group of a valued field \((F, \nu)\) by \(\nu F\),
The defect

We denote the value group of a valued field \((F, v)\) by \(vF\), and its residue field by \(Fv\).
The defect

We denote the value group of a valued field \((F, v)\) by \(vF\), and its residue field by \(Fv\).

A finite extension \((E|F, v)\) of valued fields is a **defect extension**
We denote the value group of a valued field \((F, v)\) by \(vF\), and its residue field by \(Fv\).

A finite extension \((E|F, v)\) of valued fields is a defect extension if the extension of \(v\) from \(F\) to \(E\) is unique.
We denote the value group of a valued field \((F, v)\) by \(vF\), and its residue field by \(Fv\).

A finite extension \((E|F, v)\) of valued fields is a defect extension if the extension of \(v\) from \(F\) to \(E\) is unique and

\[ [E : F] > (vE : vF)[Ev : Fv]. \]
We denote the value group of a valued field \((F, v)\) by \(vF\), and its residue field by \(Fv\).

A finite extension \((E|F, v)\) of valued fields is a **defect extension** if the extension of \(v\) from \(F\) to \(E\) is unique and

\[
[E : F] > (vE : vF)[Ev : Fv].
\]

The Lemma of Ostrowski tells us that

\[
[E : F] = p^k(vE : vF)[Ev : Fv]
\]
The defect

We denote the value group of a valued field \((F, \nu)\) by \(\nu F\), and its residue field by \(F\nu\).

A finite extension \((E|F, \nu)\) of valued fields is a defect extension if the extension of \(\nu\) from \(F\) to \(E\) is unique and

\[
[E : F] > (\nu E : \nu F)[E\nu : F\nu].
\]

The Lemma of Ostrowski tells us that

\[
[E : F] = p^k(\nu E : \nu F)[E\nu : F\nu]
\]

where \(p\) is the characteristic of the residue field of \(K\) if positive,
We denote the value group of a valued field \((F, v)\) by \(vF\), and its residue field by \(Fv\).

A finite extension \((E|F, v)\) of valued fields is a defect extension if the extension of \(v\) from \(F\) to \(E\) is unique and

\[
[E : F] > (vE : vF)[Ev : Fv].
\]

The Lemma of Ostrowski tells us that

\[
[E : F] = p^k(vE : vF)[Ev : Fv]
\]

where \(p\) is the characteristic of the residue field of \(K\) if positive, and \(p = 1\) otherwise.
As for the value groups and residue fields of rational function fields,
A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect.
A second example

As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:
As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

**Theorem**

Let $K$ be any algebraically closed field of positive characteristic.
As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

**Theorem**

Let $K$ be any algebraically closed field of positive characteristic. Then there exists a valuation $v$ on the rational function field $K(x, y)|K$ whose restriction to $K$ is trivial, such that $(K(x, y), v)$ admits an infinite chain of Galois extensions of degree $p$ and defect $p$. 

Franz-Viktor Kuhlmann  

Extensions of valuations
As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

**Theorem**

Let $K$ be any algebraically closed field of positive characteristic. Then there exists a valuation $v$ on the rational function field $K(x, y)|K$ whose restriction to $K$ is trivial,
As for the value groups and residue fields of rational function fields, bad things can also happen with respect to the defect. The following was proven in [Ku1] in 2004:

**Theorem**

Let $K$ be any algebraically closed field of positive characteristic. Then there exists a valuation $v$ on the rational function field $K(x, y)|K$ whose restriction to $K$ is trivial, such that $(K(x, y), v)$ admits an infinite chain of Galois extensions of degree $p$ and defect $p$. 

Franz-Viktor Kuhlmann

Extensions of valuations
Here is what MacLane and Schilling ([MS])

Theorem

Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Choose any integer $n > \rho + \tau$. Then the rational function field in $n$ variables over $K$ admits a valuation whose restriction to $K$ is trivial, whose value group is $\Gamma$ and whose residue field is $K'$. 
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved.
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

*Let $K$ be any field.*
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

*Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Choose any integer $n > \rho + \tau$. Then the rational function field in $n$ variables over $K$ admits a valuation whose restriction to $K$ is trivial, whose value group is $\Gamma$, and whose residue field is $K'$.*
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Then the rational function field in $n$ variables over $K$ admits a valuation whose restriction to $K$ is trivial, whose value group is $\Gamma$ and whose residue field is $K'$. 
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Choose any integer $n > \rho + \tau$. 
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Choose any integer $n > \rho + \tau$. Then the rational function field in $n$ variables over $K$ admits a valuation
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

*Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Choose any integer $n > \rho + \tau$. Then the rational function field in $n$ variables over $K$ admits a valuation whose restriction to $K$ is trivial,*
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Choose any integer $n > \rho + \tau$. Then the rational function field in $n$ variables over $K$ admits a valuation whose restriction to $K$ is trivial, whose value group is $\Gamma$. 
Here is what MacLane and Schilling ([MS]) and Zariski and Samuel ([ZS], ch. VI, §15, Examples 3 and 4) proved. The first theorem we cited is a special case.

**Theorem**

Let $K$ be any field. Take $\Gamma$ to be any non-trivial ordered abelian group of finite rational rank $\rho$, and $K'$ to be any countably generated extension of $K$ of finite transcendence degree $\tau$. Choose any integer $n > \rho + \tau$. Then the rational function field in $n$ variables over $K$ admits a valuation whose restriction to $K$ is trivial, whose value group is $\Gamma$ and whose residue field is $K'$. 
A new approach, making essential use of Krasner’s Lemma,
A new approach, making essential use of Krasner’s Lemma, was presented in [Ku1].
A new approach, making essential use of Krasner’s Lemma, was presented in [Ku1]. An important observation we use is that while $K$ is relatively algebraically closed in the rational function field $K(x)$,
A new approach, making essential use of Krasner’s Lemma, was presented in [Ku1]. An important observation we use is that while $K$ is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of $v$ from $K$ to $K(x)$.

In fact, the relative algebraic closure of $K$ in $K(x)^h$ can be much larger than the henselization $K^h$ contained in it.

**Definition**

We call the relative algebraic closure of $K$ in $K(x)^h$ the implicit constant field of $(K(x)|K,v)$ and denote it by $IC(K(x)|K,v)$.
A new approach, making essential use of Krasner’s Lemma, was presented in [Ku1]. An important observation we use is that while $K$ is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of $\nu$ from $K$ to $K(x)$. 

**Definition**

We call the relative algebraic closure of $K$ in $K(x)^h$ the implicit constant field of $(K(x)|K,\nu)$ and denote it by $\text{IC}(K(x)|K,\nu)$. 

Franz-Viktor Kuhlmann

Extensions of valuations
A new approach, making essential use of Krasner’s Lemma, was presented in [Ku1]. An important observation we use is that while $K$ is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of $v$ from $K$ to $K(x)$. In fact, the relative algebraic closure of $K$ in $K(x)^h$...
A new approach, making essential use of Krasner’s Lemma, was presented in [Ku1]. An important observation we use is that while $K$ is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of $v$ from $K$ to $K(x)$. In fact, the relative algebraic closure of $K$ in $K(x)^h$ can be much larger than the henselization $K^h$ contained in it.
A new approach, making essential use of Krasner’s Lemma, was presented in [Ku1]. An important observation we use is that while $K$ is relatively algebraically closed in the rational function field $K(x)$, this does not necessarily remain true when $K(x)$ is replaced by the henselization $K(x)^h$ with respect to some chosen extension of $v$ from $K$ to $K(x)$. In fact, the relative algebraic closure of $K$ in $K(x)^h$ can be much larger than the henselization $K^h$ contained in it.

**Definition**

*We call the relative algebraic closure of $K$ in $K(x)^h$ the implicit constant field of $(K(x)|K, v)$ and denote it by $\text{IC} (K(x)|K, v)$.*
The following result is proven in [Ku1]:
The following result is proven in [Ku1]:

**Theorem**

Let \((L|K,\nu)\) be a countable separable-algebraic extension of nontrivially valued fields.
The following result is proven in [Ku1]:

**Theorem**

Let $(L|K, v)$ be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of $v$ from $L$ to the algebraic closure $L(x)^{ac} = K(x)^{ac}$ of the rational function field $K(x)$. 
The following result is proven in [Ku1]:

**Theorem**

Let $(L|K, v)$ be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of $v$ from $L$ to the algebraic closure $L(x)^{ac} = K(x)^{ac}$ of the rational function field $K(x)$ such that, upon taking henselizations in $(K(x)^{ac}, v)$,
The following result is proven in [Ku1]:

**Theorem**

Let \( (L|K, v) \) be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of \( v \) from \( L \) to the algebraic closure \( L(x)^{\text{ac}} = K(x)^{\text{ac}} \) of the rational function field \( K(x) \) such that, upon taking henselizations in \( (K(x)^{\text{ac}}, v) \),

\[
L^h = \text{IC} (K(x)|K, v). 
\]
The following result is proven in [Ku1]:

**Theorem**

Let \((L|K, v)\) be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of \(v\) from \(L\) to the algebraic closure \(L(x)^{ac} = K(x)^{ac}\) of the rational function field \(K(x)\) such that, upon taking henselizations in \((K(x)^{ac}, v)\),

\[ L^h = \text{IC} (K(x)|K, v) \, . \]

This theorem can be used efficiently to prove the following comprehensive result.
The following result is proven in [Ku1]:

**Theorem**

Let \((L|K, v)\) be a countable separable-algebraic extension of nontrivially valued fields. Then there is an extension of \(v\) from \(L\) to the algebraic closure \(L(x)^{ac} = K(x)^{ac}\) of the rational function field \(K(x)\) such that, upon taking henselizations in \((K(x)^{ac}, v)\),

\[
L^h = \text{IC} \; (K(x)|K, v).
\]

This theorem can be used efficiently to prove the following comprehensive result for the extensions of \(v\) from \(K\) to \(K(x)\).
The case of transcendence degree 1

Theorem

Take any valued field \((K, v)\),
The case of transcendence degree 1

Theorem

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\)
The case of transcendence degree 1

Theorem

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0 / vK\) is a torsion group,
The case of transcendence degree 1

**Theorem**

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0 / vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\).
The case of transcendence degree 1

Theorem

*Take any valued field* \((K,v)\), *an ordered abelian group extension* \(\Gamma_0\) *of* \(vK\) *such that* \(\Gamma_0/vK\) *is a torsion group, and an algebraic extension* \(k_0\) *of* \(Kv\). *Further, take* \(\Gamma\) *to be the abelian group* \(\Gamma_0 \oplus \mathbb{Z}\) *endowed with any extension of the ordering of* \(\Gamma_0\) *such that* \(\Gamma_0/vK\) *and* \(k_0\) *are finite.*

*If* \(v\) *is trivial on* \(K\), *then assume in addition that* \(k_0\) *is simple.*

*Then there is an extension of* \(v\) *from* \(K\) *to the rational function field* \(K(x)\) *which has value group* \(\Gamma\) *and residue field* \(k_0\).

*If* \(v\) *is non-trivial on* \(K\), *then there is also an extension which has value group* \(\Gamma_0\) *and as residue field a rational function field in one variable over* \(k_0\).

*Now assume that* \(v\) *is non-trivial on* \(K\) *and that* \(\Gamma_0/vK\) *and* \(k_0\) *are countably generated.*

*Suppose that at least one of them is infinite or that* \((K,v)\) *admits an immediate transcendental extension.*

*Then there is an extension of* \(v\) *from* \(K\) *to* \(K(x)\) *which has value group* \(\Gamma_0\) *and residue field* \(k_0\).
The case of transcendence degree 1

Theorem

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0/vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).
The case of transcendence degree 1

**Theorem**

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0 / vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0 / vK\) and \(k_0|Kv\) are finite.

If \(v\) is trivial on \(K\), then assume in addition that \(k_0|Kv\) is simple.

Then there is an extension of \(v\) from \(K\) to the rational function field \(K(x)\) which has value group \(\Gamma\) and residue field \(k_0\).

If \(v\) is non-trivial on \(K\), then there is also an extension which has value group \(\Gamma_0\) and as residue field a rational function field in one variable over \(k_0\).

Now assume that \(v\) is non-trivial on \(K\) and that \(\Gamma_0 / vK\) and \(k_0|Kv\) are countably generated. Suppose that at least one of them is infinite or that \((K, v)\) admits an immediate transcendental extension.

Then there is an extension of \(v\) from \(K\) to \(K(x)\) which has value group \(\Gamma_0\) and residue field \(k_0\).
The case of transcendence degree 1

Theorem

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0 / vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0 / vK\) and \(k_0 | Kv\) are finite. If \(v\) is trivial on \(K\), then assume in addition that \(k_0 | Kv\) is simple.
The case of transcendence degree 1

Theorem

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0/vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0/vK\) and \(k_0|Kv\) are finite. If \(v\) is trivial on \(K\), then assume in addition that \(k_0|Kv\) is simple. Then there is an extension of \(v\) from \(K\) to the rational function field \(K(x)\).
Theorem

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0/vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0/vK\) and \(k_0|Kv\) are finite. If \(v\) is trivial on \(K\), then assume in addition that \(k_0|Kv\) is simple. Then there is an extension of \(v\) from \(K\) to the rational function field \(K(x)\) which has value group \(\Gamma\) and residue field \(k_0\).
The case of transcendence degree 1

Theorem

Take any valued field \((K, \nu)\), an ordered abelian group extension \(\Gamma_0\) of \(\nu K\) such that \(\Gamma_0 / \nu K\) is a torsion group, and an algebraic extension \(k_0\) of \(K\nu\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0 / \nu K\) and \(k_0 | K\nu\) are finite. If \(\nu\) is trivial on \(K\), then assume in addition that \(k_0 | K\nu\) is simple. Then there is an extension of \(\nu\) from \(K\) to the rational function field \(K(x)\) which has value group \(\Gamma\) and residue field \(k_0\). If \(\nu\) is non-trivial on \(K\), then there is also an extension which has value group \(\Gamma_0\).
The case of transcendence degree 1

**Theorem**

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0 \mod vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0 \mod vK\) and \(k_0 \mid Kv\) are finite. If \(v\) is trivial on \(K\), then assume in addition that \(k_0 \mid Kv\) is simple. Then there is an extension of \(v\) from \(K\) to the rational function field \(K(x)\) which has value group \(\Gamma\) and residue field \(k_0\). If \(v\) is non-trivial on \(K\), then there is also an extension which has value group \(\Gamma_0\) and as residue field a rational function field in one variable over \(k_0\).
The case of transcendence degree 1

Theorem

Take any valued field \((K, v)\), an ordered abelian group extension \(\Gamma_0\) of \(vK\) such that \(\Gamma_0 / vK\) is a torsion group, and an algebraic extension \(k_0\) of \(Kv\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0 / vK\) and \(k_0|Kv\) are finite. If \(v\) is trivial on \(K\), then assume in addition that \(k_0|Kv\) is simple. Then there is an extension of \(v\) from \(K\) to the rational function field \(K(x)\) which has value group \(\Gamma\) and residue field \(k_0\). If \(v\) is non-trivial on \(K\), then there is also an extension which has value group \(\Gamma_0\) and as residue field a rational function field in one variable over \(k_0\).

Now assume that \(v\) is non-trivial on \(K\) and that \(\Gamma_0 / vK\) and \(k_0|Kv\) are countably generated.
The case of transcendence degree 1

Theorem

Take any valued field \((K, \nu)\), an ordered abelian group extension \(\Gamma_0\) of \(\nu K\) such that \(\Gamma_0 / \nu K\) is a torsion group, and an algebraic extension \(k_0\) of \(K\nu\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0 / \nu K\) and \(k_0|K\nu\) are finite. If \(\nu\) is trivial on \(K\), then assume in addition that \(k_0|K\nu\) is simple. Then there is an extension of \(\nu\) from \(K\) to the rational function field \(K(x)\) which has value group \(\Gamma\) and residue field \(k_0\). If \(\nu\) is non-trivial on \(K\), then there is also an extension which has value group \(\Gamma_0\) and as residue field a rational function field in one variable over \(k_0\).

Now assume that \(\nu\) is non-trivial on \(K\) and that \(\Gamma_0 / \nu K\) and \(k_0|K\nu\) are countably generated. Suppose that at least one of them is infinite or that \((K, \nu)\) admits an immediate transcendental extension.
The case of transcendence degree 1

Theorem

Take any valued field \((K, \nu)\), an ordered abelian group extension \(\Gamma_0\) of \(\nu K\) such that \(\Gamma_0 / \nu K\) is a torsion group, and an algebraic extension \(k_0\) of \(K\nu\). Further, take \(\Gamma\) to be the abelian group \(\Gamma_0 \oplus \mathbb{Z}\) endowed with any extension of the ordering of \(\Gamma_0\).

Assume first that \(\Gamma_0 / \nu K\) and \(k_0 | K\nu\) are finite. If \(\nu\) is trivial on \(K\), then assume in addition that \(k_0 | K\nu\) is simple. Then there is an extension of \(\nu\) from \(K\) to the rational function field \(K(x)\) which has value group \(\Gamma\) and residue field \(k_0\). If \(\nu\) is non-trivial on \(K\), then there is also an extension which has value group \(\Gamma_0\) and as residue field a rational function field in one variable over \(k_0\).

Now assume that \(\nu\) is non-trivial on \(K\) and that \(\Gamma_0 / \nu K\) and \(k_0 | K\nu\) are countably generated. Suppose that at least one of them is infinite or that \((K, \nu)\) admits an immediate transcendental extension. Then there is an extension of \(\nu\) from \(K\) to \(K(x)\) which has value group \(\Gamma_0\) and residue field \(k_0\).
The converse

**Theorem**

Let \((K(x)|K, v)\) be a valued rational function field.

Then one and only one of the following three cases holds:

1. \(v_K(x) \cong \Gamma_0 \oplus \mathbb{Z}\), where \(\Gamma_0\) is a finite extension of ordered abelian groups, and \(K(x)|v_K\) is finite;
2. \(v_K(x)/v_K\) is finite, and \(K(x)|v\) is a rational function field in one variable over a finite extension of \(Kv\);
3. \(v_K(x)/v_K\) is a torsion group and \(K(x)|v\) is algebraic.

In all cases, \(v_K(x)/v_K\) is countable and \(K(x)|v\) is countably generated.

This theorem contains Jack Ohm's Ruled Residue Theorem as a special case.
The converse

Theorem

Let \((K(x)|K, v)\) be a valued rational function field. Then one and only one of the following three cases holds:

1) \(vK(x) \cong \Gamma_0 \otimes \mathbb{Z}\), where \(\Gamma_0\) is a finite extension of ordered abelian groups, and \(K(x)\) is finite;
2) \(vK(x)/vK\) is finite, and \(K(x)/v\) is a rational function field in one variable over a finite extension of \(Kv\);
3) \(vK(x)/vK\) is a torsion group and \(K(x)\) is algebraic.

In all cases, \(vK(x)/vK\) is countable and \(K(x)\) is countably generated.

This theorem contains Jack Ohm's Ruled Residue Theorem as a special case.
The converse

Theorem

Let \((K(x)|K, v)\) be a valued rational function field. Then one and only one of the following three cases holds:

1) \(vK(x) \cong \Gamma_0 \oplus \mathbb{Z}\), where \(\Gamma_0|vK\) is a finite extension of ordered abelian groups, and \(K(x)v|Kv\) is finite;

This theorem contains Jack Ohm's Ruled Residue Theorem as a special case.
The converse

**Theorem**

Let \((K(x) | K, v)\) be a valued rational function field. Then one and only one of the following three cases holds:

1) \(vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}\), where \(\Gamma_0|vK\) is a finite extension of ordered abelian groups, and \(K(x)v|Kv\) is finite;

2) \(vK(x)/vK\) is finite, and \(K(x)v\) is a rational function field in one variable over a finite extension of \(Kv\);

3) \(vK(x)/vK\) is a torsion group and \(K(x)v\) is algebraic.

In all cases, \(vK(x)/vK\) is countable and \(K(x)v\) is countably generated.

This theorem contains Jack Ohm’s Ruled Residue Theorem as a special case.
The converse

Theorem

Let \((K(x)|K, v)\) be a valued rational function field. Then one and only one of the following three cases holds:

1) \(vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}\), where \(\Gamma_0|vK\) is a finite extension of ordered abelian groups, and \(K(x)v|Kv\) is finite;

2) \(vK(x)/vK\) is finite, and \(K(x)v\) is a rational function field in one variable over a finite extension of \(Kv\);

3) \(vK(x)/vK\) is a torsion group and \(K(x)v|Kv\) is algebraic.

In all cases, \(vK(x)/vK\) is countable and \(K(x)v|Kv\) is countably generated.

This theorem contains Jack Ohm's Ruled Residue Theorem as a special case.
The converse

Theorem

Let \((K(x)|K,v)\) be a valued rational function field. Then one and only one of the following three cases holds:

1) \(vK(x) \cong \Gamma_0 \oplus \mathbb{Z}\), where \(\Gamma_0|vK\) is a finite extension of ordered abelian groups, and \(K(x)v|Kv\) is finite;

2) \(vK(x)/vK\) is finite, and \(K(x)v\) is a rational function field in one variable over a finite extension of \(Kv\);

3) \(vK(x)/vK\) is a torsion group and \(K(x)v|Kv\) is algebraic.

In all cases, \(vK(x)/vK\) is countable and \(K(x)v|Kv\) is countably generated.

This theorem contains Jack Ohm's Ruled Residue Theorem as a special case.

Franz-Viktor Kuhlmann
Extensions of valuations
The converse

Theorem

Let \((K(x)|K,v)\) be a valued rational function field. Then one and only one of the following three cases holds:

1) \(vK(x) \simeq \Gamma_0 \oplus \mathbb{Z}\), where \(\Gamma_0|vK\) is a finite extension of ordered abelian groups, and \(K(x)v|Kv\) is finite;

2) \(vK(x)/vK\) is finite, and \(K(x)v\) is a rational function field in one variable over a finite extension of \(Kv\);

3) \(vK(x)/vK\) is a torsion group and \(K(x)v|Kv\) is algebraic.

In all cases, \(vK(x)/vK\) is countable and \(K(x)v|Kv\) is countably generated.

This theorem contains Jack Ohm’s Ruled Residue Theorem as a special case.
The case of higher transcendence degree

When asking which value group and residue field extensions may appear when $v$ is extended from $K$ to $K(x_1, \ldots, x_n)$,
The case of higher transcendence degree

When asking which value group and residue field extensions may appear when \( v \) is extended from \( K \) to \( K(x_1, \ldots, x_n) \), it may at first appear that this question can be settled by simple induction on \( n \).
When asking which value group and residue field extensions may appear when \( \nu \) is extended from \( K \) to \( K(x_1, \ldots, x_n) \), it may at first appear that this question can be settled by simple induction on \( n \).

But consider the following case: \( (K, \nu) \) is a maximal field,
When asking which value group and residue field extensions may appear when $v$ is extended from $K$ to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on $n$.

But consider the following case: $(K, v)$ is a maximal field, i.e., it does not admit any nontrivial immediate extension,
When asking which value group and residue field extensions may appear when \( v \) is extended from \( K \) to \( K(x_1, \ldots, x_n) \), it may at first appear that this question can be settled by simple induction on \( n \).

But consider the following case: \((K, v)\) is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions \( \Gamma|vK \) and \( k|Kv \) such that
When asking which value group and residue field extensions may appear when \( v \) is extended from \( K \) to \( K(x_1, \ldots, x_n) \), it may at first appear that this question can be settled by simple induction on \( n \).

But consider the following case: \((K, v)\) is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions \( \Gamma | vK \) and \( k | Kv \) such that

\[
\text{rr} \, \Gamma / vK + \text{trdeg} \, k | Kv \leq 1.
\]
When asking which value group and residue field extensions may appear when \( v \) is extended from \( K \) to \( K(x_1, \ldots, x_n) \), it may at first appear that this question can be settled by simple induction on \( n \).

But consider the following case: \((K, v)\) is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions \( \Gamma|vK \) and \( k|Kv \) such that

\[
\operatorname{rr} \Gamma/vK + \operatorname{trdeg} k|Kv \leq 1.
\]

Is it possible to find an extension of \( v \) to \( K(x, y) \)?
When asking which value group and residue field extensions may appear when $\nu$ is extended from $K$ to $K(x_1, \ldots, x_n)$, it may at first appear that this question can be settled by simple induction on $n$.

But consider the following case: $(K, \nu)$ is a maximal field, i.e., it does not admit any nontrivial immediate extension, and we have chosen extensions $\Gamma|\nu K$ and $k|K\nu$ such that

$$\text{rr } \Gamma/\nu K + \text{trdeg } k|K\nu \leq 1.$$ 

Is it possible to find an extension of $\nu$ to $K(x, y)$ such that $\nu K(x, y) = \Gamma$ and $K(x, y) = k$?
From the extension theorem for the transcendence degree 1 case
From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of $v$ to $K(x)$ exists.
From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of $v$ to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of $v$ to $K(x, y)$. 
From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of $v$ to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of $v$ to $K(x, y)$.

More precisely, we ask for criteria on an extension $(L|K, v)$ which guarantee
The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of \( v \) to \( K(x) \) exists so that \( (K(x), v) \) admits an immediate extension of \( v \) to \( K(x, y) \).

More precisely, we ask for criteria on an extension \( (L|K, v) \) which guarantee that \( (L, v) \) admits a maximal immediate extension of infinite transcendence degree.
The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of $v$ to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of $v$ to $K(x, y)$.

More precisely, we ask for criteria on an extension $(L|K, v)$ which guarantee that $(L, v)$ admits a maximal immediate extension of infinite transcendence degree. Through a far reaching generalization of a construction method introduced by MacLane and Schilling in [MS],

Franz-Viktor Kuhlmann
Extensions of valuations
The immediate extension problem

From the extension theorem for the transcendence degree 1 case one deduces that the answer depends on the question whether a suitable extension of $v$ to $K(x)$ exists so that $(K(x), v)$ admits an immediate extension of $v$ to $K(x, y)$.

More precisely, we ask for criteria on an extension $(L|K, v)$ which guarantee that $(L, v)$ admits a maximal immediate extension of infinite transcendence degree. Through a far reaching generalization of a construction method introduced by MacLane and Schilling in [MS], such criteria were given in [BK2].
Theorem

Take an extension $(L|K, v)$ of finite transcendence degree $\geq 0$, with $v$ nontrivial on $L$. Assume that one of the following four cases holds:

- **valuation-transcendental case**: $v_L/v_K$ is not a torsion group, or $Lv|Kv$ is transcendental;
- **value-algebraic case**: $v_L/v_K$ contains elements of arbitrarily high order, or there is a subgroup $\Gamma \subseteq v_L$ containing $v_K$ such that $\Gamma / v_K$ is an infinite torsion group and the order of each of its elements is prime to the characteristic exponent of $Kv$;
- **residue-algebraic case**: $Lv$ contains elements of arbitrarily high degree over $Kv$;
- **separable-algebraic case**: $L|K$ contains a separable-algebraic subextension $L_0|K$ such that within some henselization of $L$, the corresponding extension $L_0^h|K^h$ is infinite.

Then each maximal immediate extension of $(L, v)$ has infinite transcendence degree over $L$. 
Power series vs. pseudo Cauchy sequences

In [MS], MacLane and Schilling developed a method to produce algebraically independent power series.
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem.
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead,
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski.
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky’s theorems says that if \((K(x)|K, v)\) is immediate,
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky’s theorems says that if \((K(x)|K,v)\) is immediate, then \(x\) is the limit of a pseudo Cauchy sequence
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky’s theorems says that if \((K(x)|K, v)\) is immediate, then \(x\) is the limit of a pseudo Cauchy sequence that has no limit in \(K\).
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky’s theorems says that if \((K(x)|K, v)\) is immediate, then \(x\) is the limit of a pseudo Cauchy sequence that has no limit in \(K\). Further, Kaplansky distinguishes pseudo Cauchy sequence of \textit{algebraic type}
In [MS], MacLane and Schilling developed a method to produce algebraically independent power series. But working with power series is not suitable for the proof of the previous theorem. We have worked with pseudo Cauchy sequences instead, which were introduced by Ostrowski. In [Ka], Kaplansky refined their theory.

One of Kaplansky’s theorems says that if \((K(x)|K, \nu)\) is immediate, then \(x\) is the limit of a pseudo Cauchy sequence that has no limit in \(K\). Further, Kaplansky distinguishes pseudo Cauchy sequence of algebraic type and those of transcendental type.
Valuation algebraic extensions

It should be noted that if \((K(x) | K, v)\) is an extension such that

\[ v_{K(x)}(x) / v_{K}\]

is a torsion group and \(K(x) | v_{K}\) is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure \(K_{ac}\) of \(K\), the extension \((K_{ac}(x) | K_{ac}, v)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in \(K_{ac}\).

This observation is extensively used in our main theorem on the extensions of a valuation from \(K\) to \(K(x_1, \ldots, x_n)\) (see [Ku1, BK1]). (Due to the failure of simple induction, this theorem needs several case distinctions and is too long to be put on one slide.)
It should be noted that if \((K(x)|K, \nu)\) is an extension such that 
\(\nu K(x) / \nu K\) is a torsion group
It should be noted that if \((K(x)|K, \nu)\) is an extension such that \(\nu K(x)/\nu K\) is a torsion group and \(K(x)\nu|K\nu\) is algebraic, then if we lift it up to the algebraic closure \(K_{ac}\) of \(K\), the extension \((K_{ac}(x)|K_{ac}, \nu)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type. Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in \(K_{ac}\). This observation is extensively used in our main theorem on the extensions of a valuation from \(K\) to \(K(x_1, \ldots, x_n)\) (see \([Ku1, BK1]\)). (Due to the failure of simple induction, this theorem needs several case distinctions and is too long to be put on one slide.)
It should be noted that if \((K(x)|K,v)\) is an extension such that 
\(vK(x)/vK\) is a torsion group and \(K(x)v|Kv\) is algebraic (we call such extensions \textit{valuation algebraic}),

\begin{align*}
\textit{Valuation algebraic extensions}
\end{align*}
It should be noted that if \((K(x)|K, \nu)\) is an extension such that \(\nu K(x)/\nu K\) is a torsion group and \(K(x)\nu|K\nu\) is algebraic (we call such extensions \textbf{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\),
It should be noted that if \((K(x)|K, v)\) is an extension such that 
\(vK(x)/vK\) is a torsion group and 
\(K(x)v|Kv\) is algebraic (we call such extensions \textit{valuation algebraic}),
then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\),
the extension \((K^{ac}(x)|K^{ac}, v)\) becomes immediate.
Valuation algebraic extensions

It should be noted that if \((K(x)|K, \nu)\) is an extension such that \(\nu K(x)/\nu K\) is a torsion group and \(K(x)\nu|K\nu\) is algebraic (we call such extensions \textit{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{\text{ac}}\) of \(K\), the extension \((K^{\text{ac}}(x)|K^{\text{ac}}, \nu)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.
It should be noted that if \((K(x)|K, \nu)\) is an extension such that 
\(\nu K(x)/\nu K\) is a torsion group and \(K(x)\nu|K\nu\) is algebraic (we call such extensions \textit{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\), the extension \((K^{ac}(x)|K^{ac}, \nu)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties,
It should be noted that if \((K(x)|K, \nu)\) is an extension such that \(\nu K(x)/\nu K\) is a torsion group and \(K(x)\nu|K\nu\) is algebraic (we call such extensions \textbf{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\), the extension \((K^{ac}(x)|K^{ac}, \nu)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in \(K^{ac}\).
It should be noted that if \((K(x)|K, v)\) is an extension such that 
vK(x)/vK is a torsion group and \(K(x)v|Kv\) is algebraic (we call such extensions \textit{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\), the extension \((K^{ac}(x)|K^{ac}, v)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in \(K^{ac}\).

This observation is extensively used in our main theorem
It should be noted that if $(K(x)|K, \nu)$ is an extension such that $\nu K(x)/\nu K$ is a torsion group and $K(x)\nu|K\nu$ is algebraic (we call such extensions valuation algebraic), then if we lift it up to the algebraic closure $K^{ac}$ of $K$, the extension $(K^{ac}(x)|K^{ac}, \nu)$ becomes immediate and $x$ will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in $K^{ac}$.

This observation is extensively used in our main theorem on the extensions of a valuation from $K$ to $K(x_1, \ldots, x_n)$ (see [Ku1, BK1]).
It should be noted that if \((K(x)|K, v)\) is an extension such that \(vK(x)/vK\) is a torsion group and \(K(x)v|Kv\) is algebraic (we call such extensions \textit{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\), the extension \((K^{ac}(x)|K^{ac}, v)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in \(K^{ac}\).

This observation is extensively used in our main theorem on the extensions of a valuation from \(K\) to \(K(x_1, \ldots, x_n)\) (see [Ku1, BK1]). (Due to the failure of simple induction,
It should be noted that if \((K(x)|K, v)\) is an extension such that 
\(\nu K(x)/\nu K\) is a torsion group and \(K(x)\nu|K\nu\) is algebraic (we call such extensions \textit{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\), the extension \((K^{ac}(x)|K^{ac}, \nu)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in \(K^{ac}\).

This observation is extensively used in our main theorem on the extensions of a valuation from \(K\) to \(K(x_1, \ldots, x_n)\) (see [Ku1, BK1]). (Due to the failure of simple induction, this theorem needs several case distinctions
It should be noted that if \((K(x)|K, v)\) is an extension such that 
\(vK(x)/vK\) is a torsion group and \(K(x)v|Kv\) is algebraic (we call such extensions \textit{valuation algebraic}), then if we lift it up to the algebraic closure \(K^{ac}\) of \(K\), the extension \((K^{ac}(x)|K^{ac}, v)\) becomes immediate and \(x\) will be the limit of a pseudo Cauchy sequence of transcendental type.

Hence for the construction of valuation extensions with desired properties, it is a good idea to work with pseudo Cauchy sequences in \(K^{ac}\).

This observation is extensively used in our main theorem on the extensions of a valuation from \(K\) to \(K(x_1, \ldots, x_n)\) (see [Ku1, BK1]). (Due to the failure of simple induction, this theorem needs several case distinctions and is too long to be put on one slide.)
That theorem gives an almost complete description of all extensions of value group and residue field.
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field $(K, v)$,
Almost complete description

That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field $(K, v)$, an extension $\Gamma$ of $vK$ such that $\Gamma/vK$ is torsion,
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field $(K, v)$, an extension $\Gamma$ of $vK$ such that $\Gamma / vK$ is torsion, and an algebraic extension $k|Kv$. 
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field $(K, v)$, an extension $\Gamma$ of $vK$ such that $\Gamma/vK$ is torsion, and an algebraic extension $k|Kv$. Then the theorem states:
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field $(K, v)$, an extension $\Gamma$ of $vK$ such that $\Gamma/vK$ is torsion, and an algebraic extension $k|Kv$. Then the theorem states:

If at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from $K$ to $F = K(x_1, \ldots, x_n)$. However, a gap between the existence theorem and its converse has still remained.

Take a valued field $(K, v)$, an extension $\Gamma$ of $vK$ such that $\Gamma/vK$ is torsion, and an algebraic extension $k|Kv$. Then the theorem states:

If at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite or $(K, v)$ admits an immediate extension of transcendence degree $n$, 

Franz-Viktor Kuhlmann
Extensions of valuations
That theorem gives an almost complete description of all extensions of value group and residue field that can be realized by an extension of a valuation from \( K \) to \( F = K(x_1, \ldots, x_n) \). However, a gap between the existence theorem and its converse has still remained.

Take a valued field \((K, v)\), an extension \( \Gamma \) of \( vK \) such that \( \Gamma/vK \) is torsion, and an algebraic extension \( k|Kv \). Then the theorem states:

*If at least one of the two extensions \( \Gamma|vK \) and \( k|Kv \) is infinite or \((K, v)\) admits an immediate extension of transcendence degree \( n \), then there is an extension of \( v \) from \( K \) to \( F \) with \( vF = \Gamma \) and \( Fv = k \).*
In [BK2] the following is proven:

*The converse holds if char $K_v = 0$, or if char $K_v = p$,.*
In [BK2] the following is proven:

*The converse holds if \( \text{char } Kv = 0 \), or if \( \text{char } Kv = p \), \( vK \) is \( p \)-divisible*
The converse

In [BK2] the following is proven:

*The converse holds if char $K_v = 0$, or if char $K_v = p$, $vK$ is $p$-divisible and $K_v$ is perfect.*
In [BK2] the following is proven:

*The converse holds if char \( K_v = 0 \), or if char \( K_v = p \), \( vK \) is \( p \)-divisible and \( K_v \) is perfect.*

For this class of fields (which includes the tame fields
In [BK2] the following is proven:

*The converse holds if char $Kv = 0$, or if char $Kv = p$, $vK$ is $p$-divisible and $Kv$ is perfect.*

For this class of fields (which includes the tame fields but also fields that allow nontrivial defect extensions)
In [BK2] the following is proven:

The converse holds if $\text{char } K_v = 0$, or if $\text{char } K_v = p$, $vK$ is $p$-divisible and $K_v$ is perfect.

For this class of fields (which includes the tame fields but also fields that allow nontrivial defect extensions) quite a bit of results are known (see e.g. [BK3]). Valued fields not in this class pose even harder problems.
The power series problem

Take any field $k$ and a power series $y$ in $x$.
The power series problem

Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$.
The power series problem

Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x, y), v)$?
The power series problem

Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x,y)$?

How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)? If $\text{char } k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same with pseudo Cauchy sequences in place of power series. The answer is provided by homogeneous sequences.
The power series problem

Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen?

Franz-Viktor Kuhlmann

Extensions of valuations
Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x, y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x, y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized?
The power series problem

Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x, y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x, y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\text{char } k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same with pseudo Cauchy sequences in place of power series. The answer is provided by homogeneous sequences.

Franz-Viktor Kuhlmann
Extensions of valuations
Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)? If char $k = 0$, the answer is not difficult.
The power series problem

Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x, y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x, y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If char $k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field,
Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\text{char } k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same.
The power series problem

Take any field $k$ and a power series $y$ in $x$ with coefficients in $k^{ac}$ and exponents in $\mathbb{Q}$. What are value group and residue field of $(k(x,y), v)$ where $v$ is an extension of the $x$-adic valuation of $k(x)$ to $k(x,y)$? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If $\text{char } k = 0$, the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same with pseudo Cauchy sequences in place of power series.
The power series problem

Take any field \( k \) and a power series \( y \) in \( x \) with coefficients in \( k^{ac} \) and exponents in \( \mathbb{Q} \). What are value group and residue field of \((k(x, y), v)\) where \( v \) is an extension of the \( x \)-adic valuation of \( k(x) \) to \( k(x, y) \)? How can the coefficients and the exponents be chosen such that given value groups and residue fields are realized (and the proof that they are is as easy as possible)?

If \( \text{char} \ k = 0 \), the answer is not difficult. However, for our problem of constructing extensions with given implicit constant field, we need a generalization that allows us to do the same with pseudo Cauchy sequences in place of power series. The answer is provided by homogeneous sequences.
A variant of Krasner’s Lemma

The main idea is to employ the following form of Krasner’s Lemma.

Let \( K \) be a separable-algebraic extension of \( K(a) \) and \( (K(b), v) \) a valued field extension of \( (K, v) \) such that

\[
v(b - a) > \text{kras}(a, K)
\]

Then for every extension of \( v \) from \( K(a, b) \) to its algebraic closure \( K(b, K(a, b)) = K(b) \), the element \( a \) lies in the henselization of \( (K(b), v) \) in \( (K(b), v) \).
A variant of Krasner’s Lemma

The main idea is to employ the following form of Krasner’s Lemma. We define

\[ \text{kras}(a, K) := \max\{\nu(\tau a - \sigma a) \mid \sigma, \tau \in \text{Gal } K \text{ and } \tau a \neq \sigma a\}. \]
A variant of Krasner’s Lemma

The main idea is to employ the following form of Krasner’s Lemma. We define

\[ \text{kras}(a, K) := \max\{v(\tau a - \sigma a) \mid \sigma, \tau \in \text{Gal} K \text{ and } \tau a \neq \sigma a\}. \]

**Theorem**

*Take* $K(a) | K$ *to be a separable-algebraic extension*
A variant of Krasner’s Lemma

The main idea is to employ the following form of Krasner’s Lemma. We define

\[ \text{kras}(a, K) := \max\{v(\tau a - \sigma a) \mid \sigma, \tau \in \text{Gal } K \text{ and } \tau a \neq \sigma a\}. \]

Theorem

Take \( K(a)|K \) to be a separable-algebraic extension and \( (K(a, b), v) \) to be any valued field extension of \( (K, v) \) such that

\[ v(b-a) > \text{kras}(a, K) \]
The main idea is to employ the following form of Krasner’s Lemma. We define

\[ \text{kras}(a, K) := \max\{v(\tau a - \sigma a) \mid \sigma, \tau \in \text{Gal} K \text{ and } \tau a \neq \sigma a\}. \]

**Theorem**

*Take* \( K(a)|K \) *to be a separable-algebraic extension and* \( (K(a,b), v) \) *to be any valued field extension of* \( (K, v) \) *such that*

\[ v(b - a) > \text{kras}(a, K). \]
A variant of Krasner’s Lemma

The main idea is to employ the following form of Krasner’s Lemma. We define

\[ \text{kras}(a, K) := \max \{ v(\tau a - \sigma a) \mid \sigma, \tau \in \text{Gal } K \text{ and } \tau a \neq \sigma a \}. \]

Theorem

Take \( K(a)|K \) to be a separable-algebraic extension and \( (K(a,b), v) \) to be any valued field extension of \( (K, v) \) such that

\[ v(b - a) > \text{kras}(a, K). \]

Then for every extension of \( v \) from \( K(a,b) \) to its algebraic closure \( K(a,b) = K(b)^{ac} \).
A variant of Krasner’s Lemma

The main idea is to employ the following form of Krasner’s Lemma. We define

\[ \text{kras}(a, K) := \max\{v(\tau a - \sigma a) \mid \sigma, \tau \in \text{Gal} K \text{ and } \tau a \neq \sigma a\} . \]

**Theorem**

Take \( K(a)\mid K \) to be a separable-algebraic extension and \((K(a, b), v)\) to be any valued field extension of \((K, v)\) such that

\[ v(b - a) > \text{kras}(a, K) . \]

Then for every extension of \( v \) from \( K(a, b) \) to its algebraic closure \( K(a, b) = K(b)_{\text{ac}} \), the element \( a \) lies in the henselization of \((K(b), v)\) in \((K(b)_{\text{ac}}, v)\).
Homogeneous approximations

Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\).
Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is strongly homogeneous over \((K, v)\) if \(a \in K_{\text{sep}}\), the extension of \(v\) from \(K\) to \(K(a)\) is unique, and \(va = \kappa(v)(a, K)\).

We call \(a \in L\) a homogeneous approximation of \(b\) over \(K\) if there is some \(d \in K\) such that \(a - d\) is strongly homogeneous over \(K\) and \(v(b - d - (a - d)) > v(b - d) \geq v(b)\).

It then follows that \(va = vb\) and \(v(a - d) = v(b - d)\).
Homogeneous approximations

Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is strongly homogeneous over \((K, v)\) if \(a \in K^{\text{sep}} \setminus K\), and

\[
va = vb \quad \text{and} \quad v(a - d) = v(b - d) \geq vb.
\]
Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is **strongly homogeneous over \((K, v)\)** if \(a \in K^{\text{sep}} \setminus K\), the extension of \(v\) from \(K\) to \(K(a)\) is unique,
Homogeneous approximations

Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is **strongly homogeneous over** \((K, v)\) if \(a \in K^{\text{sep}} \setminus K\), the extension of \(v\) from \(K\) to \(K(a)\) is unique, and

\[ va = \text{kras}(a, K) . \]
Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is strongly homogeneous over \((K, v)\) if \(a \in K^{\text{sep}} \setminus K\), the extension of \(v\) from \(K\) to \(K(a)\) is unique, and

\[ va = \text{kras}(a, K). \]

We call \(a \in L\) a homogeneous approximation of \(b\) over \(K\).
Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is strongly homogeneous over \((K, v)\) if \(a \in K^{\text{sep}} \setminus K\), the extension of \(v\) from \(K\) to \(K(a)\) is unique, and

\[ va = \text{kras}(a, K). \]

We call \(a \in L\) a homogeneous approximation of \(b\) over \(K\) if there is some \(d \in K\) such that \(a - d\) is strongly homogeneous over \(K\)
Homogeneous approximations

Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is strongly homogeneous over \((K, v)\) if \(a \in K^{\text{sep}} \setminus K\), the extension of \(v\) from \(K\) to \(K(a)\) is unique, and

\[ va = \text{kras}(a, K). \]

We call \(a \in L\) a homogeneous approximation of \(b\) over \(K\) if there is some \(d \in K\) such that \(a - d\) is strongly homogeneous over \(K\) and

\[ v((b - d) - (a - d)) = v(b - a) > v(b - d) \geq vb. \]
Let \((K, v)\) be any valued field and \(a, b\) elements in some valued field extension \((L, v)\) of \((K, v)\). We say that \(a\) is strongly homogeneous over \((K, v)\) if \(a \in K^{\text{sep}} \setminus K\), the extension of \(v\) from \(K\) to \(K(a)\) is unique, and

\[ va = \text{kras}(a, K). \]

We call \(a \in L\) a homogeneous approximation of \(b\) over \(K\) if there is some \(d \in K\) such that \(a - d\) is strongly homogeneous over \(K\) and

\[ v((b - d) - (a - d)) = v(b - a) > v(b - d) \geq vb. \]

It then follows that \(va = vb\) and \(v(a - d) = v(b - d)\).
Applying Krasner’s Lemma

Lemma

If $a \in L$ is a homogeneous approximation of $b$
Applying Krasner’s Lemma

Lemma

If \( a \in L \) is a homogeneous approximation of \( b \) then \( a \) lies in the henselization of \( K(b) \)
Lemma

If $a \in L$ is a homogeneous approximation of $b$ then $a$ lies in the henselization of $K(b)$ w.r.t. every extension of the valuation $\nu$ from $K(a, b)$ to $K(b)^{ac}$. 
Let \((K(x)|K, v)\) be any extension of valued fields.
Let \((K(x)|K, v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\).
Homogeneous sequences

Let \((K(x)|K, v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\). Let \(S\) be an initial segment of \(\mathbb{N}\), that is,
Homogeneous sequences

Let \((K(x)|K,v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\). Let \(S\) be an initial segment of \(\mathbb{N}\), that is, \(S = \mathbb{N}\) or \(S = \{1,\ldots,n\}\) for some \(n \in \mathbb{N}\) or \(S = \emptyset\).
Let \((K(x)\mid K, v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\). Let \(S\) be an initial segment of \(\mathbb{N}\), that is, \(S = \mathbb{N}\) or \(S = \{1, \ldots, n\}\) for some \(n \in \mathbb{N}\) or \(S = \emptyset\). A sequence 

\[ S := (a_i)_{i \in S} \]

of elements in \(K^{ac}\) will be called a **homogeneous sequence for** \(x\).
Homogeneous sequences

Let \((K(x) | K, v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\). Let \(S\) be an initial segment of \(\mathbb{N}\), that is, \(S = \mathbb{N}\) or \(S = \{1, \ldots, n\}\) for some \(n \in \mathbb{N}\) or \(S = \emptyset\). A sequence 

\[ S := (a_i)_{i \in S} \]

of elements in \(K^{ac}\) will be called a **homogeneous sequence for** \(x\) if the following condition is satisfied for all \(i \in S\):

\[(HS) \quad a_i - a_{i-1} \text{ is a homogeneous approximation of } x - a_{i-1} \text{ over } K(a_0, \ldots, a_{i-1})\]
homogeneous sequences

Let \((K(x)|K,v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\). Let \(S\) be an initial segment of \(\mathbb{N}\), that is, \(S = \mathbb{N}\) or \(S = \{1,\ldots,n\}\) for some \(n \in \mathbb{N}\) or \(S = \emptyset\). A sequence \(S := (a_i)_{i \in S}\) of elements in \(K^{ac}\) will be called a \textit{homogeneous sequence for} \(x\) if the following condition is satisfied for all \(i \in S\):

\[(HS)\]  

\(a_i - a_{i-1}\) is a homogeneous approximation of \(x - a_{i-1}\) over \(K(a_0,\ldots,a_{i-1})\)

(where we set \(a_0 := 0\)).
Homogeneous sequences

Let \((K(x)|K,v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\). Let \(S\) be an initial segment of \(\mathbb{N}\), that is, \(S = \mathbb{N}\) or \(S = \{1, \ldots, n\}\) for some \(n \in \mathbb{N}\) or \(S = \emptyset\). A sequence 

\[ S := (a_i)_{i \in S} \]

of elements in \(K^{ac}\) will be called a homogeneous sequence for \(x\) if the following condition is satisfied for all \(i \in S\):

\((HS)\) \(a_i - a_{i-1}\) is a homogeneous approximation of \(x - a_{i-1}\) over \(K(a_0, \ldots, a_{i-1})\)

(where we set \(a_0 := 0\)). Then from the definition of “strongly homogeneous” it follows that \(a_i \notin K(a_0, \ldots, a_{i-1})^h\).
Let \((K(x)|K,v)\) be any extension of valued fields. We fix an extension of \(v\) to \(K(x)^{ac}\). Let \(S\) be an initial segment of \(\mathbb{N}\), that is, \(S = \mathbb{N}\) or \(S = \{1,\ldots,n\}\) for some \(n \in \mathbb{N}\) or \(S = \emptyset\). A sequence

\[ S := (a_i)_{i \in S} \]

of elements in \(K^{ac}\) will be called a **homogeneous sequence for \(x\)** if the following condition is satisfied for all \(i \in S\):

\[ \text{(HS)} \quad a_i - a_{i-1} \text{ is a homogeneous approximation of } x - a_{i-1} \text{ over } K(a_0, \ldots, a_{i-1}) \]

(where we set \(a_0 := 0\)). Then from the definition of “strongly homogeneous” it follows that \(a_i \notin K(a_0, \ldots, a_{i-1})^h\). We set

\[ K_S := K(a_i \mid i \in S) \]
Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\).
Assume that $(a_i)_{i \in S}$ is a homogeneous sequence for $x$ over $K$. Then the following assertions hold.

• $K S \subset K(x)_h$.

• For every $n \in S$, $a_1, \ldots, a_n \in K(a_n)_h$.

• If $S = \{1, \ldots, n\}$, then $K_hS = K(a_n)_h$.

Theorem
Assume that $S = (a_i)_{i \in S}$ is a homogeneous sequence for $x$ over $K$ with $S = \mathbb{N}$. Then $(a_i)_{i \in S} \in \mathbb{N}$ is a pseudo Cauchy sequence of transcendental type in $(K_S, v)$ with pseudo limit $x$, $(K_S(x)|K_S, v)$ is immediate, and $K_hS = IC(K(x)|K, v)$. Further, $K_Sv$ is the relative algebraic closure of $Kv$ in $K(x)_v$, and $vK_S$ is the relative divisible closure of $vK$ in $vK(x)_v$. 

Franz-Victor Kuhlmann
Extensions of valuations
Assume that \((a_i)_{i \in \mathcal{S}}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_\mathcal{S} \subset K(x)^h\).
Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S, a_1, \ldots, a_n \in K(a_n)^h\).
Assume that \( (a_i)_{i \in S} \) is a homogeneous sequence for \( x \) over \( K \).
Then the following assertions hold.

- \( K_S \subset K(x)^h \).
- For every \( n \in S, a_1, \ldots, a_n \in K(a_n)^h \).
- If \( S = \{1, \ldots, n\} \), then \( K_S^h = K(a_n)^h \).
Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S, a_1, \ldots, a_n \in K(a_n)^h\).
- If \(S = \{1, \ldots, n\}\), then \(K_S^h = K(a_n)^h\).

**Theorem**

Assume that \(\mathcal{G} = (a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\) with \(S = \mathbb{N}\).
Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S\), \(a_1, \ldots, a_n \in K(a_n)^h\).
- If \(S = \{1, \ldots, n\}\), then \(K^h_S = K(a_n)^h\).

### Theorem

Assume that \(\mathcal{S} = (a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\) with \(S = \mathbb{N}\). Then \((a_i)_{i \in \mathbb{N}}\) is a pseudo Cauchy sequence of transcendental type in \((K_\mathcal{S}, v)\)
Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S\), \(a_1, \ldots, a_n \in K(a_n)^h\).
- If \(S = \{1, \ldots, n\}\), then \(K_S^h = K(a_n)^h\).

**Theorem**

Assume that \(\mathcal{S} = (a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\) with \(S = \mathbb{N}\). Then \((a_i)_{i \in \mathbb{N}}\) is a pseudo Cauchy sequence of transcendental type in \((K_\mathcal{S}, \nu)\) with pseudo limit \(x\),
Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S, a_1, \ldots, a_n \in K(a_n)^h\).
- If \(S = \{1, \ldots, n\}\), then \(K^h_S = K(a_n)^h\).

**Theorem**

Assume that \(S = (a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\) with \(S = \mathbb{N}\). Then \((a_i)_{i \in \mathbb{N}}\) is a pseudo Cauchy sequence of transcendental type in \((K_S, v)\) with pseudo limit \(x\), \((K_S(x)|K_S, v)\) is immediate,
Results

Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S\), \(a_1, \ldots, a_n \in K(a_n)^h\).
- If \(S = \{1, \ldots, n\}\), then \(K_S^h = K(a_n)^h\).

Theorem

Assume that \(\mathcal{S} = (a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\) with \(S = \mathbb{N}\). Then \((a_i)_{i \in \mathbb{N}}\) is a pseudo Cauchy sequence of transcendental type in \((K_{\mathcal{S}}, \nu)\) with pseudo limit \(x\), \((K_{\mathcal{S}}(x)|K_{\mathcal{S}}, \nu)\) is immediate, and

\[
K_{\mathcal{S}}^h = \text{IC} \left( K(x)|K, \nu \right).
\]
Results

Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S\), \(a_1, \ldots, a_n \in K(a_n)^h\).
- If \(S = \{1, \ldots, n\}\), then \(K_S^h = K(a_n)^h\).

Theorem

Assume that \(S = (a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\) with \(S = \mathbb{N}\). Then \((a_i)_{i \in \mathbb{N}}\) is a pseudo Cauchy sequence of transcendental type in \((K_S, v)\) with pseudo limit \(x\), \((K_S(x)|K_S, v)\) is immediate, and

\[K_S^h = \text{IC} (K(x)|K, v)\,.

Further, \(K_S v\) is the relative algebraic closure of \(K v\) in \(K(x)v\).
Assume that \((a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\). Then the following assertions hold.

- \(K_S \subset K(x)^h\).
- For every \(n \in S\), \(a_1, \ldots, a_n \in K(a_n)^h\).
- If \(S = \{1, \ldots, n\}\), then \(K_S^h = K(a_n)^h\).

**Theorem**

Assume that \(\mathcal{G} = (a_i)_{i \in S}\) is a homogeneous sequence for \(x\) over \(K\) with \(S = \mathbb{N}\). Then \((a_i)_{i \in \mathbb{N}}\) is a pseudo Cauchy sequence of transcendental type in \((K_\mathcal{G}, v)\) with pseudo limit \(x\), \((K_\mathcal{G}(x)|K_\mathcal{G}, v)\) is immediate, and

\[K_\mathcal{G}^h = \text{IC } (K(x)|K, v) .\]

Further, \(K_\mathcal{G} v\) is the relative algebraic closure of \(K v\) in \(K(x) v\), and \(vK_\mathcal{G}\) is the relative divisible closure of \(vK\) in \(vK(x)\).
Proposition

Suppose that \((K, v)\) is henselian.

1) If \(S\) is a homogeneous sequence over \((K, v)\), then \(K^S\) is a tame extension of \(K\).

2) An element \(b \in K_{ac}\) belongs to a tame extension of \((K, v)\) if and only if there is a finite homogeneous sequence \(a_1, \ldots, a_k\) for \(b\) over \((K, v)\) such that \(b \in K(a_k)\).
Proposition

Suppose that \((K, v)\) is henselian.

1) If \(S\) is a homogeneous sequence over \((K, v)\),
Connection with tame extensions

Proposition

Suppose that \((K, v)\) is henselian.

1) If \(\mathcal{S}\) is a homogeneous sequence over \((K, v)\), then \(K_{\mathcal{S}}\) is a tame extension of \(K\).
Proposition

Suppose that $(K, v)$ is henselian.

1) If $\mathcal{S}$ is a homogeneous sequence over $(K, v)$, then $K_\mathcal{S}$ is a tame extension of $K$.

2) An element $b \in K^{ac}$ belongs to a tame extension of $(K, v)$. 
Proposition

Suppose that \((K, v)\) is henselian.

1) If \(\mathcal{S}\) is a homogeneous sequence over \((K, v)\), then \(K_{\mathcal{S}}\) is a tame extension of \(K\).

2) An element \(b \in K^{ac}\) belongs to a tame extension of \((K, v)\) if and only if there is a finite homogeneous sequence \(a_1, \ldots, a_k\) for \(b\) over \((K, v)\).
Connection with tame extensions

Proposition

Suppose that \((K,\nu)\) is henselian.

1) If \(S\) is a homogeneous sequence over \((K,\nu)\), then \(K_S\) is a tame extension of \(K\).

2) An element \(b \in K^{ac}\) belongs to a tame extension of \((K,\nu)\) if and only if there is a finite homogeneous sequence \(a_1, \ldots, a_k\) for \(b\) over \((K,\nu)\) such that \(b \in K(a_k)\).


The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html