Two New Applications of the Abel Map to Integrable Systems *

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Introduction: integrable systems and linear spectral problems

One of the many striking discoveries made in the study of the KdV equation and integrable systems is that methods of classical algebraic geometry can be used to linearize periodic versions of many such systems; see [12-15, 17-22]. These systems include

- Toda lattice, KdV equation, AKNS-ZS systems ...

(Kac, Van Moerbeke, Novikov, Dubrovin, Lax, Its, Matveev, Krichever, ...)

These problems are integrable because they can be written in the form of Lax equation for the evolution of a linear operator:

$$\frac{d}{dt} L = [L\,,\, B], \tag{1}$$
$$L = \text{linear operator.}$$

The associated linear spectral problem is

$$L\varphi = \lambda\varphi. \tag{2}$$

The flow (1) is *isospectral* for the spectral problem (2). This provides many invariants of motion and supports the idea that the flow is "completely integrable."

In order to have a complete analysis of the flow (1), one would like to find spectral data or scattering data for (2) that flow *linearly*, and that can be used, along with the spectrum itself, to reconstruct the linear operator L. This is the purpose of the *inverse scattering method*. In the non-periodic case, the asymptotics of solutions of (2) are typically described by a scattering matrix. If the operator L is asymptotic to a constant coefficient operator, then the asymptotic data for (2) flow linearly under the evolution (1). This linearization gives a possibility of finding

- (general) explicit solutions
- special solutions: solitons, multisolitons

(Gardner–Greene–Kruskal–Miura; Zakharov–Shabat; Ablowitz–Kaup–Newell–Segur)

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Periodic problems; algebro-geometric solutions

In the periodic case the inverse scattering method, in naïve form, breaks down. The analogue of the scattering matrix is the Floquet matrix, but in the absence of a simple asymptotic limit one cannot expect to find linear flow so simply. However there may be an algebraic curve associated to (a certain family of) solutions, and invariant under the flow (1). Often the Abel map

$$\begin{aligned} A : S^g\Gamma &\rightarrow \mathbb{C}^g/\Lambda, \\ \Lambda &= \text{ period lattice of } \Gamma, \end{aligned} \tag{3}$$

for the curve Γ of genus g linearizes the flow of certain spectral data and thus (nearly) linearizes (1), leading to

- general solution (finite-dimensional): Toda lattice ...

(Flaschka, Kac–Van Moerbeke);

- "finite gap" solutions (infinite-dimensional): KdV, ...

(Novikov, Dubrovin, Lax, Matveev, Its, Krichever)

In this lecture we sketch such results for two finite-dimensional integrable systems of more recent vintage. One of these newer systems is a finite-dimensional reduction of the periodic Camassa-Holm equation. The second is the (non-periodic) Calogero-Françoise system, which generalizes the non-periodic Camassa-Holm reduction, but whose scattering problem is essentially identical to periodic Camassa-Holm. Again there are associated algebraic curves, and data that linearizes under the Abel map, leading to explicit solutions in terms of theta functions.

The Camassa-Holm equation was discovered, as an integrable equation, by Fokas and Fuchsteiner [16]. It was derived as a model shallow-water equation, based on work of Green and Naghdi, by Camassa and Holm, who were the first to study it in detail; see [9,10]. The analogues for this equation of the multisoliton solutions of KdV are weak solutions known as multipeakon solutions, which are only piecewise smooth. Explicit formulas for these solutions were found in [3,4]. The smooth periodic case has been studied by Constantin and McKean [11]; see also [1,2]. The present authors found formulas for the solution of the periodic two-peakon and peakon/antipeakon pair problem in terms of Weierstrass elliptic functions [4]. Explicit solutions for an arbitrary number of peakons and/or antipeakons may be found in terms of Riemann theta functions.

The systems studied by Calogero and Françoise generalize the finite-dimensional (multipeakon) reduction of the Camassa-Holm equation. There is no (spatially) periodic version of this problem in general, because the potential for the linear problem can have exponential growth. Nevertheless the associated scattering matrix has the same structure and the same type of time evolution as the periodic discrete Camassa-Holm problem, and the methods carry over.

Camassa-Holm equation; peakons; Calogero-Françoise flows

We use the normalization

$$4u_t - u_{xxt} = 2u_x(4u - u_{xx}) + u(4u_x - u_{xxx}), \qquad (4)$$
$$u \to 0 \quad \text{at} \quad \infty.$$

It is more convenient to write this as a system

$$m_t = u_x m + (um)_x, \quad 2m_x = 4u_x - u_{xxx}. \qquad (5)$$

The associated spectral problem

$$L(\lambda)\varphi \equiv D^2\varphi - \varphi - 2\lambda m(x)\,\varphi = 0, \quad D = \frac{d}{dx} \qquad (6)$$

is compatible with a generalized Lax evolution

$$-2\lambda m_t = \frac{d}{dt}L(\lambda) = \big[\, L(\lambda), B(\lambda) \,\big] + 2u_x L(\lambda), \qquad (7)$$

where

$$B(\lambda) = \Big\{\frac{1}{2\lambda} - u(x)\Big\}D + \frac{1}{2}u_x(x).$$

The spectral problem (6) can be transformed to the KdV spectral problem (1D Schrödinger), when either $m > 0$ or $m < 0$, by the Liouville transform. Otherwise it is more difficult, because the spectral parameter multiplies the "potential" m.

The Camassa-Holm equation admits a finite-dimensional reduction, and weak solutions with corners (peakons, antipeakons, multipeakons). In fact, take m to be a discrete real measure:

$$m(x,t) = \sum_{j=1}^{d} m_j(t)\delta\big(x - x_j(t)\big). \qquad (8)$$

Positive m_j give waves with corners, travelling to the right (peakons); negative m_j give troughs travelling to the left (antipeakons).

Suitably interpreted, in case (8) the evolution (5) is given by Hamilton's equation for dual variables

$$(x_1, \ldots, x_d; m_1, \ldots m_d)$$

with Hamiltonian

$$H(x,m) = \frac{1}{2}\sum_{j,k=1}^{d} m_j m_k G(x_j - x_k); \qquad (9)$$

G is the Green's function for the problem $(D^2 - 4D)Du = -2Dm$ with $G \to 0$ at ∞.

Calogero and Françoise proved (classical) complete integrability for the Hamiltonian flow (9) with G a general Green's function for the equation

$$(D^2 - 4\nu^2)Du = -2Dm.$$

The associated spectral problem, Lax equation, and nonlinear evolution are similar to the Camassa-Holm case.

Inverse scattering, discrete C-H, and C-F

The discrete Camassa-Holm equation (with $u \to 0$ at ∞) can be attacked by inverse scattering methods, suitably adapted: the asymptotics still evolve linearly. The solution involves

- A transformation to the interval $[-1, 1]$, with $D^2 - 1 \to D^2$;
- the Weyl function for the transformed problem

$$W(\lambda) = \frac{1}{\lambda} \frac{D\varphi(1, \lambda)}{\varphi(1, \lambda)}, \qquad \varphi(-1, \lambda) = 0;$$

- the continued fraction decomposition of W;
- the recovery of the continued fraction decomposition from residues of W by a theorem of Stieltjes; [2,3].

This inverse scattering method fails for Calogero-Françoise flows with $u \not\to 0$ at ∞: here the asymptotics evolve nonlinearly as well.

Analysis of these flows shows a formal identity:

general C-F $\iff$ *periodic* discrete C-H,

calling algebro-geometric methods into play.

[insert figure showing peakon-antipeakon interaction]

Periodic discrete Camassa-Holm

Assume positions x_j and "masses" m_j, $j \in \mathbb{Z}$, repeated with period 1, with d positions per unit length:

$$x_{j-1} < x_j < \ldots \ ; \quad x_{j+d} = x_j + 1, \ ; \quad m_{j+d} = m_j.$$

The Camassa-Holm evolution is again a Hamiltonian system with d degrees of freedom, in which the original Green's function G is replaced by its periodized version.

On an interval (x_{j-1}, x_j), a solution φ of

$$(D^2 - 1 - 2\lambda m)\varphi = 0 \tag{10}$$

has the form $a_j e^x + b_j e^{-x}$. Equation (10) implies a continuity condition and jump condition at x_j. In matrix form:

$$\begin{aligned} \begin{bmatrix} a_{j+1} \\ b_{j+1} \end{bmatrix} &= T_j(\lambda) \begin{bmatrix} a_j \\ b_j \end{bmatrix} \\ &= \begin{bmatrix} 1 + \lambda m_j & \lambda m_j \, e^{-2\nu x_j} \\ -\lambda m_j \, e^{2\nu x_j} & 1 - \lambda m_j \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix}. \end{aligned} \tag{11}$$

The associated Floquet matrix is

$$\Psi(\lambda) = T_d(\lambda) T_{d-1}(\lambda) \cdots T_1(\lambda) E, \qquad (12).$$

The factor $E = \operatorname{diag}(e, e^{-1})$ is introduced so that periodic (resp. antiperiodic) φ corresponds to an eigenvector of Ψ with eigenvalue 1 (resp. -1).

The evolution of the operator $L(\lambda, t)$ implies a corresponding evolution of the Floquet matrix $\Psi = \Psi(\lambda, t)$:

$$\dot{\Psi} = [\, \Psi \, , \, A_+], \tag{13}$$

where $A_+ = A_+(\lambda, t)$ is the matrix

$$A_+(\lambda, t) = \begin{bmatrix} \frac{1}{\lambda} & \frac{M_- e^{-1}}{e - e^{-1}} \\ -\frac{M_+ e}{e - e^{-1}} & -\frac{1}{\lambda} \end{bmatrix}, \tag{14}$$

and

$$M_+ = \sum_{j=1}^{d} m_j e^{2x_j}, \qquad M_- = \sum_{j=1}^{d} m_j e^{-2x_j}. \tag{15}$$

More generally, define $M_\pm$ from other starting points:

$$M_\pm(k) = \sum_{j=1}^{k+d-1} m_j e^{\pm x_j}.$$

The data $\{x_j, m_j\}$ can be reconstructed from d successive values of $M_\pm(k)$:

$$\begin{aligned} e^{4x_j} &= -\tanh 1 \, \frac{M_+(k+1) - M_+(k)}{M_-(k+1) - M_(k)}; \\ m_k^2 &= \frac{[M_+(k+1) - M_+(k)]\,[M_-(k+1) - M_-(k)]}{4 \sinh^2 1}. \end{aligned} \tag{16}$$

Calogero-Françoise flows

Take $\nu = 1$ for comparison; the general Green's function has the form

$$G(x) = \frac{\beta_-}{2} e^{2|x|} + \frac{\beta_+}{2} e^{-2|x|} + \gamma, \qquad \beta_- - \beta_+ = 1; \tag{17}$$

(The constant γ can be eliminated by using a moving frame.) The C-H case is $\beta_- = 1$.

Assume here that $\beta_- > 1$. The same analysis as before leads to the same transition matrices T_j and scattering matrix

$$\Phi(\lambda) = T_d(\lambda) T_{d-1}(\lambda) \cdots T_1(\lambda).$$

Define

$$\Psi(\lambda) = \Phi(\lambda)\, \mathrm{diag}\,(\beta_-, \beta_+). \tag{18}$$

This modified scattering matrix satisfies the same evolution equation, with same matrix A_+, as for the periodic Camassa-Holm problem with period $X = \log \sqrt{\beta_-/(1-\beta_-)}$. In fact the Calogero-Françoise Green's function and the Camassa-Holm Green's function with period X agree on the interval $[-X, X]$.

[insert figure showing some Green's functions: C-H, periodic C-H, C-F that agrees with periodic C-H over twice the period]

Periodic discrete C-H; hyperelliptic curve

The evolution equation for the Floquet matrix Ψ implies invariance of the degree d polynomial

$$P(\lambda) = \operatorname{tr}\Psi(\lambda). \tag{19}$$

This gives d independent constants of motion.

The zeros of $\Psi_{12}\Psi_{21}$ are eigenvalues of the spectral problem, with boundary conditions

$$\begin{aligned} D_-\varphi(x_1) + \varphi(x_1) = 0 = D_+\varphi(x_d) + \varphi(x_d) \quad (\Psi_{12} = 0); \\ D_-\varphi(x_1) - \varphi(x_1) = 0 = D_+\varphi(x_d) - \varphi(x_d) \quad (\Psi_{21} = 0). \end{aligned}$$

The nonzero roots $\{\lambda_{1j}\}$ of Ψ_{12} or $\{\lambda_{2j}\}$ of Ψ_{21} evolve according to

$$\dot{\lambda}_{ij} = \pm\Big\{ \prod_{k\neq j} \Big(1 - \frac{\lambda_{ij}}{\lambda_{ik}}\Big)\Big\}^{-1} \frac{\Psi_{11}(\lambda_{ij}) - \Psi_{22}(\lambda_{ij})}{e - e^{-1}}. \tag{20}$$

Now $\det\Psi = 1$, so

$$\Psi_{12}\Psi_{21} = 0 \ \Leftrightarrow\ (\Psi_{11} - \Psi_{22})^2 = P^2 - 4.$$

Therefore the flows (20) take place on the (invariant) hyperelliptic curve $\Gamma = \{(\lambda, z)\} \subset \mathbb{C}^2$, the set of (λ, z) with

$$z^2 = \frac{P(\lambda)^2 - 4}{(e - e^{-1})^2}. \tag{21}$$

Roots of $P^2 - 4$ and $\Psi_{12}\Psi_{21}$ are real. Assume the roots of $P^2 - 4$ are distinct; they determine $2d-1$ intervals. Non-zero roots of $\Psi_{12}\Psi_{21}$ lie in $d-1$ alternate subintervals. Slit along $d-1$ of the remaining to get cycles a_k on Γ, genus $d-1$. Choose a basis $\{\omega_j\}$ of holomorphic differentials in standard way. The evolution equations for the λ_j imply that the two maps

$$(\xi_{i1}, \ldots, \xi_{i,d-1}) \to A(\xi_{i1}, \ldots, \xi_{i.d-1}) = (\sum_j \int_{\xi_0}^{\xi_{ij}} \omega_1, \ldots, \sum_j \int_{\xi_0}^{\xi_{ij}}, \omega_{d-1}), \quad i = 1, 2,$$

where $\lambda(\xi_{ij}) = \lambda_{ij}$, linearize the Camassa-Holm flow:

$$\frac{d}{dt} A(\xi_{i1}, \ldots, \xi_{i,d-1}) = \text{constant vector}.$$

The evolution of the $M_\pm$ are given by

$$\frac{\dot{M}_-}{M_-} = M \coth 1 + \sum_{j=1}^{d-1} \frac{1}{\lambda_{1j}}; \qquad \frac{\dot{M}_+}{M_+} = -M \coth 1 - \sum_{j=1}^{d-1} \frac{1}{\lambda_{2j}},$$

where $M = \sum_{j=1}^{d} m_j$. Similar equations hold from different starting points, for the $M_\pm(x_k)$, from which we may recover x_j, m_j.

Solutions via theta functions

The recovery of x_j, m_j from Γ and the roots of $\Psi_{12}\Psi_{21}$ is completed by finding $M_{\pm}(k)$, $k = 1\ldots, d$. Each of the latter is a quotient of theta functions: for $k = 1$,

$$M_-(t) = C_-\, e^{ct}\, \frac{\theta\big(A(\zeta_+) - e_-(t)\big)}{\theta\big(A(\zeta_-) - e_-(t)\big)}; \qquad M_+(t) = C_+\, e^{-ct}\, \frac{\theta\big(A(\zeta_-) - e_+(t)\big)}{\theta\big(A(\zeta_+) - e_+(t)\big)}.$$

Here

$$\zeta_\pm = (0, \pm 1) \in \Gamma, \qquad c = \sum_{j=1}^{g} \int_{a_j} \lambda^{-1}\omega_j + M \coth 1;$$
$$e_-(t) = A(\xi_{11}(t), \ldots, \xi_{1g}(t)) + K; \qquad e_+(t) = A(\xi_{21}(t), \ldots, \xi_{2g}(t)) + K,$$

where $K \in \mathbb{C}^{g-1}$ is the Riemann vector for the curve Γ, and $\xi_{ij}(t)$ projects to $=\lambda_{1j}(t)$ from the "upper sheet" of the representation of Γ as two-sheeted cover of the plane obtained by slitting along the d real subintervals determined by roots of $P^2 - 4$ that do not contain roots of $\Psi_{12}\Psi_{21}$. The upper sheet is taken to be the one on which $z \to P$ at ∞.

Camassa-Holm to Calogero-Françoise; dynamics

Introduce the parameter $\nu > 0$, and the period $X > 0$ into periodic Camassa-Holm. The Calogero-Françoise flow with $\nu > 0$, $\beta_- > 1$ corresponds to one such periodic C-H flow. If initial positions for the C-F flow lie in a period interval $(a, a + X)$, then the solutions, restricted to a (moving) window, coincide with those of the Camassa-Holm flow.

All formulas are analytic in ν, X, and initial conditions. Therefore, solutions in general case given by analytic continuation, so long as formulas remain non-singular.

The exact dynamics depend critically on the parameters:

- For periodic C-H, if all m_j have the same sign, solutions are globally analytic in time.

- For periodic C-H, if not all m_j have the same sign, then at times there will be collisions:

$$m_j, m_{j+1} = O((t - t_0)^{-1}), \quad m_{j+1} - m_j = O(1);$$
$$x_{j+1} - x_j = O((t - t_0)^2),$$

  but $t \to u(x, t)$ is continuous to $C(\mathbb{R})$; no triple collisions.

- For C-F with $\nu > 0$, $\beta_- < 1$, one may have blow-up: $x_j \to \infty$ in finite time.

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