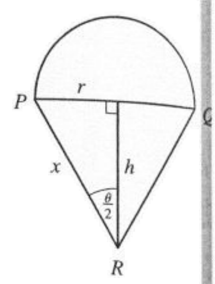
46. Let $|PR| = x$. Then we get the following formulas for r and h in terms of θ and x:

$$\sin\frac{\theta}{2} = \frac{r}{x} \Rightarrow r = x\sin\frac{\theta}{2} \text{ and } \cos\frac{\theta}{2} = \frac{h}{x} \Rightarrow h = x\cos\frac{\theta}{2}.$$



Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \frac{1}{2}\pi\lim_{\theta\to 0^+}\frac{x\sin(\theta/2)}{x\cos(\theta/2)}$$
$$= \frac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0.$$

47. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ, we can see that $\sin\frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r\sin\frac{\theta}{2}$.

So $\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$. [This is just the reciprocal of the li

$\lim_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta \to 0$, $\frac{\theta}{2} \to 0$ also.]

3.5 The Chain Rule

1. Let $u = g(x) = 4x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(4) = 4\cos 4x$.

2. Let $u = g(x) = 4 + 3x$ and $y = f(u) = \sqrt{u} = u^{1/2}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2}(3) = \frac{3}{2\sqrt{u}} = \frac{3}{2\sqrt{4+3}}$

3. Let $u = g(x) = 1 - x^2$ and $y = f(u) = u^{10}$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (10u^9)(-2x) = -20x(1-x^2)^9$

4. Let $u = g(x) = \sin x$ and $y = f(u) = \tan u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\sec^2 u)(\cos x) = \sec^2(\sin x)\cdot\cos x$, or

equivalently, $[\sec(\sin x)]^2\cos x$.

5. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = e^u$.

Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (e^u)\left(\frac{1}{2}x^{-1/2}\right) = e^{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

6. Let $u = g(x) = e^x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(e^x) = e^x\cos e^x$.

7. $F(x) = (x^3+4x)^7 \Rightarrow F'(x) = 7(x^3+4x)^6(3x^2+4)$ [or $7x^6(x^2+4)^6(3x^2+4)$]

8. $F(x) = (x^2-x+1)^3 \Rightarrow F'(x) = 3(x^2-x+1)^2(2x-1)$

9. $F(x) = \sqrt[4]{1+2x+x^3} = (1+2x+x^3)^{1/4} \Rightarrow$

$$F'(x) = \frac{1}{4}(1+2x+x^3)^{-3/4}\cdot\frac{d}{dx}(1+2x+x^3) = \frac{1}{4(1+2x+x^3)^{3/4}}\cdot(2+3x^2)$$
$$= \frac{2+3x^2}{4(1+2x+x^3)^{3/4}} = \frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

10. $f(x) = (1+x^4)^{2/3} \Rightarrow f'(x) = \frac{2}{3}(1+x^4)^{-1/3}(4x^3) = \frac{8x^3}{3\sqrt[3]{1+x^4}}$

11. $g(t) = \frac{1}{(t^4+1)^3} = (t^4+1)^{-3} \Rightarrow g'(t) = -3(t^4+1)^{-4}(4t^3) = -12t^3(t^4+1)^{-4} = \frac{-12t^3}{(t^4+1)^4}$

12. $f(t) = \sqrt[3]{1 + \tan t} = (1 + \tan t)^{1/3} \Rightarrow f'(t) = \frac{1}{3}(1 + \tan t)^{-2/3} \sec^2 t = \dfrac{\sec^2 t}{3\sqrt[3]{(1 + \tan t)^2}}$

13. $y = \cos(a^3 + x^3) \Rightarrow y' = -\sin(a^3 + x^3) \cdot 3x^2$ [a^3 is just a constant] $= -3x^2 \sin(a^3 + x^3)$

14. $y = a^3 + \cos^3 x \Rightarrow y' = 3(\cos x)^2(-\sin x)$ [a^3 is just a constant] $= -3 \sin x \cos^2 x$

15. $y = e^{-mx} \Rightarrow y' = e^{-mx} \dfrac{d}{dx}(-mx) = e^{-mx}(-m) = -me^{-mx}$

16. $y = 4 \sec 5x \Rightarrow y' = 4 \sec 5x \tan 5x(5) = 20 \sec 5x \tan 5x$

17. $g(x) = (1 + 4x)^5(3 + x - x^2)^8 \Rightarrow$

$$\begin{aligned} g'(x) &= (1 + 4x)^5 \cdot 8(3 + x - x^2)^7(1 - 2x) + (3 + x - x^2)^8 \cdot 5(1 + 4x)^4 \cdot 4 \\ &= 4(1 + 4x)^4(3 + x - x^2)^7 \left[2(1 + 4x)(1 - 2x) + 5(3 + x - x^2)\right] \\ &= 4(1 + 4x)^4(3 + x - x^2)^7 \left[(2 + 4x - 16x^2) + (15 + 5x - 5x^2)\right] \\ &= 4(1 + 4x)^4(3 + x - x^2)^7 (17 + 9x - 21x^2) \end{aligned}$$

18. $h(t) = (t^4 - 1)^3(t^3 + 1)^4 \Rightarrow$

$$\begin{aligned} h'(t) &= (t^4 - 1)^3 \cdot 4(t^3 + 1)^3(3t^2) + (t^3 + 1)^4 \cdot 3(t^4 - 1)^2(4t^3) \\ &= 12t^2(t^4 - 1)^2(t^3 + 1)^3 \left[(t^4 - 1) + t(t^3 + 1)\right] = 12t^2(t^4 - 1)^2(t^3 + 1)^3 (2t^4 + t - 1) \end{aligned}$$

19. $y = (2x - 5)^4(8x^2 - 5)^{-3} \Rightarrow$

$$\begin{aligned} y' &= 4(2x - 5)^3(2)(8x^2 - 5)^{-3} + (2x - 5)^4(-3)(8x^2 - 5)^{-4}(16x) \\ &= 8(2x - 5)^3(8x^2 - 5)^{-3} - 48x(2x - 5)^4(8x^2 - 5)^{-4} \end{aligned}$$

[This simplifies to $8(2x - 5)^3(8x^2 - 5)^{-4}(-4x^2 + 30x - 5)$.]

20. $y = (x^2 + 1)(x^2 + 2)^{1/3} \Rightarrow$

$$y' = 2x(x^2 + 2)^{1/3} + (x^2 + 1)\left(\tfrac{1}{3}\right)(x^2 + 2)^{-2/3}(2x) = 2x(x^2 + 2)^{1/3}\left[1 + \frac{x^2 + 1}{3(x^2 + 2)}\right]$$

21. $y = xe^{-x^2} \Rightarrow y' = xe^{-x^2}(-2x) + e^{-x^2} \cdot 1 = e^{-x^2}(-2x^2 + 1) = e^{-x^2}(1 - 2x^2)$

22. $y = e^{-5x} \cos 3x \Rightarrow y' = e^{-5x}(-3 \sin 3x) + (\cos 3x)(-5e^{-5x}) = -e^{-5x}(3 \sin 3x + 5 \cos 3x)$

23. $y = e^{x \cos x} \Rightarrow y' = e^{x \cos x} \cdot \dfrac{d}{dx}(x \cos x) = e^{x \cos x}\left[x(-\sin x) + (\cos x) \cdot 1\right] = e^{x \cos x}(\cos x - x \sin x)$

24. Using Formula 5 and the Chain Rule, $y = 10^{1 - x^2} \Rightarrow$

$$y' = 10^{1 - x^2}(\ln 10) \cdot \frac{d}{dx}(1 - x^2) = -2x(\ln 10)10^{1 - x^2}$$

25. $F(z) = \sqrt{\dfrac{z - 1}{z + 1}} = \left(\dfrac{z - 1}{z + 1}\right)^{1/2} \Rightarrow$

$$\begin{aligned} F'(z) &= \frac{1}{2}\left(\frac{z - 1}{z + 1}\right)^{-1/2} \cdot \frac{d}{dz}\left(\frac{z - 1}{z + 1}\right) = \frac{1}{2}\left(\frac{z + 1}{z - 1}\right)^{1/2} \cdot \frac{(z + 1)(1) - (z - 1)(1)}{(z + 1)^2} \\ &= \frac{1}{2}\frac{(z + 1)^{1/2}}{(z - 1)^{1/2}} \cdot \frac{z + 1 - z + 1}{(z + 1)^2} = \frac{1}{2}\frac{(z + 1)^{1/2}}{(z - 1)^{1/2}} \cdot \frac{2}{(z + 1)^2} = \frac{1}{(z - 1)^{1/2}(z + 1)^{3/2}} \end{aligned}$$

3.6 Implicit Differentiation

1. (a) $\frac{d}{dx}(xy + 2x + 3x^2) = \frac{d}{dx}(4) \Rightarrow (x \cdot y' + y \cdot 1) + 2 + 6x = 0 \Rightarrow xy' = -y - 2 - 6x \Rightarrow$

$y' = \frac{-y - 2 - 6x}{x}$ or $y' = -6 - \frac{y+2}{x}$.

(b) $xy + 2x + 3x^2 = 4 \Rightarrow xy = 4 - 2x - 3x^2 \Rightarrow y = \frac{4 - 2x - 3x^2}{x} = \frac{4}{x} - 2 - 3x$, so $y' = -\frac{4}{x^2} - 3$.

(c) From part (a), $y' = \frac{-y - 2 - 6x}{x} = \frac{-(4/x - 2 - 3x) - 2 - 6x}{x} = \frac{-4/x - 3x}{x} = -\frac{4}{x^2} - 3$.

2. (a) $\frac{d}{dx}(4x^2 + 9y^2) = \frac{d}{dx}(36) \Rightarrow 8x + 18y \cdot y' = 0 \Rightarrow y' = -\frac{8x}{18y} = -\frac{4x}{9y}$

(b) $4x^2 + 9y^2 = 36 \Rightarrow 9y^2 = 36 - 4x^2 \Rightarrow y^2 = \frac{4}{9}(9 - x^2) \Rightarrow y = \pm\frac{2}{3}\sqrt{9 - x^2}$, so

$y' = \pm\frac{2}{3} \cdot \frac{1}{2}(9 - x^2)^{-1/2}(-2x) = \mp\frac{2x}{3\sqrt{9 - x^2}}$

(c) From part (a), $y' = -\frac{4x}{9y} = -\frac{4x}{9\left(\pm\frac{2}{3}\sqrt{9 - x^2}\right)} = \mp\frac{2x}{3\sqrt{9 - x^2}}$.

3. (a) $\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{d}{dx}(1) \Rightarrow -\frac{1}{x^2} - \frac{1}{y^2}y' = 0 \Rightarrow -\frac{1}{y^2}y' = \frac{1}{x^2} \Rightarrow y' = -\frac{y^2}{x^2}$

(b) $\frac{1}{x} + \frac{1}{y} = 1 \Rightarrow \frac{1}{y} = 1 - \frac{1}{x} = \frac{x-1}{x} \Rightarrow y = \frac{x}{x-1}$, so $y' = \frac{(x-1)(1) - (x)(1)}{(x-1)^2} = \frac{-1}{(x-1)^2}$.

(c) $y' = -\frac{y^2}{x^2} = -\frac{[x/(x-1)]^2}{x^2} = -\frac{x^2}{x^2(x-1)^2} = -\frac{1}{(x-1)^2}$

4. (a) $\frac{d}{dx}(\sqrt{x} + \sqrt{y}) = \frac{d}{dx}(4) \Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{y} = 4 - \sqrt{x} \Rightarrow y = (4 - \sqrt{x})^2 = 16 - 8\sqrt{x} + x \Rightarrow y' = -\frac{4}{\sqrt{x}} + 1$

(c) $y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{4 - \sqrt{x}}{\sqrt{x}} = -\frac{4}{\sqrt{x}} + 1$

5. $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(1) \Rightarrow 2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow y' = -\frac{x}{y}$

6. $\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(1) \Rightarrow 2x - 2yy' = 0 \Rightarrow 2x = 2yy' \Rightarrow y' = \frac{x}{y}$

7. $\frac{d}{dx}(x^3 + x^2y + 4y^2) = \frac{d}{dx}(6) \Rightarrow 3x^2 + (x^2y' + y \cdot 2x) + 8yy' = 0 \Rightarrow x^2y' + 8yy' = -3x^2 - 2xy$

$\Rightarrow (x^2 + 8y)y' = -3x^2 - 2xy \Rightarrow y' = -\frac{3x^2 + 2xy}{x^2 + 8y} = -\frac{x(3x + 2y)}{x^2 + 8y}$

8. $\frac{d}{dx}(x^2 - 2xy + y^3) = \frac{d}{dx}(c) \Rightarrow 2x - 2(xy' + y \cdot 1) + 3y^2y' = 0 \Rightarrow 2x - 2y = 2xy' - 3y^2y' \Rightarrow$

$2x - 2y = y'(2x - 3y^2) \Rightarrow y' = \frac{2x - 2y}{2x - 3y^2}$

9. $\frac{d}{dx}(x^2y + xy^2) = \frac{d}{dx}(3x) \Rightarrow (x^2y' + y \cdot 2x) + (x \cdot 2yy' + y^2 \cdot 1) = 3 \Rightarrow$

$x^2y' + 2xyy' = 3 - 2xy - y^2 \Rightarrow y'(x^2 + 2xy) = 3 - 2xy - y^2 \Rightarrow y' = \frac{3 - 2xy - y^2}{x^2 + 2xy}$

10. $\frac{d}{dx}(y^5 + x^2y^3) = \frac{d}{dx}\left(1 + ye^{x^2}\right) \Rightarrow 5y^4y' + (x^2 \cdot 3y^2y' + y^3 \cdot 2x) = 0 + y \cdot e^{x^2} \cdot 2x + e^{x^2} \cdot y' \Rightarrow$

$y'\left(5y^4 + 3x^2y^2 - e^{x^2}\right) = 2xye^{x^2} - 2xy^3 \Rightarrow y' = \frac{2xy\left(e^{x^2} - y^2\right)}{5y^4 + 3x^2y^2 - e^{x^2}}$

11. $\frac{d}{dx}(x^2y^2 + x\sin y) = \frac{d}{dx}(4) \Rightarrow x^2 \cdot 2yy' + y^2 \cdot 2x + x\cos y \cdot y' + \sin y \cdot 1 = 0 \Rightarrow$

$2x^2yy' + x\cos y \cdot y' = -2xy^2 - \sin y \Rightarrow (2x^2y + x\cos y)y' = -2xy^2 - \sin y \Rightarrow y' = \frac{-2xy^2 - \sin y}{2x^2y + x\cos y}$

12. $\frac{d}{dx}(1 + x) = \frac{d}{dx}[\sin(xy^2)] \Rightarrow 1 = [\cos(xy^2)](x \cdot 2yy' + y^2 \cdot 1) \Rightarrow 1 = 2xy\cos(xy^2)y' + y^2\cos(xy^2)$

$\Rightarrow 1 - y^2\cos(xy^2) = 2xy\cos(xy^2)y' \Rightarrow y' = \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}$

13. $\frac{d}{dx}(4\cos x \sin y) = \frac{d}{dx}(1) \Rightarrow 4[\cos x \cdot \cos y \cdot y' + \sin y \cdot (-\sin x)] = 0 \Rightarrow$

$y'(4\cos x\cos y) = 4\sin x \sin y \Rightarrow y' = \frac{4\sin x \sin y}{4\cos x \cos y} = \tan x \tan y$

14. $\frac{d}{dx}[y\sin(x^2)] = \frac{d}{dx}[x\sin(y^2)] \Rightarrow y\cos(x^2) \cdot 2x + \sin(x^2) \cdot y' = x\cos(y^2) \cdot 2yy' + \sin(y^2) \cdot 1 \Rightarrow$

$y'[\sin(x^2) - 2xy\cos(y^2)] = \sin(y^2) - 2xy\cos(x^2) \Rightarrow y' = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}$

15. $\frac{d}{dx}\left(e^{x^2y}\right) = \frac{d}{dx}(x + y) \Rightarrow e^{x^2y}(x^2y' + y \cdot 2x) = 1 + y' \Rightarrow x^2e^{x^2y}y' + 2xye^{x^2y} = 1 + y' \Rightarrow$

$x^2e^{x^2y}y' - y' = 1 - 2xye^{x^2y} \Rightarrow y'(x^2e^{x^2y} - 1) = 1 - 2xye^{x^2y} \Rightarrow y' = \frac{1 - 2xye^{x^2y}}{x^2e^{x^2y} - 1}$

16. $\frac{d}{dx}(\sqrt{x + y}) = \frac{d}{dx}(1 + x^2y^2) \Rightarrow \frac{1}{2}(x + y)^{-1/2}(1 + y') = x^2 \cdot 2yy' + y^2 \cdot 2x \Rightarrow$

$\frac{1}{2\sqrt{x + y}} + \frac{y'}{2\sqrt{x + y}} = 2x^2yy' + 2xy^2 \Rightarrow 1 + y' = 4x^2y\sqrt{x + y}\,y' + 4xy^2\sqrt{x + y} \Rightarrow$

$y' - 4x^2y\sqrt{x + y}\,y' = 4xy^2\sqrt{x + y} - 1 \Rightarrow y'(1 - 4x^2y\sqrt{x + y}) = 4xy^2\sqrt{x + y} - 1 \Rightarrow$

$y' = \frac{4xy^2\sqrt{x + y} - 1}{1 - 4x^2y\sqrt{x + y}}$

17. $\sqrt{xy} = 1 + x^2y \Rightarrow \frac{1}{2}(xy)^{-1/2}(xy' + y \cdot 1) = 0 + x^2y' + y \cdot 2x \Rightarrow \frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} = x^2y' + 2xy$

$\Rightarrow y'\left(\frac{x}{2\sqrt{xy}} - x^2\right) = 2xy - \frac{y}{2\sqrt{xy}} \Rightarrow y'\left(\frac{x - 2x^2\sqrt{xy}}{2\sqrt{xy}}\right) = \frac{4xy\sqrt{xy} - y}{2\sqrt{xy}} \Rightarrow y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$

67. (a) If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x, we get $f'(y)\,\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \Rightarrow (f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

(b) $f(4) = 5 \Rightarrow f^{-1}(5) = 4$. By part (a), $(f^{-1})'(5) = 1/f'(f^{-1}(5)) = 1/f'(4) = 1/\left(\frac{2}{3}\right) = \frac{3}{2}$.

68. (a) $f(x) = 2x + \cos x \Rightarrow f'(x) = 2 - \sin x > 0$ for all x. Thus, f is increasing for all x and is therefore one-to-one.

(b) Since f is one-to-one, $f^{-1}(1) = k \Leftrightarrow f(k) = 1$. By inspection, we see that $f(0) = 2(0) + \cos 0 = 1$, so $k = f^{-1}(1) = 0$.

(c) $(f^{-1})'(1) = 1/f'(f^{-1}(1)) = 1/f'(0) = 1/(2 - \sin 0) = \frac{1}{2}$

69. $x^2 + 4y^2 = 5 \Rightarrow 2x + 4(2yy') = 0 \Rightarrow y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$ or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \Leftrightarrow$ $4b^2 = -a^2 - 5a \Leftrightarrow a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \Leftrightarrow$ $a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \Rightarrow b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \Rightarrow h = 2$. So the lamp is located 2 units above the x-axis.

3.7 Higher Derivatives

1. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

2. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

3. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

4. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

5. $f(x) = x^5 + 6x^2 - 7x \Rightarrow f'(x) = 5x^4 + 12x - 7 \Rightarrow f''(x) = 20x^3 + 12$

6. $f(t) = t^8 - 7t^6 + 2t^4 \Rightarrow f'(t) = 8t^7 - 42t^5 + 8t^3 \Rightarrow f''(t) = 56t^6 - 210t^4 + 24t^2$

7. $y = \cos 2\theta \Rightarrow y' = -2\sin 2\theta \Rightarrow y'' = -4\cos 2\theta$

8. $y = \theta \sin\theta \Rightarrow y' = \theta\cos\theta + \sin\theta \Rightarrow y'' = \theta(-\sin\theta) + \cos\theta \cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$

9. $F(t) = (1-7t)^6 \quad \Rightarrow \quad F'(t) = 6(1-7t)^5(-7) = -42(1-7t)^5 \quad \Rightarrow$
$F''(t) = -42 \cdot 5(1-7t)^4(-7) = 1470(1-7t)^4$

10. $g(x) = \dfrac{2x+1}{x-1} \quad \Rightarrow \quad g'(x) = \dfrac{(x-1)(2)-(2x+1)(1)}{(x-1)^2} = \dfrac{2x-2-2x-1}{(x-1)^2} = \dfrac{-3}{(x-1)^2}$ or $-3(x-1)^{-2}$
$\Rightarrow \quad g''(x) = -3(-2)(x-1)^{-3} = 6(x-1)^{-3}$ or $\dfrac{6}{(x-1)^3}$

11. $h(u) = \dfrac{1-4u}{1+3u} \quad \Rightarrow \quad h'(u) = \dfrac{(1+3u)(-4)-(1-4u)(3)}{(1+3u)^2} = \dfrac{-4-12u-3+12u}{(1+3u)^2} = \dfrac{-7}{(1+3u)^2}$ or
$-7(1+3u)^{-2} \quad \Rightarrow \quad h''(u) = -7(-2)(1+3u)^{-3}(3) = 42(1+3u)^{-3}$ or $\dfrac{42}{(1+3u)^3}$

12. $H(s) = a\sqrt{s} + \dfrac{b}{\sqrt{s}} = as^{1/2} + bs^{-1/2} \quad \Rightarrow$
$H'(s) = a \cdot \frac{1}{2}s^{-1/2} + b\left(-\frac{1}{2}s^{-3/2}\right) = \frac{1}{2}as^{-1/2} - \frac{1}{2}bs^{-3/2} \quad \Rightarrow$
$H''(s) = \frac{1}{2}a\left(-\frac{1}{2}s^{-3/2}\right) - \frac{1}{2}b\left(-\frac{3}{2}s^{-5/2}\right) = -\frac{1}{4}as^{-3/2} + \frac{3}{4}bs^{-5/2}$

13. $h(x) = \sqrt{x^2+1} \quad \Rightarrow \quad h'(x) = \dfrac{1}{2}(x^2+1)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+1}} \quad \Rightarrow$
$$h''(x) = \frac{\sqrt{x^2+1}\cdot 1 - x\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right]}{\left(\sqrt{x^2+1}\right)^2} = \frac{(x^2+1)^{-1/2}\left[(x^2+1)-x^2\right]}{(x^2+1)^1} = \frac{1}{(x^2+1)^{3/2}}$$

14. $y = xe^{cx} \quad \Rightarrow \quad y' = x \cdot e^{cx} \cdot c + e^{cx} \cdot 1 = e^{cx}(cx+1) \quad \Rightarrow$
$y'' = e^{cx}(c) + (cx+1)e^{cx} \cdot c = ce^{cx}(1 + cx + 1) = ce^{cx}(cx+2)$

15. $y = (x^3+1)^{2/3} \quad \Rightarrow \quad y' = \frac{2}{3}(x^3+1)^{-1/3}(3x^2) = 2x^2(x^3+1)^{-1/3} \quad \Rightarrow$
$y'' = 2x^2\left(-\frac{1}{3}\right)(x^3+1)^{-4/3}(3x^2) + (x^3+1)^{-1/3}(4x) = 4x(x^3+1)^{-1/3} - 2x^4(x^3+1)^{-4/3}$

16. $y = \dfrac{4x}{\sqrt{x+1}} \quad \Rightarrow$
$$y' = \frac{\sqrt{x+1}\cdot 4 - 4x \cdot \frac{1}{2}(x+1)^{-1/2}}{\left(\sqrt{x+1}\right)^2} = \frac{4\sqrt{x+1} - 2x/\sqrt{x+1}}{x+1} = \frac{4(x+1)-2x}{(x+1)^{3/2}} = \frac{2x+4}{(x+1)^{3/2}} \quad \Rightarrow$$
$$y'' = \frac{(x+1)^{3/2}\cdot 2 - (2x+4)\cdot \frac{3}{2}(x+1)^{1/2}}{\left[(x+1)^{3/2}\right]^2} = \frac{(x+1)^{1/2}[2(x+1) - 3(x+2)]}{(x+1)^3}$$
$$= \frac{2x+2-3x-6}{(x+1)^{5/2}} = \frac{-x-4}{(x+1)^{5/2}}$$

17. $H(t) = \tan 3t \quad \Rightarrow \quad H'(t) = 3\sec^2 3t \quad \Rightarrow$
$H''(t) = 2 \cdot 3 \sec 3t \dfrac{d}{dt}(\sec 3t) = 6 \sec 3t\,(3 \sec 3t \tan 3t) = 18 \sec^2 3t \tan 3t$

18. $g(s) = s^2 \cos s \quad \Rightarrow \quad g'(s) = 2s \cos s - s^2 \sin s \quad \Rightarrow$
$g''(s) = 2\cos s - 2s \sin s - 2s \sin s - s^2 \cos s = (2 - s^2)\cos s - 4s \sin s$

19. $g(t) = t^3 e^{5t} \quad \Rightarrow \quad g'(t) = t^3 e^{5t} \cdot 5 + e^{5t} \cdot 3t^2 = t^2 e^{5t}(5t+3) \quad \Rightarrow$
$g''(t) = (2t)e^{5t}(5t+3) + t^2\left(5e^{5t}\right)(5t+3) + t^2 e^{5t}(5)$
$= te^{5t}\left[2(5t+3) + 5t(5t+3) + 5t\right] = te^{5t}\left(25t^2 + 30t + 6\right)$

3.8 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $f(x) = \ln(x^2 + 10) \Rightarrow f'(x) = \frac{1}{x^2+10}\frac{d}{dx}(x^2+10) = \frac{2x}{x^2+10}$

3. $f(\theta) = \ln(\cos\theta) \Rightarrow f'(\theta) = \frac{1}{\cos\theta}\frac{d}{d\theta}(\cos\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$

4. $f(x) = \cos(\ln x) \Rightarrow f'(x) = -\sin(\ln x)\cdot\frac{1}{x} = \frac{-\sin(\ln x)}{x}$

5. $f(x) = \log_2(1-3x) \Rightarrow f'(x) = \frac{1}{(1-3x)\ln 2}\frac{d}{dx}(1-3x) = \frac{-3}{(1-3x)\ln 2}$ or $\frac{3}{(3x-1)\ln 2}$

6. $f(x) = \log_{10}\left(\frac{x}{x-1}\right) = \log_{10} x - \log_{10}(x-1) \Rightarrow f'(x) = \frac{1}{x\ln 10} - \frac{1}{(x-1)\ln 10}$ or $-\frac{1}{x(x-1)\ln 10}$

7. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \Rightarrow f'(x) = \frac{1}{5}(\ln x)^{-4/5}\frac{d}{dx}(\ln x) = \frac{1}{5(\ln x)^{4/5}}\cdot\frac{1}{x} = \frac{1}{5x\sqrt[5]{(\ln x)^4}}$

8. $f(x) = \ln\sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5}\ln x \Rightarrow f'(x) = \frac{1}{5}\cdot\frac{1}{x} = \frac{1}{5x}$

9. $f(x) = \sqrt{x}\ln x \Rightarrow f'(x) = \sqrt{x}\left(\frac{1}{x}\right) + (\ln x)\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} = \frac{2+\ln x}{2\sqrt{x}}$

10. $f(t) = \frac{1+\ln t}{1-\ln t} \Rightarrow$

$$f'(t) = \frac{(1-\ln t)(1/t) - (1+\ln t)(-1/t)}{(1-\ln t)^2} = \frac{(1/t)[(1-\ln t)+(1+\ln t)]}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$$

11. $F(t) = \ln\frac{(2t+1)^3}{(3t-1)^4} = \ln(2t+1)^3 - \ln(3t-1)^4 = 3\ln(2t+1) - 4\ln(3t-1) \Rightarrow$

$$F'(t) = 3\cdot\frac{1}{2t+1}\cdot 2 - 4\cdot\frac{1}{3t-1}\cdot 3 = \frac{6}{2t+1} - \frac{12}{3t-1}, \text{ or combined, } \frac{-6(t+3)}{(2t+1)(3t-1)}.$$

12. $h(x) = \ln\left(x + \sqrt{x^2-1}\right) \Rightarrow$

$$h'(x) = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

13. $g(x) = \ln\frac{a-x}{a+x} = \ln(a-x) - \ln(a+x) \Rightarrow$

$$g'(x) = \frac{1}{a-x}(-1) - \frac{1}{a+x} = \frac{-(a+x)-(a-x)}{(a-x)(a+x)} = \frac{-2a}{a^2-x^2}$$

14. $F(y) = y\ln(1+e^y) \Rightarrow F'(y) = y\cdot\frac{1}{1+e^y}\cdot e^y + \ln(1+e^y)\cdot 1 = \frac{ye^y}{1+e^y} + \ln(1+e^y)$

15. $f(u) = \frac{\ln u}{1+\ln(2u)} \Rightarrow$

$$f'(u) = \frac{[1+\ln(2u)]\cdot\frac{1}{u} - \ln u\cdot\frac{1}{2u}\cdot 2}{[1+\ln(2u)]^2} = \frac{\frac{1}{u}[1+\ln(2u) - \ln u]}{[1+\ln(2u)]^2}$$
$$= \frac{1+(\ln 2+\ln u)-\ln u}{u[1+\ln(2u)]^2} = \frac{1+\ln 2}{u[1+\ln(2u)]^2}$$

16. $y = \ln(x^4 \sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4\ln x + 2\ln\sin x \Rightarrow y' = 4 \cdot \frac{1}{x} + 2 \cdot \frac{1}{\sin x} \cdot \cos x = \frac{4}{x} + 2\cot x$

17. $y = \ln|2 - x - 5x^2| \Rightarrow y' = \frac{1}{2 - x - 5x^2} \cdot (-1 - 10x) = \frac{-10x - 1}{2 - x - 5x^2}$ or $\frac{10x + 1}{5x^2 + x - 2}$

18. $G(u) = \ln\sqrt{\frac{3u+2}{3u-2}} = \frac{1}{2}[\ln(3u+2) - \ln(3u-2)] \Rightarrow G'(u) = \frac{1}{2}\left(\frac{3}{3u+2} - \frac{3}{3u-2}\right) = \frac{-6}{9u^2 - 4}$

19. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \Rightarrow$

$y' = -1 + \frac{1}{1+x} = \frac{-1 - x + 1}{1+x} = -\frac{x}{1+x}$

20. $y = [\ln(1+e^x)]^2 \Rightarrow y' = 2[\ln(1+e^x)] \cdot \frac{1}{1+e^x} \cdot e^x = \frac{2e^x \ln(1+e^x)}{1+e^x}$

21. $y = x\ln x \Rightarrow y' = x(1/x) + (\ln x) \cdot 1 = 1 + \ln x \Rightarrow y'' = 1/x$

22. $y = \frac{\ln x}{x^2} \Rightarrow y' = \frac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \frac{x(1 - 2\ln x)}{x^4} = \frac{1 - 2\ln x}{x^3} \Rightarrow$

$y'' = \frac{x^3(-2/x) - (1 - 2\ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2 - 3 + 6\ln x)}{x^6} = \frac{6\ln x - 5}{x^4}$

23. $y = \log_{10} x \Rightarrow y' = \frac{1}{x\ln 10} = \frac{1}{\ln 10}\left(\frac{1}{x}\right) \Rightarrow y'' = \frac{1}{\ln 10}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2 \ln 10}$

24. $y = \ln(\sec x + \tan x) \Rightarrow y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x \Rightarrow y'' = \sec x \tan x$

25. $f(x) = \frac{x}{1 - \ln(x-1)} \Rightarrow$

$f'(x) = \frac{[1 - \ln(x-1)] \cdot 1 - x \cdot \frac{-1}{x-1}}{[1 - \ln(x-1)]^2} = \frac{\frac{(x-1)[1 - \ln(x-1)] + x}{x-1}}{[1 - \ln(x-1)]^2} = \frac{x - 1 - (x-1)\ln(x-1) + x}{(x-1)[1 - \ln(x-1)]^2}$

$= \frac{2x - 1 - (x-1)\ln(x-1)}{(x-1)[1 - \ln(x-1)]^2}$

$\text{Dom}(f) = \{x \mid x - 1 > 0$ and $1 - \ln(x-1) \neq 0\} = \{x \mid x > 1$ and $\ln(x-1) \neq 1\}$

$= \{x \mid x > 1$ and $x - 1 \neq e^1\} = \{x \mid x > 1$ and $x \neq 1 + e\} = (1, 1+e) \cup (1+e, \infty)$

26. $f(x) = \frac{1}{1 + \ln x} \Rightarrow f'(x) = -\frac{1/x}{(1 + \ln x)^2}$ [Reciprocal Rule] $= -\frac{1}{x(1 + \ln x)^2}$.

$\text{Dom}(f) = \{x \mid x > 0$ and $\ln x \neq -1\} = \{x \mid x > 0$ and $x \neq 1/e\} = (0, 1/e) \cup (1/e, \infty)$.

27. $f(x) = x^2 \ln(1 - x^2) \Rightarrow f'(x) = 2x\ln(1 - x^2) + \frac{x^2(-2x)}{1 - x^2} = 2x\ln(1 - x^2) - \frac{2x^3}{1 - x^2}$.

$\text{Dom}(f) = \{x \mid 1 - x^2 > 0\} = \{x \mid |x| < 1\} = (-1, 1)$.

28. $f(x) = \ln\ln\ln x \Rightarrow f'(x) = \frac{1}{\ln\ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$.

$\text{Dom}(f) = \{x \mid \ln\ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty)$.

29. $f(x) = \frac{x}{\ln x} \Rightarrow f'(x) = \frac{\ln x - x(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \frac{1 - 1}{1^2} = 0$

30. $f(x) = x^2 \ln x \Rightarrow f'(x) = 2x\ln x + x^2\left(\frac{1}{x}\right) = 2x\ln x + x \Rightarrow f'(1) = 2\ln 1 + 1 = 1$