**UNIVERSITY OF SASKATCHEWAN**

**Department of Mathematics**

**MATHEMATICS 327.3 - FINAL EXAMINATION**

*Instructor: C. Soteros*

*This is a closed book examination - no calculators*

*Handouts of definitions and theorems may be used*

December 21, 2000 Time: 3hrs

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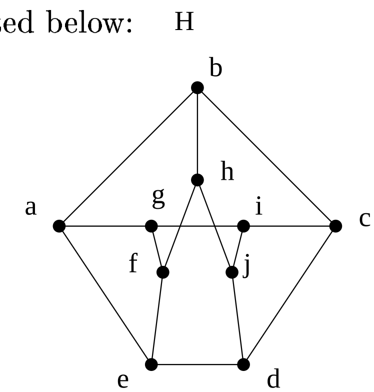
[25] 1. Consider the graph $H$ depicted below:




2 i. Is $H$ bipartite? Justify your answer.

2 ii. Give an example of a graph with more than 10 vertices which is homeomorphic to $H$.

2 iii. Is $H$ planar? Justify your answer.

2 iv. Does $H$ have either an euler cycle or an euler trail? Justify your answers.

1 v. Find the closure of $H$, $c(H)$.

2 vi. Find a breadth first search spanning tree of $H$.

3 vii. Does $H$ have a strongly connected orientation? If it does, find one.

1 viii. Find a maximum matching for $H$. Justify your answer.

2 ix. Let $H_1$ be the subgraph of $H$ which contains all the edges of $H$ except for the edges $\{h, f\}$ and $\{h, j\}$. Suppose $H_1$ represents the streets in a postal worker's route and suppose vertex $b$ indicates the location of the post office. Find an optimal postal route. Justify your answer.

2 x. Let $H_2$ be the subgraph of $H$ induced (or generated) by the vertex set $\{a, b, f, g, h, d\}$. What are the connected components of $H_2$?

2 xi. Find the characteristic polynomial of $H_2$ (defined in x.).

2 xii. Give an adjacency matrix for $H_2$ (defined in x.).

2 xiii. Give an incidence matrix for $H_2$ (defined in x.).

[10] 2. Determine whether the following statements are true or false. Give a counter example for those that are false and give some justification for those that are true.

2 i. There exists a simple graph which has all its vertices having different degrees (i.e. its degree sequence consists of distinct numbers).

2 ii. All simple graphs with degree sequence (5,5,4,4,3,3) are isomorphic.

2 iii. For every graph the number of edges in a maximum matching equals the number of vertices in a minimum covering.

2 iv. A graph $G$ with $n$ vertices such that every vertex has degree at least $n/2$ has a Hamiltonian circuit.

2 v. There is no $(5,3)$ de Bruijn sequence.

[10] 3. The following questions concern bridges.

3 i. Prove that an edge $\{x, y\}$ in a connected graph $G$ is a bridge if and only if it belongs to no circuit in the graph.

2 ii. What is the maximum number of bridges in a graph on $n$ vertices? Justify your answer.

3 iii. Prove that every cycle in a graph contains a circuit.

2 iv. Prove that a graph in which every vertex has even degree can have no bridge.

[7] 4. The following question concerns trees and graph colourings.

3 i. Prove by Mathematical Induction, using the properties of trees and the properties of chromatic polynomials, that the chromatic polynomial of a tree $T$ with $n$ vertices is given by $P(T, x) = x(x-1)^{n-1}$.

1 ii. Use the result in (i.) to determine the chromatic number of a tree $T$.

2 iii. For any graph $G = (V, E)$, let $P_x(G) = a_p x^p + a_{p-1}x^{p-1} + ... + a_1 x + a_0$ be its chromatic polynomial. Based on a result given in class what are $p$, $a_p$, $a_{p-1}$, and $a_0$?

1 iv. Prove using the properties given in (iii.) that if $P_x(G) = x(x-1)^n$ and $G$ is connected, then $G$ is a tree.

[8] 5. The following question concerns the adjacency matrix for graphs and digraphs.

1 i. What conclusion can be made about a graph with the following adjacency matrix:

$$\begin{bmatrix} 0_{n\times n} & A_{n\times m} \\ B_{m\times n} & 0_{m\times m} \end{bmatrix}$$

where $0_{k\times k}$ is a $k \times k$ matrix of zeroes and in this case $B_{m\times n}$ is an $m \times n$ matrix which is the transpose of $A_{n\times m}$.

1 ii. What if instead the adjacency matrix has the following form:

$$\begin{bmatrix} A_{n\times n} & 0_{n\times m} \\ 0_{m\times n} & C_{m\times m} \end{bmatrix}$$

where $A$ and $C$ are symmetric matrices.

3 iii. Given any digraph $D$ with adjacency matrix $A$, prove using Mathematical induction that the $i,j$th entry of $A^k$ (the $k$th power of the adjacency matrix of $D$) represents the number of walks of length $k$ from vertex $i$ to vertex $j$ in the digraph $D$.

3 iv. Prove that a tournament $D$ is transitive if and only if the sum of the diagonal entries of $A^3$ is zero. (You may use any known results for tournaments.)

[5] 6. Prove Corollary 15 on the handout: A bipartite graph $G = (X, Y, E)$ has a matching of size $t$ if and only if for all $S \subseteq X$, $|N(S)| \geq |S| + t - |X|$. (Hint: Add $|X| - t$ new vertices to $Y$ and join each new $Y$-vertex to each $X$-vertex.)

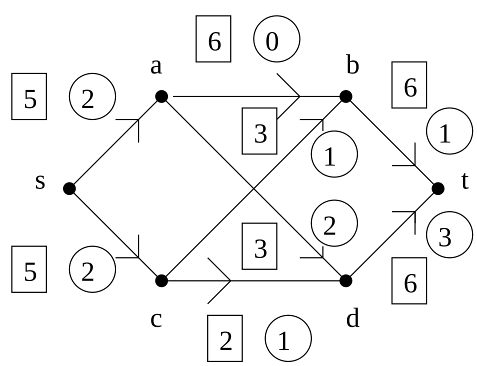
[12] 7. Consider the following network, $N_2$, where the capacities are shown in squares. Let $\{x_{ij}\}$ be the flow shown in circles.

2 i. Find the value of the flow.

3 ii. List two distinct cuts for $N_2$ and give their capacities.

3 iii. Find two flow augmenting paths relative to the given flow which each use the arc $(a, d)$. Use one of these paths to find a new flow whose value is greater than the current flow.

4 iv. What is the maximum value for a flow in $N_2$? Justify your answer by finding an $(s,t)$-cut whose capacity is the same as the value of the flow.




[13] 8. Consider again the graph $H$ in question 1 and construct the subgraph $G_1$ induced by the vertex set $\{a, b, c, e, f, g\}$.

2 i. List four different matchings for $G_1$.

1 ii. List two different coverings for $G_1$

2 iii. Without listing all possible matchings, find the size of a maximum matching of $G_1$. Justify your answer.

2 iv. Let $Y = \{g, e, b\}$ and construct the associated network $N_1$ for $G_1$.

1 v. Consider the matching $M = \{\{a, g\}, \{b, c\}\}$ of $G_1$. Give a flow $\{x_{ij}\}$ on $N_1$ which corresponds to this matching.

1 vi. Find an $M$-augmenting path for the matching in v.).

1 vii. Using the flow defined in v.), find a flow augmenting path relative to $\{x_{ij}\}$ in $N_1$.

3 viii. Find a maximum flow of the network and a minimum cut. Find the associated maximum matching and minimum covering for $G_1$. Justify your answers.

[90] TOTAL

*Good Luck and Happy Holidays*