Wild and even points in global fields
and beyond

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joint work with:

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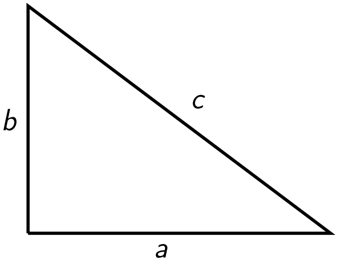
Problem

To what extend
geometry is determined by arithmetic?



$$a^2 + b^2 = c^2$$

For this to work, one needs an arithmetic property:

$$\sum \square = \square$$

Problem

To what extend
geometry is determined by arithmetic?

- linear structures = affine geometry — the same over any field;
- hence look at orthogonality, angles, lengths...

- R domain $1/2 \in R$ (resp. K field, char $K \neq 2$);
- M finitely generated projective R-module;
- $\xi : M \times M \to R$ symmetric, bilinear and non-degenerate

  i.e. $\hat{\xi} : M \xrightarrow{\sim} \mathrm{Hom}_R(M, R), \qquad \big(\hat{\xi}(a)\big)(b) := \xi(a, b);$

- ξ defines orthogonal geometry on M.

Standard dot product determines:

- lengths

$$\|v\| = \sqrt{v \bullet v};$$

- angels

$$\cos\alpha = \frac{v \bullet w}{\|v\| \cdot \|w\|};$$

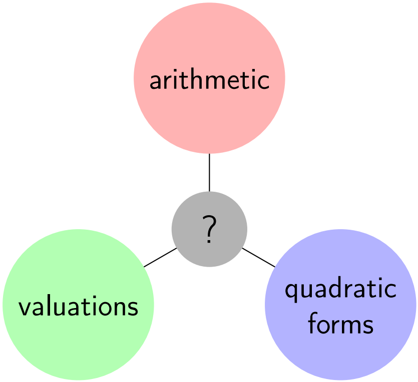
- in particular orthogonality

$$v \perp w \iff v \bullet w = 0.$$

- orthogonality is well defined

$$v \perp w \iff \xi(v, w) = 0;$$

- "magnitude" of an element $\sim$ valuation.


arithmetic
?
valuations
quadratic forms

Problems:

- P1: Compare two quadratic (orthogonal) spaces.
- P2: Describe all possible orthogonal geometries.
- P3: Compare classes of orthogonal geometries over two distinct rings.

Isometry $(V, \xi) \cong (W, \zeta)$:

$$t : V \xrightarrow{\sim} W, \qquad \xi(u, v) = \zeta(tu, tv);$$

Similarity $(V, \xi) \sim (W, \zeta)$:

$$(V, \xi) \perp \text{hyperbolic} \cong (W, \zeta) \perp \text{hyperbolic};$$

Weak Hasse principle:

K global field, X_K set of all primes:

$$(V,\xi) \underset{K}{\cong} (W,\zeta) \iff \forall \mathfrak{p} \in X_K : \ (V,\xi) \otimes K_{\mathfrak{p}} \underset{K_{\mathfrak{p}}}{\cong} (W,\zeta) \otimes K_{\mathfrak{p}}$$

Weak Witt theorem:

- K function field over a real closed field $\Bbbk$,
- γ_K real points of K, trivial on $\Bbbk$

$$(V,\xi) \underset{K}{\cong} (W,\zeta) \iff \forall \mathfrak{p} \in \gamma_K : \ (V,\xi) \otimes K_{\mathfrak{p}} \underset{K_{\mathfrak{p}}}{\cong} (W,\zeta) \otimes K_{\mathfrak{p}}$$

Strong Hasse principle:

K global field, X_K set of all primes:

$$(V,\xi) \underset{K}{\sim} (W,\zeta) \iff \forall \mathfrak{p} \in X_K :\ (V,\xi) \otimes K_{\mathfrak{p}} \underset{K_{\mathfrak{p}}}{\sim} (W,\zeta) \otimes K_{\mathfrak{p}}$$

Witt theorem:

- K function field over a real closed field $\mathbb{k}$,
- γ_K real points of K, trivial on $\mathbb{k}$

$$(V,\xi) \underset{K}{\sim} (W,\zeta) \iff \forall \mathfrak{p} \in \gamma_K :\ (V,\xi) \otimes K_{\mathfrak{p}} \underset{K_{\mathfrak{p}}}{\sim} (W,\zeta) \otimes K_{\mathfrak{p}}$$

for $\dim \xi, \dim \zeta \geq 3$.

In both cases we used valuations to compare quadratic spaces.

Witt ring

Set of similarity classes of non-degenarte bilinear R-modules with

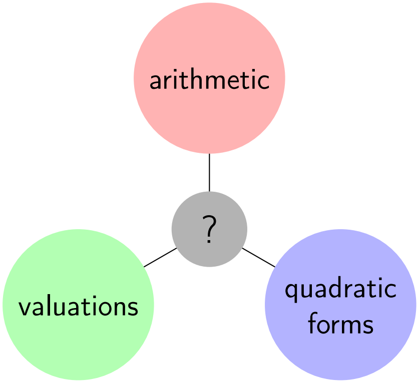
- $\perp$ orthogonal sum
- $\otimes$ tensor product

is a ring called *Witt ring* of R, denoted WR.

Witt functor

$R \mapsto WR$ is a covariant endofunctor on the category of commutative rings.

Find criteria for fields/rings for Witt functor to take equal values.


arithmetic
?
valuations
quadratic forms

In full generality they do not match!

	$\mathbb{C}((x))$	$\mathbb{F}_5$
arithmetics	very different	
valuations	lots	trivial only
Witt rings	$W\mathbb{C}((x)) \cong W\mathbb{F}_5$	

Hasse principles and Witt theorems suggest to concentrate on certain classes of fields: global fields, real function fields.

Perlis, Szymiczek, Conner, Litherland; 1994

Theorem

- *K, L global fields,* char K, char $L \neq 2$,
- *X_K, X_L sets of all primes of K, L.*

Then $WK \cong WL$ iff there are:

$$T : X_K \xrightarrow{\sim} X_L, \qquad t : {}^{K}/_{\square} \xrightarrow{\sim} {}^{L}/_{\square}$$

such that

$$(a, b)_{\mathfrak{p}} = (ta, tb)_{T\mathfrak{p}} \qquad \textit{for all } a, b \in {}^{K}/_{\square} \textit{ and } \mathfrak{p} \in X_K$$

Theorem (K., 2002)

- $\Bbbk, \Bbbk'$ *real closed fields,*
- K, L *function fields (over* $\Bbbk$, $\Bbbk'$*),*
- γ_K, γ_L *sets (curves) of real points.*

Then $WK \cong WL$ *iff there are:*

- $T : \gamma_K \setminus \{\textit{finite set}\} \xrightarrow{\textit{homeo}} \gamma_L \setminus \{\textit{finite set}\}$,
- $t : {}^K\!/\square \xrightarrow{\sim} {}^L\!/\square$

such that

$$\left(\frac{a, b}{K_{\mathfrak{p}}}\right) = 1 \iff \left(\frac{ta, tb}{L_{T\mathfrak{p}}}\right) = 1 \quad \textit{for all } a, b \textit{ and } \mathfrak{p}$$

In both cases Witt equivalence depends on matching valuation on K and L

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PROPOSITION (K., 2002)

At every point $\mathfrak{p}$ where T is defined, we have

$$\operatorname{ord}_{\mathfrak{p}} a \equiv \operatorname{ord}_{T\mathfrak{p}} ta \pmod 2.$$

PROPOSITION (K., 2009)

If T is defined on the whole γ_K, then there is $\varphi : WK \xrightarrow{\sim} WL$ s.t.

$$\varphi(WR_K) = WR_L,$$

where R_K, R_L are rings of regular functions.

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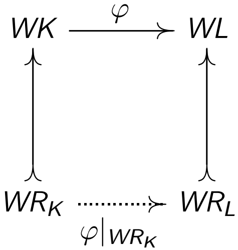


- K, L global fields ($\operatorname{char} K, \operatorname{char} L \neq 2$);
- (t, T) as above.

Definition

A point $\mathfrak{p} \in X$ is

TAME if $\operatorname{ord}_{\mathfrak{p}} a \equiv \operatorname{ord}_{T\mathfrak{p}} ta$ for every $a \in {}^{K}/\square$,

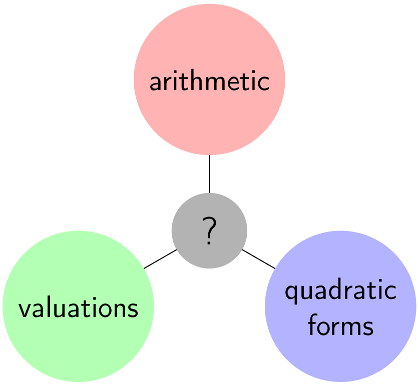
WILD otherwise.

Theorem (Czogała, 2001)

- *K, L global fields (char K, char $L \neq 2$);*
- *$\mathcal{O}_K$, $\mathcal{O}_L$ integral closures of either $\mathbb{Z}$ or $\mathbb{F}_q[x]$;*
- *(t, T) as above.*

It every point of K is tame w.r.t (t, T), then there is $\varphi : WK \xrightarrow{\sim} WL$ s.t.

$$\varphi(W\mathcal{O}_K) = W\mathcal{O}_L.$$



Tame points — valuations and quadratic forms "cooperate",
Wild points — things get... ehm... "wild".

- All possible Witt equivalences of a given field K,
- "null object" = all *self-equivalences* (i.e. Witt equivalences of K with itself).

- K-admissible sets of wild points in X_K,
- "null objects" = sets of wild points of self-equivalences.

Definition

A finite set $\mathcal{W} \subset X_K$ is called *wild*, if it is a set of wild points of some self-equivalence of K.

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Explicitly, $\mathcal{W}$ is wild if there are:

- $T : X_K \xrightarrow{\sim} X_L$;
- $t : \mathrm{Aut}({}^{K}/\square)$,
- $c_\mathfrak{p} \in {}^{K}/\square$ for every $\mathfrak{p} \in \mathcal{W}$

such that

$$(a, b)_\mathfrak{p} = (ta, tb)_{T\mathfrak{p}}, \qquad \text{for all } a, b \in {}^{K}/\square,\ \mathfrak{p} \in X_K$$

and

$$\mathrm{ord}_\mathfrak{p} c_\mathfrak{p} \equiv 1 + \mathrm{ord}_{T\mathfrak{p}} tc_\mathfrak{p} \pmod 2, \qquad \text{for all } \mathfrak{p} \in \mathcal{W}.$$

Small question

Describe wild sets/points in global fields:

- number fields;
- function fields.

Theorem (Somodi, 2006)

A finite set $\mathcal{W} \subset X_{\mathbb{Q}}$ is wild iff for every $(p) \in \mathcal{W}$, either $p = 2$ or $p \equiv 1 \pmod 4$.

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Every finite set $\mathcal{W} \subset X_{\mathbb{Q}(i)}$ is wild.

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Theorem (Somodi, 2008)

Every finite set $\mathcal{W} \subset X_{\mathbb{Q}(i)}$ is wild.

- $[K : \mathbb{Q}] < \infty$;
- K has a unique dyadic prime $\mathfrak{d}$;
- $\mathfrak{d} \in 2C_K$
- $\mathrm{rk}_2\, C_k = \mathrm{rk}_2\, C_K^+$

(C_K^+ narrow ideal class group, i.e. fractional ideals modulo totally positive ones).

Theorem (Czogała, Rothkegel; 2014)

- *K number field as above,*
- $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ *primes of K;*
- $\mathfrak{p}_i \in 2C_K$ *for every i;*
- -1 *is a local square at every* $\mathfrak{p}_i$

Then $\mathcal{W} := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is a wild set.

- K global function field;
- $\mathbb{k} = \mathbb{F}_q$ full field of constants ($2 \nmid q$);
- X_K set of all primes of K
  $=$ smooth, irreducible complete curve over $\mathbb{k}$;
- (t, T) always denotes some self-equivalence of K, i.e.

$$t \in \mathrm{Aut}(K/\square), \quad T : X \xrightarrow{\sim} X, \quad (\cdot,\cdot)\mathfrak{p} = (t\cdot, t\cdot)_{T\mathfrak{p}}$$

Theorem

Let $\mathfrak{p} \in X_K$. The following conditions are equivalent:

- $\mathfrak{p} \in 2 \cdot \operatorname{Pic} K$ (*$\mathfrak{p}$ is even*);
- *$\mathfrak{p}$ is a unique wild point of some self-equivalence of K.*

If $K = \Bbbk(x)$, then $\deg : \operatorname{Pic} K \xrightarrow{\sim} \mathbb{Z}$, hence:

OBSERVATION

Over $\Bbbk(x)$, the following conditions are equivalent:

- $\{\mathfrak{p}\}$ *is a wild set;*
- $\mathfrak{p}$ *is an even point;*
- $\deg \mathfrak{p} \in 2\mathbb{Z}$.

If $K = \mathbb{k}(x)$, then $\deg : \operatorname{Pic} K \xrightarrow{\sim} \mathbb{Z}$, hence:

Observation

Over $\mathbb{k}(x)$, the following conditions are equivalent:

$$\{\mathfrak{p}\} \text{ wild set} \iff \mathfrak{p} \in 2 \operatorname{Pic} K \iff \deg \mathfrak{p} \in 2\mathbb{Z}.$$

In general, $\deg : \operatorname{Pic} K \twoheadrightarrow \mathbb{Z}$, thus:

Observation

Every even point has an even degree.

The opposite implication fails in general:

- $\mathbb{k} = \mathbb{F}_3$,
- X elliptic curve $y^2 - x^3 + x = 0$;
- X has:
  - 6 point of degree 2, none of them even,
  - 12 points of degree 4, none of them even.

Proposition

- *X smooth (hyper)elliptic curve $y^2 - f(x) = 0$,*
- $\deg f \notin 2\mathbb{Z}$,
- *K associated function field,*
- $\mathfrak{p} \in X$, $\deg \mathfrak{p} \in 2\mathbb{Z}$.

Then $\mathfrak{p}$ is even iff there is $\lambda \in K$ s.t.

$$\mathrm{Norm}_{K/\mathbb{k}(x)}\, \mathfrak{p} \xmapsto{\mathrm{Div}\, \mathbb{k}(x) \to \mathbb{k}(x)^*} \mathrm{Norm}_{K/\mathbb{k}(x)}\, \lambda$$

Proposition

- *X smooth (hyper)elliptic curve $y^2 - f(x) = 0$,*
- *f monic, $\deg f \notin 2\mathbb{Z}$,*
- *K associated function field.*

Then there are infinitely many even points (wild singletons) of K.

Theorem

- *K global function field;*
- *X_K associated smooth and complete curve;*
- *$\mathcal{W}_1 := \mathcal{W}(T_1, t_1)$ a wild set of some self-equivalence (T_1, t_1);*
- *$\mathcal{W}_2 := T_1\big(\mathcal{W}(T_2, t_2)\big)$ an image of a wild set of $T_2, t_2)$.*

Then $\mathcal{W}_1 \cup \mathcal{W}_2$ is a wild set.

(*The associated self-equivalence being* $(T_2 \circ T_1, t_2 \circ t_1)$.)

Theorem

- *K global function field;*
- *X_K associated smooth curve;*
- *$\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in X$ even points.*

Then $\mathcal{W} := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is a wild set.

- $\mathbb{k} = \mathbb{F}_5$;
- X elliptic curve $y^2 + x^3 + x + 2 = 0$;
- $\mathfrak{p}, \mathfrak{q} \in X$, $\mathfrak{p} \sim (1,1)$, $\mathfrak{q} \sim (1,4)$.

Then

1. neither $\mathfrak{p}$ nor $\mathfrak{q}$ is even,
2. but $\mathcal{W} := \{\mathfrak{p}, \mathfrak{q}\}$ is a wild set!

OBSERVATION

There are irreducible wild sets, which are not singletons!

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OBSERVATION

There are irreducible wild sets, which are not singletons!

THEOREM

- *K global function field;*
- *X_K associated smooth curve;*
- *$\mathcal{W} \subset X$ a finite wild set.*

Then

$$|\mathcal{W}| \geq 2 \cdot \mathrm{rk}_2 \langle \mathfrak{p}_1 + 2 \operatorname{Pic} K, \ldots, \mathfrak{p}_n + 2 \operatorname{Pic} K \rangle$$

PROPOSITION

- *K global function field;*
- *X_K associated smooth and complete curve;*
- $\mathfrak{p}, \mathfrak{q} \in X_K$;
- $\mathrm{rk}_2 \langle \mathfrak{p} + 2\,\mathrm{Pic}\,K, \mathfrak{q} + 2\,\mathrm{Pic}\,K \rangle = 1$.

Then $\{\mathfrak{p}, \mathfrak{q}\}$ is a wild set.

NC is sufficient for:

- triples (providing that -1 is a local square);
- sets $\mathcal{W} \subset X_K$ s.t. $\mathrm{rk}_2\langle \mathcal{W} + 2\,\mathrm{Pic}\,K \rangle \leq 1$;
- sets displaying certain kind of symmetry.

Conjecture

NC is a necessary and sufficient condition for wildness.

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Conjecture

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[Sharif; 2013]

Divisibility in Picard groups of curves over over local fields.

[Czogała, Rothkegel, Sładek; 2016]

Wild sets w.r.t. higher degree forms in number fields.

[Marshall, Gładki; 2017]

Witt equivalence of function fields over global field.
Tame/wild points play a crucial role, again.

Thank you for your attention