The closedness theorem over Henselian valued fields and its applications

The Twentieth Colloquiumfest
University of Szczecin
May 19-22, 2017

Krzysztof Jan Nowak
Institute of Mathematics
Jagiellonian University, Cracow
E-mail: nowak@im.uj.edu.pl

The closedness theorem over Henselian valued fields and its applications

The Twentieth Colloquiumfest
University of Szczecin
May 19-22, 2017

Krzysztof Jan Nowak
Institute of Mathematics
Jagiellonian University, Cracow
E-mail: nowak@im.uj.edu.pl

Table of contents

Table of contents

Table of contents

Table of contents

Table of contents

Aim of the talk

The aim is to develop geometry of algebraic subvarieties of K^n over arbitrary Henselian valued fields K of equicharacteristic zero, including my recent preprints [N3, N4, N5, N6]. This is a continuation of my previous article [N2], devoted to algebraic geometry over rank one valued fields, which in general requires more involved techniques and to some extent new treatment.

Aim of the talk

The aim is to develop geometry of algebraic subvarieties of K^n over arbitrary Henselian valued fields K of equicharacteristic zero, including my recent preprints [N3, N4, N5, N6]. This is a continuation of my previous article [N2], devoted to algebraic geometry over rank one valued fields, which in general requires more involved techniques and to some extent new treatment.

Again, at the center of my approach is the closedness theorem that the projections $K^n \times \mathbb{P}^m(K) \to K^n$ are definably closed maps. Hence we obtain a descent property for blow-ups, which enables applications of resolution of singularities in much the same way as over the locally compact ground field. As before, my approach relies on quantifier elimination for Henselian valued fields (in the language $\mathcal{L}$ of Denef–Pas) and certain preparation cell decomposition due to Pas.

But now I also rely on relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok.

But now I also rely on relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok.

In the proof of the closedness theorem I apply i.a. the local behaviour of definable functions of one variable and fiber shrinking, being a relaxed version of curve selection. To achieve the former result over arbitrary Henselian valued fields, I first examine functions given by algebraic power series ([N5]).

But now I also rely on relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok.

In the proof of the closedness theorem I apply i.a. the local behaviour of definable functions of one variable and fiber shrinking, being a relaxed version of curve selection. To achieve the former result over arbitrary Henselian valued fields, I first examine functions given by algebraic power series ([N5]).

The results from my previous article [N2] will be established in the general settings: several versions of curve selection (by resolution of singularities) and of the Łojasiewicz inequality (by two instances of quantifier elimination indicated above), extending continuous hereditarily rational functions and the theory of regulous functions, sets and sheaves (including regulous versions of Nullstellensatz and Cartan's theorems A and B).

But now I also rely on relative quantifier elimination for ordered abelian groups (in a many-sorted language with imaginary auxiliary sorts) due to Cluckers–Halupczok.

In the proof of the closedness theorem I apply i.a. the local behaviour of definable functions of one variable and fiber shrinking, being a relaxed version of curve selection. To achieve the former result over arbitrary Henselian valued fields, I first examine functions given by algebraic power series ([N5]).

The results from my previous article [N2] will be established in the general settings: several versions of curve selection (by resolution of singularities) and of the Łojasiewicz inequality (by two instances of quantifier elimination indicated above), extending continuous hereditarily rational functions and the theory of regulous functions, sets and sheaves (including regulous versions of Nullstellensatz and Cartan's theorems A and B).

Closedness Theorem

Other applications of the closedness theorem are piecewise continuity of definable functions ([N4]) and Hölder continuity of functions on closed bounded subsets of K^n ([N3]).

Closedness Theorem

Other applications of the closedness theorem are piecewise continuity of definable functions ([N4]) and Hölder continuity of functions on closed bounded subsets of K^n ([N3]).

Theorem. ([N6]) *Let D be an $\mathcal{L}$-definable subset of K^n. Then the canonical projection*

$$\pi : D \times R^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times R^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

Closedness Theorem

Other applications of the closedness theorem are piecewise continuity of definable functions ([N4]) and Hölder continuity of functions on closed bounded subsets of K^n ([N3]).

Theorem. ([N6]) *Let D be an $\mathcal{L}$-definable subset of K^n. Then the canonical projection*

$$\pi : D \times R^m \longrightarrow D$$

is definably closed in the K-topology, i.e. if $B \subset D \times R^m$ is an $\mathcal{L}$-definable closed subset, so is its image $\pi(B) \subset D$.

Corollary 1. *Let A be a closed $\mathcal{L}$-definable subset of R^m or of $\mathbb{P}^m(K)$. Then every continuous $\mathcal{L}$-definable map $f : A \to K^n$ is definably closed in the K-topology.*

Corollary 2. *Let X be a smooth K-variety, ϕ_i, $i = 0, \dots, m$, regular functions on X, D be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \longrightarrow X$ the blow-up of the ideal $(\phi_0, \dots, \phi_m)$. Then the restriction*

$$\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$$

is a definably closed quotient map.

Corollary 2. *Let X be a smooth K-variety, ϕ_i, $i = 0, \dots, m$, regular functions on X, D be an $\mathcal{L}$-definable subset of $X(K)$ and $\sigma : Y \longrightarrow X$ the blow-up of the ideal $(\phi_0, \dots, \phi_m)$. Then the restriction*

$$\sigma : Y(K) \cap \sigma^{-1}(D) \longrightarrow D$$

is a definably closed quotient map.

Corollary 3. (Descent property) *Under the assumptions of the above corollary, every continuous $\mathcal{L}$-definable function*

$$g : Y(K) \cap \sigma^{-1}(D) \longrightarrow K$$

that is constant on the fibers of the blow-up σ descends to a (unique) continuous $\mathcal{L}$-definable function $f : D \longrightarrow K$.

Functions given by algebraic power series

A version of the Artin–Mazur theorem (cf. [AM, BCR] for the classical versions).

Theorem. ([N5]) *Let $\phi \in (X)K[[X]]$ be an algebraic formal power series. Then there exist polynomials*

$$p_1, \ldots, p_r \in K[X, Y], \quad Y = (Y_1, \ldots, Y_r),$$

and formal power series $\phi_2, \ldots, \phi_r \in K[[X]]$ such that

$$e := \frac{\partial(p_1, \ldots, p_r)}{\partial(Y_1, \ldots, Y_r)}(\mathbf{0}) = \det \left[\frac{\partial p_i}{\partial Y_j}(\mathbf{0}) : \ i, j = 1, \ldots, r \right] \neq 0,$$

and

$$p_i(X_1, \ldots, X_n, \phi_1(X), \ldots, \phi_r(X)) = 0, \quad i = 1, \ldots, r,$$

where $\phi_1 := \phi$.

Corollary. *Let $\phi \in (X)K[[X]]$ be an algebraic power series with irreducible polynomial $p(X,T) \in K[X,T]$. Then there is an $a \in K$, $a \neq 0$, and a unique continuous function*

$$\widetilde{\phi} : a \cdot R^n \longrightarrow K$$

which is definable in the language of valued fields and such that $\widetilde{\phi}(0) = 0$ and $p(x, \widetilde{\phi}(x)) = 0$ for all $x \in a \cdot R^n$. $\square$

Corollary. *Let $\phi \in (X)K[[X]]$ be an algebraic power series with irreducible polynomial $p(X,T) \in K[X,T]$. Then there is an $a \in K$, $a \neq 0$, and a unique continuous function*

$$\widetilde{\phi} : a \cdot R^n \longrightarrow K$$

which is definable in the language of valued fields and such that $\widetilde{\phi}(0) = 0$ and $p(x, \widetilde{\phi}(x)) = 0$ for all $x \in a \cdot R^n$. □

This corollary also relies on the following version of the implicite function theorem from [N5]; cf. the versions from [P-Z, Kuhl, G-G-MB].

Corollary. *Let $\phi \in (X)K[[X]]$ be an algebraic power series with irreducible polynomial $p(X,T) \in K[X,T]$. Then there is an $a \in K$, $a \neq 0$, and a unique continuous function*

$$\widetilde{\phi} : a \cdot R^n \longrightarrow K$$

which is definable in the language of valued fields and such that $\widetilde{\phi}(0) = 0$ and $p(x, \widetilde{\phi}(x)) = 0$ for all $x \in a \cdot R^n$. □

This corollary also relies on the following version of the implicite function theorem from [N5]; cf. the versions from [P-Z, Kuhl, G-G-MB].

Let $(R, \mathfrak{m})$ be a Henselian ring, $0 \leqslant r < n$, $p = (p_{r+1}, \dots, p_n)$ be an $(n-r)$-tuple of polynomials $p_{r+1}, \dots, p_n \in R[X]$, $X = (X_1, \dots, X_n)$, and

$$J := \frac{\partial(p_{r+1}, \dots, p_n)}{\partial(X_{r+1}, \dots, X_n)}, \quad e := J(\mathbf{0}).$$

Proposition. *Suppose that*

$$\mathbf{0} \in V := \{x \in R^n : p_{r+1}(x) = \ldots = p_n(x) = 0\}.$$

If $e \neq 0$, then there exists a unique continuous map

$$\phi : (e^2 \cdot \mathfrak{m})^{\times r} \longrightarrow (e \cdot \mathfrak{m})^{\times (n-r)}$$

which is definable in the language of valued fields and such that $\phi(0) = 0$ and the graph map

$$(e^2 \cdot \mathfrak{m})^{\times r} \ni u \longrightarrow (u, \phi(u)) \in (e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)}$$

is an open embedding into the zero locus V of the polynomials p and, more precisely, onto

$$V \cap \left[(e^2 \cdot \mathfrak{m})^{\times r} \times (e \cdot \mathfrak{m})^{\times (n-r)} \right].$$

.

A non-archimedean Abhyankar–Jung theorem

Let K be an algebraically closed field of characteristic zero. Consider a henselian $K[X]$-subalgebra $K\langle X\rangle$ of the formal power series ring $K[[X]]$ which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For a positive integer r put

$$K\langle X^{1/r}\rangle = K\langle X_1^{1/r}, \ldots, X_n^{1/r}\rangle := \left\{ a(X_1^{1/r}, \ldots, X_n^{1/r}) : a(X) \in K\langle X\rangle \right\}.$$

A non-archimedean Abhyankar–Jung theorem

Let K be an algebraically closed field of characteristic zero. Consider a henselian $K[X]$-subalgebra $K\langle X\rangle$ of the formal power series ring $K[[X]]$ which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For a positive integer r put

$$K\langle X^{1/r}\rangle = K\langle X_1^{1/r}, \dots, X_n^{1/r}\rangle := \left\{ a(X_1^{1/r}, \dots, X_n^{1/r}) : a(X) \in K\langle X\rangle \right\}.$$

Theorem. ([N1]) *Every quasiordinary (i.e. whose discriminant is a normal crossing) polynomial*

$$f(X; T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

has all its roots in $K\langle X^{1/r}\rangle$ *for some* $r \in \mathbb{N}$*; actually, one can take* $r = s!$.

A non-archimedean Newton–Puiseux theorem

Corollary. ([N1]) *Every polynomial*

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

in one variable X has all its roots in $K\langle X^{1/r}\rangle$ for some $r \in \mathbb{N}$; one can take $r = s!$. Equivalently, the polynomial $f(X^r, T)$ splits into T-linear factors. If $f(X, T)$ is irreducible, then $r = s$ will do and

$$f(X^s, T) = \prod_{i=1}^{s} (T - \phi(\epsilon^i X)),$$

where $\phi(X) \in K\langle X\rangle$ and ϵ is a primitive root of unity.

A non-archimedean Newton–Puiseux theorem

Corollary. ([N1]) *Every polynomial*

$$f(X;T) = T^s + a_{s-1}(X)T^{s-1} + \cdots + a_0(X) \in K\langle X\rangle[T]$$

in one variable X has all its roots in $K\langle X^{1/r}\rangle$ for some $r \in \mathbb{N}$; one can take $r = s!$. Equivalently, the polynomial $f(X^r, T)$ splits into T-linear factors. If $f(X, T)$ is irreducible, then $r = s$ will do and

$$f(X^s, T) = \prod_{i=1}^{s} (T - \phi(\epsilon^i X)),$$

where $\phi(X) \in K\langle X\rangle$ and ϵ is a primitive root of unity.

Remark. If the field K is not algebraically closed, the theorem remain valid for the Henselian subalgebra $\overline{K} \otimes_K K\langle X\rangle$ of $\overline{K}[[X]]$; $\overline{K}$ is the algebraic closure of K. The case of the algebra of algebraic power series will be applied in our analysis of definable functions of one variable.

Definable functions of one variable

Theorem. ([N6]) *Let $f : A \longrightarrow K$ be an $\mathcal{L}$-definable function on a subset A of K and suppose 0 is an accumulation point of A. Then there is a finite partition of A into $\mathcal{L}$-definable sets $A_1, \ldots, A_r$ and points $w_1 \ldots, w_r \in \mathbb{P}^1(K)$ such that*

$$\lim_{x \longrightarrow 0} f|A_j(x) = w_j \quad \text{for } j = 1, \ldots, r.$$

Moreover, there is a neighbourhood U of 0 such that each definable set

$$\{(v(x), v(f(x))) : \ x \in (A_j \cap U) \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\}), \ j = 1, \ldots, r,$$

is contained in an affine line

$$l = \frac{p_j}{q} \cdot k + \beta_j, \ j = 1, \ldots, r,$$

with rational slope, $p_j, q \in \mathbb{Z}$, $q > 0$, $\beta_j \in \Gamma$, or in $\Gamma \times \{\infty\}$.

QE for Henselian valued fields

The field K is considered in the 3-sorted language $\mathcal{L}$ of Denef-Pas; main sort: the valued field K with valuation

$$v : K \to \Gamma \cup \{\infty\};$$

two auxiliary sorts: the value group Γ and the residue field $\Bbbk$.

QE for Henselian valued fields

The field K is considered in the 3-sorted language $\mathcal{L}$ of Denef-Pas; main sort: the valued field K with valuation

$$v : K \to \Gamma \cup \{\infty\};$$

two auxiliary sorts: the value group Γ and the residue field $\Bbbk$.

The only symbols of $\mathcal{L}$ connecting the sorts are: the valuation map v and the angular component map $\overline{ac} : K \to \Bbbk$ which is multiplicative, sends 0 to 0 and coincides with the residue map on units of the valuation ring R of K.

QE for Henselian valued fields

The field K is considered in the 3-sorted language $\mathcal{L}$ of Denef-Pas; main sort: the valued field K with valuation

$$v : K \to \Gamma \cup \{\infty\};$$

two auxiliary sorts: the value group Γ and the residue field $\Bbbk$.

The only symbols of $\mathcal{L}$ connecting the sorts are: the valuation map v and the angular component map $\overline{ac} : K \to \Bbbk$ which is multiplicative, sends 0 to 0 and coincides with the residue map on units of the valuation ring R of K.

Theorem. ([Pas]) *If the valued field K is Henselian and of equicharacteristic zero, then $(K, \Gamma, \Bbbk)$ admits elimination of K-quantifiers in the language $\mathcal{L}$.*

Cells

Consider an $\mathcal{L}$-definable subset D of $K^n \times \Bbbk^m$, three $\mathcal{L}$-definable funkctions

$$a(x,\xi), b(x,\xi), c(x,\xi) : D \to K$$

and a positive integer ν. For $\xi \in \Bbbk^m$ put

$$C(\xi) := \{(x,y) \in K_x^n \times K_y : \ (x,\xi) \in D,$$

$$v(a(x,\xi)) \lhd_1 v((y-c(x,\xi))^\nu) \lhd_2 v(b(x,\xi)), \ \overline{ac}(y-c(x,\xi)) = \xi_1\},$$

where $\lhd_1, \lhd_2$ stand for $<, \leqslant$ or no condition.

Cells

Consider an $\mathcal{L}$-definable subset D of $K^n \times \Bbbk^m$, three $\mathcal{L}$-definable funkctions

$$a(x,\xi), b(x,\xi), c(x,\xi) : D \to K$$

and a positive integer ν. For $\xi \in \Bbbk^m$ put

$$C(\xi) := \{(x,y) \in K_x^n \times K_y : \ (x,\xi) \in D,$$

$$v(a(x,\xi)) \lhd_1 v((y-c(x,\xi))^\nu) \lhd_2 v(b(x,\xi)), \ \overline{ac}(y-c(x,\xi)) = \xi_1\},$$

where $\lhd_1, \lhd_2$ stand for $<, \leqslant$ or no condition.

Definition. If the sets $C(\xi)$, $\xi \in \Bbbk^m$, are pairwise disjoint, the union

$$C := \bigcup_{\xi \in \Bbbk^m} C(\xi)$$

is called a cell in $K^n \times K$ with parameters ξ and center $c(x,\xi)$; $C(\xi)$ is called a fiber of the cell C.

Preparation Theorem

Theorem. ([Pas]) *Let*

$$f_1(x,y),\ldots,f_r(x,y)$$

be polynomials in one variable y with coefficients being $\mathcal{L}$-definable functions on K^n_x. Then $K^n \times K$ admits a finite partition into cells such that on each cell C with parameters ξ and center $c(x,\xi)$ and for all $i = 1,\ldots,r$ we have:

$$v(f_i(x,y)) = v\left(\tilde{f}_i(x,\xi)(y-c(x,\xi))^{\nu_i}\right),$$

$$\overline{ac}\, f_i(x,y) = \xi_{\mu(i)},$$

where $\tilde{f}_i(x,\xi)$ are $\mathcal{L}$-definable functions, $\nu_i \in \mathbb{N}$ for all $i = 1,\ldots,r$, and the map $\mu : \{1,\ldots,r\} \to \{1,\ldots,m\}$ does not depend on x, y, ξ.

QE for ordered abelian groups

It is well known that archimedean ordered abelian groups admit quantifier elimination in the Presburger language. Much more complicated is quantifier elimination for non-archimedean groups (especially of infinite rank), going back as far as Gurevich [Gur].

QE for ordered abelian groups

It is well known that archimedean ordered abelian groups admit quantifier elimination in the Presburger language. Much more complicated is quantifier elimination for non-archimedean groups (especially of infinite rank), going back as far as Gurevich [Gur].

He established a transfer of sentences from ordered abelian groups to so-called coloured chains (i.e. linearly ordered sets with additional unary predicates), enhanced later to allow arbitrary formulas (his doctoral dissertation "The decision problem for some algebraic theories", Sverdlovsk, 1968), and next also by Schmitt (habilitation thesis "Model theory of ordered abelian groups", Heidelberg, 1982; see also [Sch]).

QE for ordered abelian groups

It is well known that archimedean ordered abelian groups admit quantifier elimination in the Presburger language. Much more complicated is quantifier elimination for non-archimedean groups (especially of infinite rank), going back as far as Gurevich [Gur].

He established a transfer of sentences from ordered abelian groups to so-called coloured chains (i.e. linearly ordered sets with additional unary predicates), enhanced later to allow arbitrary formulas (his doctoral dissertation "The decision problem for some algebraic theories", Sverdlovsk, 1968), and next also by Schmitt (habilitation thesis "Model theory of ordered abelian groups", Heidelberg, 1982; see also [Sch]).

This is a kind of relative quantifier elimination, which allowed them [Gur-Sch] in their study of the NIP property to lift model theoretic properties from ordered sets to ordered abelian groups.

Instead Cluckers–Halupczok [C-H] introduce a suitable many-sorted language $\mathcal{L}_{qe}$ with main group sort Γ and auxiliary imaginary sorts carrying the structure of a linearly ordered set with some additional unary predicates. They provide quantifier elimination relative to the auxiliary sorts, where each definable set in the group is a union of a family of quantifier free definable sets with parameter running a definable (with quantifiers) set of the auxiliary sorts.

Instead Cluckers–Halupczok [C-H] introduce a suitable many-sorted language $\mathcal{L}_{qe}$ with main group sort Γ and auxiliary imaginary sorts carrying the structure of a linearly ordered set with some additional unary predicates. They provide quantifier elimination relative to the auxiliary sorts, where each definable set in the group is a union of a family of quantifier free definable sets with parameter running a definable (with quantifiers) set of the auxiliary sorts.

Fortunately, sometimes it is possible to directly deduce information about ordered abelian groups without any knowledge of the auxiliary sorts. This may be illustrated by their theorem on piecewise linearity of definable functions [C-H, Corollary 1.10] as well as by our application of quantifier elimination in the proofs of fiber shrinking and of the closedness theorem.

Fiber shrinking

Let A and E be $\mathcal{L}$-definable subsets of K^n and K with accumulation points $a = (a_1, \ldots, a_n) \in K^n$ and a_1, respectively. We call an $\mathcal{L}$-definable family of sets

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$$

an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \to a_1} \Phi_t = (a_2, \ldots, a_n),$$

i.e. for any neighbourhood U of $(a_2, \ldots, a_n) \in K^{n-1}$, there is a neighbourhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. This is a relaxed version of curve selection.

Fiber shrinking

Let A and E be $\mathcal{L}$-definable subsets of K^n and K with accumulation points $a = (a_1, \ldots, a_n) \in K^n$ and a_1, respectively. We call an $\mathcal{L}$-definable family of sets

$$\Phi = \bigcup_{t \in E} \{t\} \times \Phi_t \subset A$$

an $\mathcal{L}$-definable x_1-fiber shrinking for the set A at a if

$$\lim_{t \to a_1} \Phi_t = (a_2, \ldots, a_n),$$

i.e. for any neighbourhood U of $(a_2, \ldots, a_n) \in K^{n-1}$, there is a neighbourhood V of $a_1 \in K$ such that $\emptyset \neq \Phi_t \subset U$ for every $t \in V \cap E$, $t \neq a_1$. This is a relaxed version of curve selection.

Theorem. ([N6]) *Every $\mathcal{L}$-definable subset A of K^n with accumulation point $a \in K^n$ has, after a permutation of the coordinates, an $\mathcal{L}$-definable x_1-fiber shrinking at a.*

The Łojasiewicz inequalities

Theorem 1. ([N2]) *Let U and F be two $\mathcal{L}$-definable subsets of K^m, suppose U is open and F closed in the K-topology and consider two continuous $\mathcal{L}$-definable functions $f, g : A \to K$ on the locally closed subset $A := U \cap F$ of K^m. If*

$$\{x \in A : g(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a continuous $\mathcal{L}$-definable function h on A such that $f^s(x) = h(x) \cdot g(x)$ for all $x \in A$.

Theorem 2. *Let $f, g_1, \ldots, g_m : A \to K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If*

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geqslant v(g_1(x), \ldots, g_m(x)) \quad \textit{for all} \quad x \in A.$$

Theorem 2. *Let $f, g_1, \ldots, g_m : A \longrightarrow K$ be continuous $\mathcal{L}$-definable functions on a closed (in the K-topology) bounded subset A of K^m. If*

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer s and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geqslant v(g_1(x), \ldots, g_m(x)) \ \text{ for all } \ x \in A.$$

Corollary. (Hölder continuity) *Let $f : A \longrightarrow K$ be a continuous $\mathcal{L}$-definable function $f : A \longrightarrow K$ on a closed bounded subset $A \subset K^n$. Then f is Hölder continuous with a positive integer s and a constant $\beta \in \Gamma$, i.e.*

$$s \cdot v(f(x) - f(z)) + \beta \geqslant v(x - z)$$

for all $x, z \in A$.

Curve selection

By a (valuative) semialgebraic subset of K^n we mean a (finite) Boolean combination of elementary (valuative) semialgebraic subsets, i.e. sets of the form

$$\{x \in K^n : \ v(f(x)) \leqslant v(g(x))\},$$

where f and g are regular functions on K^n. We call a map φ semialgebraic if its graph is a semialgebraic set.

Curve selection

By a (valuative) semialgebraic subset of K^n we mean a (finite) Boolean combination of elementary (valuative) semialgebraic subsets, i.e. sets of the form

$$\{x \in K^n : \; v(f(x)) \leqslant v(g(x))\},$$

where f and g are regular functions on K^n. We call a map φ semialgebraic if its graph is a semialgebraic set.

Theorem 1. *Let A be a semialgebraic subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there is a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series such that*

$$\varphi(0) = a \quad \textit{and} \quad \varphi(R \setminus \{0\}) \subset A \setminus \{a\}.$$

Theorem 2. *Let A be an $\mathcal{L}$-definable set subset of K^n. If a point $a \in K^n$ lies in the closure (in the K-topology) of $A \setminus \{a\}$, then there exist a semialgebraic map $\varphi : R \longrightarrow K^n$ given by algebraic power series and an $\mathcal{L}$-definable subset E of R with accumulation point 0 such that*

$$\varphi(0) = a \quad \textit{and} \quad \varphi(E \setminus \{0\}) \subset A \setminus \{a\}.$$

Piecewise continuity of definable functions

Theorem. *Let $A \subset K^n$ and $f : A \to \mathbb{P}^1(K)$ be an $\mathcal{L}^P$-definable function in the three-sorted language of Denef–Pas. Then f is piecewise continuous, i.e. there is a finite partition of A into $\mathcal{L}^P$-definable locally closed subsets $A_1, \ldots, A_s$ of K^n such that the restriction of f to each A_i is continuous.*

References

[AM] M. Artin, B. Mazur, *On periodic points*, Ann. Math. **81** (1965), 82–99.

[BCR] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36, Springer-Verlag, Berlin, 1998.

[C-H] R. Cluckers, E. Halupczok, *Quantifier elimination in ordered abelian groups*, Confluentes Math. **3** (2011), 587–615.

[FHMM] G. Fichou, J. Huisman, F. Mangolte, J.-P. Monnier, *Fonctions régulues*, J. Reine Angew. Math. **718** (2016), 103–151.

[G-G-MB] O. Gabber, P. Gille, L. Moret-Bailly, *Fibrés principaux sur les corps valués henséliens*, Algebraic Geometry **1** (2014), 573–612.

[Gur] Y. Gurevich, *Elementary properties of ordered abelian groups*, Algebra i Logika Seminar, **3** (1964), 5–39 (in Russian); Amer. Math. Soc. Transl., II Ser. **46** (1965), 165–192 (in English).

[Gur-Sch] Y. Gurevich, P.H. Schmitt, *The theory of ordered abelian groups does not have the independence property*, Trans. Amer. Math. Soc. **284** (1984), 171–182.

[K-N] J. Kollár, K. Nowak, *Continuous rational functions on real and p-adic varieties*, Math. Zeitschrift **279** (2015), 85–97.

[Kuhl] F.-V. Kuhlmann, *Maps on ultrametric spaces, Hensel's lemma and differential equations over valued fields*, Comm. in Algebra **39** (2011), 1730–1776.

[N1] K.J. Nowak, *Supplement to the paper "Quasianalytic perturbation of multiparameter hyperbolic polynomials and symmetric matrices" (Ann. Polon. Math. 101 (2011), 275–291)*, Ann. Polon. Math. **103** (2012), 101-107.

[N2] K.J. Nowak, *Some results of algebraic geometry over Henselian rank one valued fields*, Sel. Math. New Ser. **23** (2017), 455–495.

[N3] K.J. Nowak, *Hölder and Lipschitz continuity of functions definable over Henselian rank one valued fields*, arXiv:1702.03463 [math.AG].

[N4] K.J. Nowak, *Piecewise continuity of functions definable over Henselian rank one valued fields*, arXiv:1702.07849 [math.AG].

[N5] K.J. Nowak, *On functions given by algebraic power series over Henselian valued fields*, arXiv:1703.08203 [math.AG].

[N6] K.J. Nowak, *The closedness theorem over Henselian valued fields*, arXiv:1704.01093 [math.AG].

[Pas] J. Pas, *Uniform p-adic cell decomposition and local zeta functions*, J. Reine Angew. Math. **399** (1989), 137–172.

[P-Z] A. Prestel, M. Ziegler, *Model theoretic methods in the theory of topological fields*, J. Reine Angew. Math. **299–300** (1978), 318–341.

[Sch] P.H. Schmitt, *Model and substructure complete theories of ordered abelian groups*; In: *Models and Sets* (Proceedings of Logic Colloquium '83), Lect. Notes Math. **1103**, Springer-Verlag, Berlin, 1984, 389–418.