# $C(M, S^n)$: RESULTS AND OPEN QUESTIONS

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Let $K$ denote a formally real field, meaning that $-1$ is not a sum of squares in $K$. The field $K$ admits real places $\lambda : K \to \mathbb{R} \cup \infty =: \mathbb{P}^1$. These are functions satisfying the following conditions:

$$V_\lambda := \{a \in K \mid \lambda(a) \neq \infty\} \text{ is a valuation ring}, \lambda|V_\lambda \text{ is a ring homomorphism into } \mathbb{R},$$
$$\text{and for all } a \in K : \lambda(a) = \infty \Rightarrow \lambda(a^{-1}) = 0.$$

**M** denotes the set of all real places of $K$. By evaluation, each $a \in K$ induces a function $\hat{a} : M \to \mathbb{P}^1, \lambda \mapsto \lambda(a)$. We endow $M$ with the coarsest topology such that all these functions are continuous. $M$ is a compact space, so we get a map

$$\Phi : K \to C(M, \mathbb{P}^1), a \mapsto \hat{a}$$

where all function spaces in this talk are endowed with the compact-open topology. The subring

$$\mathbf{H} := \{a \in K \mid \hat{a}(M) \subseteq \mathbb{R}\}$$

is called the **real holomorphy ring of** $K$. The restriction $\Phi : H \to C(M, \mathbb{R})$ is a $\mathbb{Q}$-algebra homomorphism, its image is dense as can be shown by the Stone-Weierstraß Theorem. While studying sums of squares and higher powers in algebraic function fields in one variable over $\mathbb{R}$ the author was led to find out whether the map $\Phi : K \to C(M, \mathbb{P}^1)$ has a dense image as well.

This problem leads naturally to a series of other questions, i.e. whether each map

$$\Phi_n : S^n(H) \to C(M, S^n), (a_0, \dots, a_n) \mapsto (\widehat{a_0}, \dots, \widehat{a_n}),$$

$$S^n \text{ the standard n-sphere and } S^n(H) = \{(a_0, \dots, a_n) \in H^{n+1} \mid \sum_0^{n+1} a_i^2 = 1\}.$$

has a dense image, $n \in \mathbb{N}$. Using the stereographic projection, the map $\Phi_1$ gets translated into the map $\Phi : K \to C(M, \mathbb{P}^1)$.

The talk will focus on three topics:

1. General results on the maps $\Phi, \Phi_n$, a sketch of the proof that $\Phi$ has dense image in the function field case above,
2. Application to sums of powers in fields: the higher Pythagoras numbers,
3. Making use of Bochnak-Kucharz' work on regular maps $V \to S^n, V$ a real algebraic variety.