On immediate extensions of valued fields

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immediate extensions

(K, v) a valued field
vK the value group,
Kv the residue field.

If $(L|K, v)$ is a finite extension of valued fields, such that the extension of v from K to L is unique, then

$$[L : K] = p^n (vL : vK)[Lv : Kv]$$

where $p = \text{char} Kv$ if it is positive and $p = 1$ otherwise.

If $p^n > 1$, then $(L|K, v)$ is called a **defect extension**.

An extension $(F|K, v)$ of valued fields is called **immediate** if

$$(vF : vK) = [Fv : Kv] = 1.$$

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Fact: Every valued field admits a maximal immediate extension.

Goals:

- Describe the structure of maximal immediate extensions of certain classes of valued fields.
- Describe the structure of maximal immediate **algebraic** extensions of certain classes of valued fields.
- Determine the classes of valued fields which admit unique (up to isomorphism) maximal immediate extensions.

Problem: Describing all possible extensions of a valuation from a given field (K, v) to a rational function field $L|K$.

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Do the maximal immediate extensions of a given valued field have finite or infinite transcendence degree?

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maximal immediate extensions

Theorem 1

Take a henselian field (K, v) and an extension $(L|K, v)$ of finite transcendence degree. Assume that v is nontrivial on L and at least one of the following cases holds:

1) vL/vK is not a torsion group, or $Lv|Kv$ is transcendental;

2) vL/vK contains elements of arbitrarily high order,

3) Lv contains elements of arbitrarily high degree over Kv;

4) $L|K$ contains an infinite separable-algebraic subextension.

Then each maximal immediate extension of (L, v) has infinite transcendence degree over L. If in addition the cofinality of vL is countable, then already $(L, v)^c$ has infinite transcendence degree over L.

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A valued field is called **maximal** if it admits no proper immediate extensions.

Every maximal field (M, v) is:

- henselian,
- complete,
- defectless, i.e., $[L : M] = (vL : vM)[Lv : Mv]$ for every finite extension $L|M$,

A finite extension of maximal field is again a maximal field.

Theorem 2

Take a maximal field (K, v) of characteristic 0 or of positive characteristic p and finite p-degree. If $(L|K, v)$ is an algebraic extension, then the field (L, v) is maximal if and only if $L|K$ is finite.

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A valued field (K, v) of residue characteristic p is called a Kaplansky field if it satisfies the following conditions:

(K1) if $p > 0$ then vK is p-divisible,

(K2) the residue field Kv is perfect,

(K3) the residue field Kv admits no finite separable extension of degree divisible by p.

Theorem 3 (I. Kaplansky)

The maximal immediate extension of a Kaplansky field (K, v) is unique up to valuation preserving isomorphism over K.

- There are valued fields admitting non-isomorphic maximal immediate extensions.

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Theorem 4

Take a henselian field (K, v) of residue characteristic p. Assume that the condition $(K3)$ does not hold (i.e., the residue field Kv admits a finite separable extension of degree divisible by p) and (K, v) is not separable-algebraically maximal. Then there is a finite tame extension E of K such that (E, v) admits two maximal immediate algebraic extensions which are not isomorphic over E.

General assumption:
(K, v) is a henselian field of residue characteristic p such that:
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(K, v) is a henselian field of residue characteristic p such that:

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Take a valued field (F, v) of characteristic $p > 0$ and an Artin-Schreier defect extension $(F(\vartheta)|F, v)$ with $\vartheta^p - \vartheta \in K$.

We call $(F(\vartheta)|F, v)$ a dependent Artin-Schreier defect extension if there exists a purely inseparable defect extension $(F(\eta)|F, v)$ of degree p, such that

$$v(\eta - \vartheta) > v(\vartheta - c) \text{ for all } c \in K.$$

Suppose that $(F(a^{1/p})|F, v)$ is a purely inseparable defect extension of degree p and $a^{1/p} \notin F^c$.

$$Y^p - a \quad \longrightarrow \quad Y^p - b^{p-1}Y - a \xrightarrow[Y = bX]{} X^p - X - \frac{a}{b^p}$$

For $b \in F^\times$ of large enough value, $X^p - X - \frac{a}{b^p}$ induces a dependent Artin-Schreier defect extension of F.

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(K1) if $p > 0$ then vK is p-divisible,
(K2) the residue field Kv is perfect.

Theorem 5

If $\mathrm{char} K = p$ and K admits at least one dependent Artin-Schreier defect extension, then it admits an infinite tower of such extensions.

Theorem 6

Assume that (K, v) admits a maximal immediate extension of finite transcendence degree. Then

- *(K, v) admits no immediate separable-algebraic extensions,*
- *the perfect hull of K is contained in the completion of K,*
- *the maximal immediate extension of (K, v) is unique up to isomorphism.*

defect and maximal immediate extensions

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Theorem 7

Suppose that (M, v) is maximal, of finite transcendence degree over K and v is nontrivial on M. Take $L|K$ to be the maximal separable-algebraic subextension of $M|K$. Then we have:

- *K is a separably tame field,*
- *$L|K$ is a finite tame extension,*
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Take a valued field (L, v), $n \geq 1$, an ordered abelian group extension Γ of vL and a field extension k of Lv.

When do we have an extension of v to the rational function field $L(x_1, \ldots, x_n)$ such that

$$vL(x_1, \ldots, x_n) = \Gamma \text{ and } L(x_1, \ldots, x_n)v = k\,? \tag{1}$$

Theorem 8

Assume that Γ/vL is a torsion group and $k|Lv$ is an algebraic extension, both countably generated. Suppose that at least one of the following cases holds:

- *the group Γ/vL is infinite or the extension $k|Lv$ is infinite,*
- *(L, v) admits an immediate extension of trdeg $\geq n$*
- *K^h admits an infinite separable-algebraic extension (L, v) with $vL \subseteq \Gamma$ and $Lv \subseteq k$.*

Then there is an extension of v to the rational function field $L(x_1, \ldots, x_n)$ such that (1) holds.

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Assume that Γ/vL is a torsion group and $k|Lv$ is an algebraic extension, both countably generated. Suppose that at least one of the following cases holds:

- *the group Γ/vL is infinite or the extension $k|Lv$ is infinite,*
- *(L, v) admits an immediate extension of trdeg $\geq n$*
- *K^h admits an infinite separable-algebraic extension (L, v) with $vL \subseteq \Gamma$ and $Lv \subseteq k$.*

Then there is an extension of v to the rational function field $L(x_1, \ldots, x_n)$ such that (1) holds.

Take a valued field (L, v), $n \geq 1$, an ordered abelian group extension Γ of vL and a field extension k of Lv.
When do we have an extension of v to the rational function field $L(x_1, \ldots, x_n)$ such that

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Theorem 9

Take a valued field (K, v) of residue characteristic exponent p. Assume that vK is p-divisible and Kv is perfect. Further, take an ordered abelian group extension Γ of vK such that Γ/vK is a torsion group, and an algebraic extension k of Kv, both countably generated. Then there is an extension of v from K to the rational function field $K(x_1, \ldots, x_n)$ with

$$vK(x_1, \ldots, x_n) = \Gamma \text{ and } K(x_1, \ldots, x_n)v = k$$

if and only if at least one of the two extensions $\Gamma|vK$ and $k|Kv$ is infinite or (K, v) admits an immediate extension of transcendence degree n.

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Take a maximal field (K, v) of characteristic $p > 0$ for which $[K : K^p]$ is infinite.

Then $(vK : pvK)$ or $[Kv : (Kv)^p]$ is infinite.

Consider $K \subseteq L \subseteq K^{1/p}$ such that $K^{1/p}|L$ is immediate and infinite.

- $(K^{1/p}, v)$ is a maximal immediate extension of (L, v).

Take infinitely many elements $\eta_i \in K^{1/p}$ which are p-independent over L. Under additional assumptions on vK we can choose η_i to not lie in the completion of (L, v).

$$X^p - \eta_i^p \quad \longrightarrow \quad X^p - X - \frac{\eta_i^p}{b_i^p}$$

$$\eta_i \quad \longrightarrow \quad \vartheta_i$$

$(L(\vartheta_1, \ldots, \vartheta_n)|L(\vartheta_1, \ldots, \vartheta_{n-1}, v)$ is an Artin-Schreier defect extension, $n \in \mathbb{N}$.

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Theorem 10

Take a henselian field (L, v) and an extension $(F|L, v)$ of finite transcendence degree. Assume that v is nontrivial on L and one of the following cases holds:

...

4) $F|L$ contains an infinite separable-algebraic subextension.

Then each maximal immediate extension of (F, v) has infinite transcendence degree over F.

Set $F = L(\vartheta_n : n \in \mathbb{N})$. Then (F, v) admits a maximal immediate extension of infinite transcendence degree. Since $(F|L, v)$ is also immediate, we obtain that

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Theorem 11

There are valued fields which admit an algebraic maximal immediate extension as well as one of infinite transcendence degree.

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