A generalization of the Abhyankar Jung Theorem to associated graded rings of valuations

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K a field with a valuation ν.

(V_ν, m_ν) the valuation ring of ν.

Γ_ν the value group of ν.

(R, m_R) a local ring with quot. field K which is dominated by ν.

For $\gamma \in \Gamma_\nu$, we have valuation ideals

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geq \gamma\}$$

$$\mathcal{P}^+_\gamma(R)(R) = \{f \in R \mid \nu(f) > \gamma\}.$$

The associated graded ring of R along ν is

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma \in \Gamma_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R).$$

This ring is generally not Noetherian even when R is.

Ramification

Suppose that $K \to K^*$ is a finite field extension, ν^* is a valuation of K^*, $\nu = \nu^*|K$.

reduced ramification index:

$$e = [\Gamma_{\nu^*} : \Gamma_\nu]$$

Residue degree:

$$f = [V_{\nu^*}/m_{\nu^*} : V_\nu/m_\nu]$$

Defect δ

If ν^* is the unique extension of ν to K^* then

$$[K^* : K] = efp^{\delta}$$

where p is the characteristic of V_ν/m_ν ($p = 1$ if the characteristic is zero).

$\delta = 0$ always if the characteristic is 0 ($p = 1$).

We will be concerned with commutative diagrams

$$\begin{array}{ccc}
K & \stackrel{\text{finite}}{\rightarrow} & K^* \\
\uparrow & & \uparrow \\
V_\nu & \rightarrow & V_{\nu^*} \\
\uparrow & & \uparrow \\
R & \rightarrow & S = S(R)
\end{array}$$

where R is a normal local ring with quotient field K and $S = S(R)$ is the localization of the integral closure of R in K^* at the center of ν^*.

We always have

$$[\mathrm{QF}(\mathrm{gr}_{\nu^*}(S)) : \mathrm{QF}(\mathrm{gr}_{\nu}(R))] = ef.$$

The defect disappears!

For now on, assume that rank $\nu^* = 1$ (there exists an order preserving embedding $\Gamma_{\nu^*} < \mathbb{R}$).

Theorem A. Suppose that $K \to K^*$ are algebraic function fields over a field k and R is essentially of finite type over k (an algebraic local ring of K which is dominated by ν). Then there exists a birational extension $R \to R'$ where R' is a normal algebraic local ring of K which is dominated by ν giving a commutative diagram

$$\begin{array}{ccc} V_\nu & \to & V_{\nu^*} \\ \uparrow & & \uparrow \\ R' & \to & S' = S'(R') \\ \uparrow & & \\ R & & \end{array}$$

such that $\mathrm{gr}_{\nu^*}(S')$ is integral over $\mathrm{gr}_\nu(R)$.

Blowing up is necessary: there exist examples of $R \to S = S(R)$ such that $\mathrm{gr}_{\nu^*}(S')$ is not integral over $\mathrm{gr}_\nu(R)$.

Theorem B. Suppose ν^* is the unique extension of ν to K^*, K^* is Galois over K, $e = [\Gamma_{\nu^*} : \Gamma_\nu]$ is a power of p and V_{ν^*}/m_{ν^*} is purely inseparable over V_ν/m_ν. Then $[K^* : K] = p^n$ is a power of p and

$$\mathrm{gr}_{\nu^*}(S)^{p^n} \subset \mathrm{gr}_\nu(R).$$

In general, we cannot obtain that $\mathrm{gr}_{\nu^*}(R) \to \mathrm{gr}_{\nu}(S)$ is a finite extension after blowing up, as shown by the following example.

Example C. (Piltant, —) There exists an infinite sequence of sequences of quadratic transforms of 2 dimensional algebraic regular local rings in a defect extension

$$\begin{array}{ccc}
V_\nu & \rightarrow & V_{\nu^*}\\
\vdots & & \vdots\\
\uparrow & & \uparrow\\
R_i & \rightarrow & S_i\\
\vdots & & \vdots\\
\uparrow & & \uparrow\\
R & \rightarrow & S
\end{array}$$

such that for all i, the extension $\mathrm{gr}_\nu(R_i) \rightarrow \mathrm{gr}_{\nu^*}(S_i)$ is integral but not finite.

The extensions are

$$
\begin{array}{c}
\mathrm{gr}_\nu(R_i) = k[U_0, U_1, \ldots]/(U_1^{p^2} - U_0, \{U_j^{p^2} - U_0^{p^{2j-2}} U_{j-1}\}_{2\le j})\\
\downarrow\\
\mathrm{gr}_\nu(S_i) = k[X_0, X_1, \ldots]/(X_1^{p^2} - X_0, \{X_j^{p^2} - X_0^{p^{2j-2}} X_{j-1}\}_{2\le j})
\end{array}
$$

with $U_j = X_j^p$ for all j.

We have that $\mathrm{gr}_{\nu^*}(S_i)^p = \mathrm{gr}_\nu(R_i)$.

The Abhyankar Jung Theorem

Theorem (Abhyankar, 1956) Suppose that $K \to K^*$ is a finite extension of algebraic function fields over an algebraically closed field k of characteristic 0, R is an algebraic regular local ring of K (with residue field k), and S is a local ring of the integral closure of R in K^*. Suppose there exist regular parameters $x_1, \ldots, x_n$ in R such that the discriminant of $R \to K^*$ is a monomial in $x_1, \ldots, x_n$. Then there exists $d > 0$ such that

$$\hat{R} = k[[x_1, \ldots, x_n]] \subset \hat{S} \subset k[[x_1^{\frac{1}{d}}, \ldots, x_n^{\frac{1}{d}}]]$$

where $\hat{S}$ is an invariant ring of the action of a subgroup of $\mathbb{Z}_d^n$ on $k[[x_1^{\frac{1}{d}}, \ldots, x_n^{\frac{1}{d}}]]$.

Abhyankar Jung for associated graded rings of valuations

Theorem D. Suppose that $K \rightarrow K^*$ is a finite extension of algebraic function fields over an algebraically closed field k of characteristic zero, ν^* is a valuation of K^* with restriction ν to K and residue field k. Suppose that R' is a local ring of K which is dominated by ν. Then there exists a sequence of monoidal transforms (blow ups of regular primes) $R' \rightarrow R$ along ν such that, with $S = S(R)$ being the local ring of the integral closure of R in K^* which is dominated by ν^*, we have

1) $\mathrm{gr}_{\nu^*}(S)$ is a free $\mathrm{gr}_{\nu}(R)$-module of finite rank $e = [\Gamma_{\nu^*} : \Gamma_{\nu}]$.
2) There is a natural action of $\Gamma_{\nu^*}/\Gamma_{\nu}$ on $\mathrm{gr}_{\nu^*}(S)$ such that

$$\mathrm{gr}_{\nu^*}(S)^{\Gamma_{\nu^*}/\Gamma_{\nu}} \cong \mathrm{gr}_{\nu}(R).$$

Some comments about the theorem and proof.

The condition (of the classical Abhyankar Jung Theorem) that R is regular and the discriminant of R is a monomial is not enough for Theorem D to hold.

Theorem D is proven when K has dimension 2 by Ghezzi, Ha and Kascheyeva.

Both conclusions 1) and 2) of Theorem D are not true if k has characteristic $p > 0$ (or if you prefer this notation, $\neq 1$), as is shown by Example C above.

The main point of the proof is to find $R \to S$ such that there exist $w_i \in S$, for $1 \leq i \leq e$ such that $\{\nu^*(w_i)\}$ is a complete set of representatives of Γ_ν in Γ_{ν^*} and

$$\hat{S} = \bigoplus_{i=1}^{e} w_i \hat{R}.$$

The local monomialization theorem of "Local monomialization and factorization of morphisms", its refinement in "Ramification of valuations" with Olivier Piltant, and the methods of Abyankar's proof of the original Abhyankar Jung Theorem play a major role in the proof.