Residue fields of arbitrary convex valuations on restricted analytic fields with exponentiation I*

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1 Introduction

In their paper [D–M–M2], van den Dries, Macintyre and Marker give an explicit construction of a nonarchimedean model of the theory of the reals with restricted analytic functions and exponentiation. This model, called the logarithmic exponential power series field, lies in a generalized power series field. They use the results of Ressayre and Mourgues about truncation-closed embeddings to answer a problem raised by Hardy. They also show that certain functions, including the Gamma-function and the Riemann Zeta-function, cannot be defined using exponential function, logarithm and restricted analytic functions.

This paper answers a question raised by Angus Macintyre in a talk for the Algebraic Model Theory Programme at the Fields Institute, November 1996. He asked whether the results of [D–M–M2] can be deduced by a "more invariant" version of truncation. Indeed, we derive the results of [D–M–M2] without using embeddings in the logarithmic exponential power series field. We replace truncation results by an intrinsic property, which is an assertion about the residue fields of arbitrary convex valuations. It is invariant because it does not depend on an embedding in logarithmic exponential power series fields. Our result about the residue fields is proved using some comparingly simple lemmas which build on the "Valuation Property of restricted analytic functions" (cf. Corollary 3.7 of [D–M–M1]). In addition to that, we just use the knowledge of how to build up exponential fields from subfields.

The following fact is well known: Take a real closed field L and a convex valuation w on L (that is, its valuation ring $\mathcal{O}_w$ is convex in L). Then the residue field Lw is a real closed field, and it can be embedded in L in such a way that the composition of the residue map with the embedding yields the identity on Lw. Indeed, the embedding can be constructed by use of Hensel's Lemma (a convex valuation on a real closed field is always henselian), since the residue field has characteristic 0. The image under every such embedding is a maximal subfield of $\mathcal{O}_w$, and conversely, every maximal subfield K of $\mathcal{O}_w$ is isomorphic to Lw via the residue map. Therefore, we will always write $Lw = K \subset \mathcal{O}_w$, where the equality is to be understood modulo the isomorphism induced by the residue

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map. Note that the maximal subfields of $\mathcal{O}_w$ are characterized by the property that for all $y \in \mathcal{O}_w$ there is a unique $z \in K$ such that $w(y - z) > 0$.

More generally: Take any ordered field L with convex valuation w. We denote the real closure of L by L^{r}. Then every embedding of Lw in L has a unique extension to an embedding of $(Lw)^{\mathrm{r}}$ in L^{r}.

In view of these facts, the following natural question arises: suppose we have additional structure on our ordered fields, to which extent can this structure be preserved under the embedding of the residue fields? We wish to answer this for the case where the additional structure is given by restricted analytic functions and exponentiation.

Let $\mathcal{F}$ be any set of restricted analytic functions, closed under partial derivations and containing the restricted exp and log. In [D–M–M1], van den Dries, Macintyre and Marker have explained in detail the connection between the restricted analytic functions and convergent power series representing them ("convergent" means "convergent near 0"). Therefore, following [D–M–M2], we will rather consider $\mathcal{F}$ as a set of convergent power series; for instance, $\mathcal{F}_{LE}$ will denote the set consisting of the power series expansions of $\exp x$, $\log(1+x)$ and $\frac{1}{1+x}$. Similarly, $\mathcal{F}_{\mathrm{an}}$ will denote the set of all convergent power series (in finitely many variables), thus representing all restricted analytic functions.

Let $\mathbb{R}_{\mathrm{an,exp}}$ denote the reals with all restricted analytic functions and exponentiation. Throughout this paper, except for Section 3, we work in a model M of $T_{\mathrm{an,exp}}$, the elementary theory of $\mathbb{R}_{\mathrm{an,exp}}$. We assume that M properly contains $\mathbb{R}$, and take $x \in M$ to be a positive infinite element, that is, $x > \mathbb{R}$. We are interested in the smallest subfield of M containing $\mathbb{R}(x)$ which is

- real closed,
- exp-closed, i.e., $\exp a \in F$ for every $a \in F$,
- log-closed, i.e., $\log a \in F$ for every *positive* $a \in F$,
- $\mathcal{F}$-closed, i.e., closed under all functions from $\mathcal{F}$.

We will denote this closure of $\mathbb{R}(x)$ by $LE_{\mathcal{F}}(x)$. From [D–M–M1] we know that a field $L \subset M$ containing $\mathbb{R}$ is closed under the restricted analytic functions represented by $\mathcal{F}$ if and only if for every convergent power series $f(X_1, \ldots, X_n) \in \mathcal{F}$ and all infinitesimals $\varepsilon_1, \ldots, \varepsilon_n \in L$, the element $f(\varepsilon_1, \ldots, \varepsilon_n) \in M$ lies in L. (The nonzero infinitesimals are the multiplicative inverses of the infinite elements.)

For positive infinite elements $z \in M$, we set $\log^0 z = z$ and $\log^{m+1} z = \log(\log^m z)$ for all integers $m \geq 0$; note that every $\log^m z$ is again positive infinite. Similarly, we define $\exp^m z$ for every $z \in M$.

For every subfield K of $\mathcal{O}_w$, its multiplicative group $K^\times$ is contained in the multiplicative group $\mathcal{O}_w^\times$ of all units of $\mathcal{O}_w$. We will say that K is **relatively exp-closed in** $\mathcal{O}_w^\times$ if $a \in K$ and $\exp(a) \in \mathcal{O}_w^\times$ implies that $\exp(a) \in K$. For example, $\mathbb{R}$ is relatively exp-closed in $\mathcal{O}_w^\times$ for every convex valuation w of M. Our main theorem is:

Theorem 1.1 *Let w be an arbitrary convex valuation of $LE_{\mathcal{F}}(x)$, and denote its valuation ring by $\mathcal{O}_w$. Then there exists a real closed subfield $K \subset \mathcal{O}_w$ which is log-closed and $\mathcal{F}$-closed, relatively exp-closed in $\mathcal{O}_w^\times$ and satisfies $LE_{\mathcal{F}}(x)w = K$. If w is not the natural valuation, then there is some integer $m_0 \geq 0$ such that K can be chosen to be the uniquely determined smallest subfield of $\mathcal{O}_w$ which is real closed, log- and $\mathcal{F}$-closed, relatively exp-closed in $\mathcal{O}_w^\times$ and contains $\mathbb{R}(\log^{m_0} x)$. If $wx = 0$, then we can choose $m_0 = 0$, so that K contains x.*

After giving some preliminaries in Section 2, we will determine in Section 3 the value groups and residue fields of $\mathcal{F}$-closures of special subfields of models of $T_{\rm an}$, the elementary theory of the reals with all restricted analytic functions. In Section 4, we build up $LE_{\mathcal{F}}(x)$ from $\mathbb{R}(x)$. From this construction and the results of Section 3, we derive Theorems 4.7 and 4.8, which imply Theorem 1.1.

Let $H(\mathbb{R}_{\rm an,exp})$ denote the field of the germs at $+\infty$ of all functions on $\mathbb{R}$ which are definable in $\mathbb{R}_{\rm an,exp}$, that is, definable using restricted analytic functions and the exponential function. Denote by x the germ of the identity function. Then LE is defined to be the smallest subfield of $H(\mathbb{R}_{\rm an,exp})$ which is real closed, exp- and log-closed and contains $\mathbb{R}(x)$. It is the field of the germs of all compositions of semialgebraic functions, exp and log. From [D–M–M1] it is known that $H(\mathbb{R}_{\rm an,exp})$ is a Hardy field and a model of $T_{\rm an,exp}$. So we can take $M = H(\mathbb{R}_{\rm an,exp})$. With this choice, $H(\mathbb{R}_{\rm an,exp})$ is equal to $LE_{\mathcal{F}_{\rm an}}(x)$ (cf. Section 5 of [D–M–M1]), and the Hardy field LE is equal to $LE_{\mathcal{F}_{LE}}(x)$ (cf. Section 3 of [D–M–M2]). In Section 5 we derive the solution of the Hardy problem from Theorem 1.1, and prove the undefinability of certain functions, working directly in $H(\mathbb{R}_{\rm an,exp})$. We do not need to embed it in logarithmic exponential power series.

Note that we do *not* define $LE_{\mathcal{F}}(x)$ to be the definable closure of $\mathbb{R}(x)$ inside of M. We know from [D–M–M1] that it will coincide with the definable closure in the case of $\mathcal{F} = \mathcal{F}_{\rm an}$. But in general, it will be properly contained in the definable closure. In fact, the compositional inverse of $(\log x)(\log\log x)$ appearing in the Hardy problem, is definable over LE, but not an element of it (as is shown in [D–M–M2]). We will prove the analogue of Theorem 1.1 for definable closures of arbitrarily large subfields of M in a subsequent paper ([K–K3]), building on recent work of van den Dries [D].

In Section 6, we introduce an intrinsic form of power series expansions for the elements of $LE_{\mathcal{F}}(x)$. For this, we use monomials (which are obtained from elements in the image of an arbitrary cross-section by multiplication with reals) together with coefficients from significant residue fields $LE_{\mathcal{F}}(x)w$. From such an expansion of a function $h \in H(\mathbb{R}_{\rm an,exp})$, one can define the **principal part** of h, which turns out to carry information about the asymptotic behaviour of the function $\exp h(x)$ (Theorem 6.4). This puts the particular solution of the Hardy problem in a more general framework (Corollary 6.5).

We will apply the results of Section 6 in a subsequent paper ([K–K4]), where we will refine our construction given in Section 4. With this refinement, we will be able to present explicitly exponential integer parts of the fields $LE_{\mathcal{F}}(x)$ and to discuss their irreducible and prime elements.

Although we are explicitly treating only the case of $\mathcal{F}$ a set of restricted analytic functions, Theorem 1.1 and some other results on the valuation theoretical properties of $LE_{\mathcal{F}}(x)$ hold more generally. We may replace $T_{\rm an}$ by the theory of an arbitrary polynomially bounded o-minimal expansion of the reals, since L. van den Dries has shown that the Valuation Property remains true after a certain modification (invoking the fields of exponents). Then we may take $\mathcal{F}$ to be any set of definable functions for which an analogue of our key Lemma 3.1 can be proved (and require in addition that $\mathcal{F}_{LE} \subset \mathcal{F}$). For example, Lemma 3.1 and its proof also hold if $\mathcal{F}$ is a set of Gevrey functions, closed under partial derivatives. Another interesting example is obtained if one replaces $T_{\rm an}$ by the theory of the reals with convergent generalized power series. In [D–S], this theory is shown to be model complete and o-minimal. Take $\mathcal{F}$ to be a set of generalized power series for which the exponents of each variable form a sequence cofinal in $\mathbb{R}$ (indexed by the natural numbers). Then an analogue of Lemma 3.1 can be proved if $\mathcal{F}$ is closed under formal derivatives in the sense of [D–S]. Although the condition on the exponents is quite restrictive, it holds for the presently known applications of interest. In particular, the function $\zeta(-\log x) = \sum_{n=1}^{\infty} x^{\log n}$ on $[0, e^{-2}]$ (with ζ the Riemann zeta function) satisfies the condition.

In a later version of this paper, we will use this more general approach to show that if T is the theory of an o-minimal polynomially bounded expansion of the reals, then T_{exp} is exponentially bounded and levelled, in the sense of [M–M].

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2 Some preliminaries

If (K, w) is a valued field, then we write wa for the value of $a \in K$ and wK for its value group $\{wa \mid 0 \neq a \in K\}$. Further, we write aw for the residue of a, and Kw for the residue field. The valuation ring is denoted by $\mathcal{O}_w$. For generalities on valuation theory, see [R] or [KF].

Lemma 2.1 *A valuation w on an ordered field K is convex (i.e., $\mathcal{O}_w$ is convex), if and only if the following holds for all $a, b \in K$:*

$$(a < b < 0 \ \vee \ 0 < b < a) \ \Rightarrow \ wa \leq wb \, .$$

The convex valuation rings of an ordered field are linearly ordered by inclusion. If $\mathcal{O}_w \subsetneq \mathcal{O}_{w'}$ then w is said to be **finer** than w'. There is always a finest convex valuation, called the **natural valuation**. It is characterized by the fact that its residue field is archimedean. A valuation w on an ordered field is convex if and only if the natural valuation is finer or equal to w. **Throughout this paper, v will always denote the natural valuation,** unless stated otherwise.

If a, b are elements of an ordered group or an ordered field, then we write $a \ll b < 0$ if $a < b < 0$ and $\forall n \in \mathbb{N} : a < nb$. Similarly, $a \gg b > 0$ if $a > b > 0$ and $\forall n \in \mathbb{N} : a > nb$. We set $|a| := \max\{a, -a\}$. Then the natural valuation is characterized by:

$$va < vb \ \Leftrightarrow \ |a| \gg |b| \, . \qquad (1)$$

Note that if $\mathbb{R} \subset K$ and $a \in K$ with $va = 0$, then there is some $r \in \mathbb{R}$ such that $v(a - r) > 0$. Further, $wr = 0$ for every $r \in \mathbb{R}$ and every convex valuation w.

Lemma 2.2 *Let v, w be arbitrary valuations on some field K. Suppose that v is finer than w. Then for all $a, b \in K$,*

$$va \leq vb \ \Rightarrow \ wa \leq wb \, . \qquad (2)$$

In particular, $wa > 0 \Rightarrow va > 0$. Further, $H_w := \{vz \mid z \in K \wedge wz = 0\}$ is a convex subgroup of the value group vK of v. We have that $vz \in H_w \Leftrightarrow z \in \mathcal{O}_w^\times$. There is a *canonical isomorphism $wK \simeq vK/H_w$. Conversely, every convex subgroup of vK is of the form H_w for some valuation w such that v finer or equal to w.*

The valuation v of K induces a valuation v/w on Kw. There are canonical isomorphisms $v/w(Kw) \simeq H_w$ and $(Kw)v/w \simeq Kv$. If Kw is embedded in $\mathcal{O}_w$ such that the restriction of the residue map is the identity on Kw, then $v/w = v|_{Kw}$ (up to equivalence). Writing v instead of $v|_{Kw}$, we then have that $v(Kw) = H_w$ and $(Kw)v = Kv$.

We will call H_w the **convex subgroup associated with** w and w the **valuation associated with** H_w. Since the isomorphism is canonical, we will write $wK = vK/H_w$.

The order type of the chain of nontrivial convex subgroups of an ordered abelian group G is called the **rank** of G. If finite, then the rank is not bigger than the maximal number of rationally independent elements in G. In particular, G has finite rank if it is finitely generated or its divisible hull is a $\mathbb{Q}$-vector space of finite dimension.

From (1) and (2) it follows that for every convex valuation w,

$$|a| \leq |b| \;\Rightarrow\; wa \geq wb\,. \tag{3}$$

Lemma 2.3 *Let w be any valuation on $K(x_i \mid i \in I_1 \cup I_2)$ such that the values wx_i, $i \in I_1$, are rationally independent over wK, and the residues $x_i w$, $i \in I_2$, are algebraically independent over Kw. Then the elements x_i, $i \in I_1 \cup I_2$ are algebraically independent over K. Moreover,*

$$wK(x_i \mid i \in I_1 \cup I_2) \;=\; wK \oplus \bigoplus_{i\in I_1} \mathbb{Z}wx_i \quad \textit{and} \quad K(x_i \mid i \in I_1 \cup I_2)w \;=\; Kw(x_i w \mid i \in I_2)\,. \tag{4}$$

For the proof, see [KF] or [B], chapter VI, §10.3, Theorem 1.

Corollary 2.4 *Suppose that $\mathbb{R}(x_i \mid i \in I)$ is an ordered field such that the values vx_i, $i \in I$ are rationally independent. Let w be a convex valuation on $\mathbb{R}(x_i \mid i \in I)$. Assume that there is a subset $I_w \subset I$ such that $wx_i = 0$ for all $i \in I_w$ and that the values wx_i, $i \in I \setminus I_w$ are rationally independent. Then*

$$w\mathbb{R}(x_i \mid i \in I) \;=\; \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i \quad \textit{and} \quad \mathbb{R}(x_i \mid i \in I)w \;=\; \mathbb{R}(x_i \mid i \in I_w)\,.$$

Proof: For $i \in I_w$, $wx_i = 0$ implies that $vx_i \in H_w$. By the foregoing lemma, $v\mathbb{R}(x_i \mid i \in I_w) = \bigoplus_{i\in I_w} \mathbb{Z}vx_i \subset H_w$. This proves that w is trivial on $\mathbb{R}(x_i \mid i \in I_w)$. So we can assume that the residue map is the identity on $\mathbb{R}(x_i \mid i \in I_w)$. Now apply the foregoing lemma with $K = \mathbb{R}(x_i \mid i \in I_w)$ (then $Kw = K$), $I_1 = I \setminus I_w$ and $I_2 = \emptyset$. □

A sequence of elements $a_\nu \in K$, $\nu < \lambda$ (λ some limit ordinal), is called a **pseudo Cauchy sequence** in (K, w) if $w(a_\rho - a_\sigma) < w(a_\sigma - a_\tau)$ for all ρ, σ, τ with $\rho < \sigma < \tau < \lambda$. It follows from the ultrametric triangle law that $w(a_\nu - a_\tau) = w(a_\nu - a_{\nu+1})$ whenever $\nu < \tau < \lambda$. The element a is called a (pseudo) limit of this pseudo Cauchy sequence if $w(a_\nu - a) = w(a_\nu - a_{\nu+1})$ for all $\nu < \lambda$. In general, there may be several distinct limits:

Lemma 2.5 *Let a be a limit of $(a_\nu)_{\nu<\lambda}$. Then b is also a limit of $(a_\nu)_{\nu<\lambda}$ if and only if $w(a-b) > w(a_\nu - a_{\nu+1})$ for all $\nu < \lambda$.*

An extension $(K,w) \subset (L,w)$ of valued fields is called **immediate** if the canonical embedding of wK in wL and the canonical embedding of Kw in Lw are surjective (we then write $wK = wL$ and $Kw = Lw$). The henselization of a valued field is an immediate extension.

Lemma 2.6 *Assume that $(K,w) \subset (L,w)$ is immediate and that $a \in L \setminus K$. Then there is a pseudo Cauchy sequence in (K,w) with limit a, but not having a limit in K.*

The next lemma follows from the Lemma of Ostrowski (cf. [R], [KF]) and the results of Kaplansky's important paper [KA] (cf. also [KF]):

Lemma 2.7 *Let K be any real closed field and w a convex valuation on K. Assume that $(a_\nu)_{\nu<\lambda}$ is a pseudo Cauchy sequence in (K,w), not having a limit in K. Assume further that in some extension of (K,w), there exists a limit a. Then the extension of w to $K(a)$ is uniquely determined and immediate.*

If $(K_1,w) \subset (K_2,w)$ is an immediate algebraic extension of ordered fields with convex valuation w, then their henselizations (in a fixed henselian extension field) are equal.

If the values $w(a_\nu - a_{\nu+1})$ are cofinal in wK, then $(a_\nu)_{\nu<\lambda}$ is called a **Cauchy sequence** in (K,w). Lemma 2.5 shows that if this sequence has a limit in K, then this limit is uniquely determined. Indeed, if $a,b \in K$ are limits, then $w(a-b) > wK$, that is, $w(a-b) = \infty$, or in other words, $a = b$. All elements in the completion of a valued field are limits of Cauchy sequences (and in particular, the completion is an immediate extension). Conversely:

Lemma 2.8 *Let the situation be as in Lemma 2.7, with $(a_\nu)_{\nu<\lambda}$ a Cauchy sequence. Then there is a unique embedding of $(K(a),w)$ over K in the completion of (K,w).*

Note that if wK is archimedean, then it follows from Newton's method together with this lemma that the henselization of (K,w) is embeddable in the completion of (K,w). If w and v are arbitrary valuations such that v is finer than w and $Kw \subset K$, then (K,v) is henselian if and only if (K,w) and (Kw,v) are henselian (cf. [R] or [KF]). From these facts, one obtains:

Lemma 2.9 *Let K be an ordered field with convex valuation w. Suppose that $Kw \subset K$ and that (Kw,v) is henselian. Then the henselization of K with respect to v is equal to the henselization of K with respect to w. If in addition wK is archimedean, this henselization is embeddable in the completion of (K,w).*

For the proof of the next lemma, see [P] or [KF].

Lemma 2.10 *Let K be an ordered field with convex valuation w. Then K is real closed if and only if (K,w) is henselian, wK is divisible and Kw is real closed. Further, $wK^{\mathrm{r}} = \mathbb{Q} \otimes wK$ (the divisible hull of wK), and $K^{\mathrm{r}}w = (Kw)^{\mathrm{r}}$. If wK is divisible and Kw is real closed, then the real closure of K is equal to the henselization of K with respect to w (and embeddable in the completion of (K,w) if wK is archimedean).*

If x is a positive element in the real closed field K, then it has a unique positive k-th root, for every $k \in \mathbb{N}$. So if K contains the real closure of a field $\mathbb{R}(x_i \mid i \in I)$, with all x_i positive, then $x_i^q \in K$ for all $i \in I$ and all $q \in \mathbb{Q}$. This can be used to show that every real closed field K, with its natural (or any convex) valuation v, admits a **cross-section**, i.e., an embedding π of the group vK in the multiplicative group $K^\times$ such that $v\pi\alpha = \alpha$ for all $\alpha \in vK$. Indeed, take any maximal set $\mathcal{X} = \{x_i \mid i \in I\} \subset K$ such that the values vx_i are rationally independent. By the maximality of the set, together with Lemma 2.10, it follows that vK is the divisible hull of $v\mathbb{R}(x_i \mid i \in I) = \bigoplus_{i\in I} \mathbb{Z}vx_i$. For every $\alpha \in v\mathbb{R}(x_i \mid i \in I)$ there is a unique element x of the multiplicative group $\langle \mathcal{X} \rangle$ generated by the x_i, such that $vx = \alpha$. Consequently, there is a unique cross-section π of (K, v) whose image contains $\mathcal{X}$, and this image πvK is the divisible hull $\widetilde{\langle \mathcal{X} \rangle} = \{\prod_{i\in I_0} x_i^{q_i} \mid I_0 \subset I \text{ finite}, q_i \in \mathbb{Q}\}$ of $\langle \mathcal{X} \rangle$. If we have fixed a cross-section π, or a set $\mathcal{X}$ and take π to be the associated cross-section, then we call $\mathbb{R}^\times \cdot \pi vK$ the set of **monomials** of K. Hence the monomials are the elements of the form

$$d = r \prod_{i\in I_0} x_i^{q_i} \text{ with } 0 \neq r \in \mathbb{R},\ I_0 \subset I \text{ finite, and } q_i \in \mathbb{Q} \text{ for every } i \in I_0 .$$

Assume that M is a model of $T_{\mathrm{an,exp}}$. (Note that it follows that $\mathbb{R} \subset M$.) Then the exponential exp of M is an order preserving isomorphism from the additive group of M onto its multiplicative group of positive elements. Its inverse is the logarithm log; it is order preserving and defined for all positive elements. Consequently, if $z \in M$ is positive infinite, that is, $z > \mathbb{R}$, then $\log z > \log(\{r \in \mathbb{R} \mid r > 0\}) = \mathbb{R}$. In other words,

$$vz < 0 \wedge z > 0 \Rightarrow v \log z < 0 \wedge \log z > 0 . \tag{5}$$

Further, exp satisfies the Taylor axiom scheme:

(TA) $\qquad |z| \leq 1 \Rightarrow |\exp z - \sum_{n=0}^{m} \frac{z^n}{n!}| < |z^m| \qquad (m \in \mathbb{N})$.

In order to derive a valuation theoretical property from this axiom, we need the following simple lemma:

Lemma 2.11 *Let K be an ordered field and w a convex valuation on K. Suppose that $h \in K$ satisfies*

$$\left| h - \sum_{n=0}^{m} r_n z_n \right| < |r'_m z_m| \quad \textit{for all } m \in \mathbb{N}, \tag{6}$$

where $r_n, r'_n \in \mathbb{R} \setminus \{0\}$, and $z_n \in K$ are such that $wz_{n+1} > wz_n$. Write

$$S_m := \sum_{n=0}^{m} r_n z_n .$$

Then $(S_m)_{m\in\mathbb{N}}$ is a pseudo Cauchy sequence in (K, w). Further,

$$w(h - S_m) = wz_{m+1} = w(S_{m+1} - S_m) , \tag{7}$$

which shows that h is a limit of this sequence.

Proof: Recall that $wr = 0$ for $0 \neq r \in \mathbb{R}$, and that $w|a| = wa$ for every a in K. By (6) and (3), we have that

$$\begin{aligned} w(h - S_m - r_{m+1}z_{m+1} - r_{m+2}z_{m+2}) &= w(h - S_{m+2}) \geq wr'_{m+2}z_{m+2} = wz_{m+2} \\ &> wz_{m+1} = wr_{m+1}z_{m+1} \,. \end{aligned}$$

By the ultrametric triangle law,

$$w(r_{m+1}z_{m+1} + r_{m+2}z_{m+2}) = \min\{wr_{m+1}z_{m+1}, wr_{m+2}z_{m+2}\} = wr_{m+1}z_{m+1} \,.$$

Hence, again by the ultrametric triangle law,

$$\begin{aligned} w(h - S_m) &= \min\{w(h - S_m - r_{m+1}z_{m+1} - r_{m+2}z_{m+2}), w(r_{m+1}z_{m+1} + r_{m+2}z_{m+2})\} \\ &= wr_{m+1}z_{m+1} = w(S_{m+1} - S_m) \,. \end{aligned}$$

□

Lemma 2.12 *Assume that M is a model of $T_{\text{an,exp}}$, and let w be a convex valuation on M. Then for every $z \in M$,*

$$wz > 0 \;\Rightarrow\; w\exp z = 0 \wedge w(\exp z - 1) = wz \tag{8}$$

$$vz = 0 \;\Rightarrow\; v\exp z = 0 \,. \tag{9}$$

Proof: By Lemma 2.2, $wz > 0$ implies $vz > 0$, that is, z is infinitesimal. In particular, $|z| < 1$, and (TA) holds. Applying (7) of Lemma 2.11 with $m = 1$ and $z_m = z^m$, we find that $w(\exp z - 1 - z) = wz^2 = 2wz > wz$. By the ultrametric triangle law, this implies that $w\exp z = w(1 + z) = w1 = 0$ and $w(\exp z - 1) = wz$. This proves (8).

Now assume that $vz = 0$. Then there is some $r \in \mathbb{R} \subset M$ such that $v(z - r) > 0$. We have that $\exp r \in \mathbb{R}$, hence $v\exp r = 0$. By (8) with $w = v$, $v\exp(z - r) = 0$. Thus, $v\exp z = v\exp r \exp(z - r) = v\exp r + v\exp(z - r) = 0$. This proves (9). □

With M as before, exp also satisfies the following growth axiom scheme (which is part of Ressayre's Axioms):

(GA) $\quad z > m^2 \implies \exp z > z^m \qquad (m \in \mathbb{N})$.

From this, we derive:

Lemma 2.13 *Assume that M is a model of $T_{\text{an,exp}}$. Then*

$$vz < 0 \wedge z > 0 \;\Rightarrow\; v\exp z \ll vz \ll v\log z < 0 \tag{10}$$

$$wz = 0 \wedge z > 0 \;\Rightarrow\; w\log z \geq 0 \tag{11}$$

$$vz \geq 0 \;\Leftrightarrow\; v\exp z = 0 \,. \tag{12}$$

Proof: If $vz < 0$ and $z > 0$, then $z > \mathbb{R}$ and thus, $z > m^2$ for every $m \in \mathbb{N}$. So by (GA), $\exp z > z^m > 0$ for all m. Hence by (3), $v \exp z \leq mvz$ for all m, i.e., $v \exp z \ll vz < 0$. In view of (5), we can replace z by $\log z$ to get that $vz \ll v \log z < 0$. This proves (10).

Now assume that $wz = 0$ and $z > 0$. If $vz < 0$, then by (10), $vz < v \log z < 0$. If $vz > 0$, then $vz^{-1} < 0$ and by (10), $-vz = vz^{-1} < v \log z^{-1} = v(-\log z) = v \log z < 0$. In both cases, it follows from Lemma 2.2 that $0 = wz = wz^{-1} \leq w \log z \leq 0$, i.e., $w \log z = 0$. Now let $vz = 0$. If $v \log z < 0$, then by (10), $vz = v \exp \log z < 0$ if $\log z > 0$, and $vz = -vz^{-1} = -v \exp(-\log z) > 0$ if $\log z < 0$. Hence, $v \log z \geq 0$, and again by Lemma 2.2, $w \log z \geq 0$. This proves (11).

Implication "⇒" of (12) follows from (8) with $w = v$, together with (9). The converse implication follows from (11), where we take $w = v$ and replace z by $\exp z$. □

The valuation v is a homomorphism from the multiplicative group $M^{>0}$ of positive elements onto the value group vM. Its kernel is $\mathcal{U}^{>0} = \{z \in M \mid vz = 0 \wedge z > 0\}$, the subgroup of positive units. So v induces an isomorphism $M^{>0}/\mathcal{U}^{>0} \simeq vM$. (3) shows that it is order reversing. The exponential exp is an order preserving isomorphism from the additive group of M onto the multiplicative group $M^{>0}$. By (12), the preimage of $\mathcal{U}^{>0}$ under exp is precisely $\mathcal{O}_v$. Hence, the map $z \mapsto v \exp z$ induces an order reversing isomorphism $M/\mathcal{O}_v \simeq vM$ of ordered abelian groups. This gives:

Lemma 2.14 *If the elements a_j, $j \in J$, are rationally independent over $\mathcal{O}_v$ in the additive group of M, then the values $v \exp a_j$, $j \in J$, are rationally independent in vM. In particular, if $va < 0$ and $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathbb{R}$, then the values $v \exp ra$, $r \in \mathcal{B}$, are rationally independent.*

For further details on the valuation theory of exponential fields, see [KS] and [K–K1].

Assume that M is a model of T_{an}, and take an arbitrary subfield $F \subset M$. The real closure F^{r} of F can be taken to lie in M since M is real closed. We denote by F^{h} the henselization of (F, v). It can be taken to lie in M since by Lemma 2.10, (M, v) is henselian. For $\mathcal{F}$ as discussed in the introduction, we let $F^{\mathcal{F}}$ denote the smallest subfield of M which is $\mathcal{F}$-closed. Further, we let $F^{\mathrm{h}\mathcal{F}}$ denote the smallest subfield of M which is $\mathcal{F}$-closed, henselian for v and contains F. Similarly, we denote by $F^{\mathrm{r}\mathcal{F}}$ the smallest subfield of M which is real closed, $\mathcal{F}$-closed and contains F. We will say that F is **$\mathcal{F}$-closed** if $F = F^{\mathcal{F}}$, and **r$\mathcal{F}$-closed** if $F = F^{\mathrm{r}\mathcal{F}}$. Note that $F^{\mathcal{F}} \subset F^{\mathrm{h}\mathcal{F}} \subset F^{\mathrm{r}\mathcal{F}}$.

In [D–M–M1] it is shown that every real closed substructure of a model of T_{an} is itself a model of T_{an}. Hence, $F^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} \models T_{\mathrm{an}}$. In [D–M–M1] it is also shown that T_{an} is o-minimal.

3 Residue fields of $\mathcal{F}$-closures

Throughout this section, let M be a model of T_{an}, and $\mathcal{F}$ an arbitrary set of convergent power series representing restricted analytic functions, closed under partial derivations, but not necessarily containing $\mathcal{F}_{LE}$.

Lemma 3.1 *Let $F \subset M$ and w a convex valuation on M. Assume that $Fw \subset F$ is $\mathcal{F}$-closed. Assume further that the value group wF is archimedean. Then either $F^{\mathcal{F}}$ is* *embeddable in the completion of* (F,w) *(and in particular,* $wF^{\mathcal{F}} = wF$ *and* $F^{\mathcal{F}}w = Fw$*), or there is some* $y \in F^{\mathcal{F}}$*,* $y \neq 0$*, such that* $wy > wF$ *(and in particular,* $wF^{\mathcal{F}}$ *is not archimedean).*

Proof: By Zorn's Lemma, we find a maximal subfield F_0 of $F^{\mathcal{F}}$ containing F and embeddable in the completion of (F,w). Suppose that $F^{\mathcal{F}}$ is not embeddable in the completion of (F,w). Then $F_0 \neq F^{\mathcal{F}}$, that is, F_0 is not $\mathcal{F}$-closed. So let $f(X_1,\ldots,X_k) \in \mathcal{F}$ and $a = (a_1,\ldots,a_k) \in F_0^k$ with $va_i > 0$ such that $f(a) \in F^{\mathcal{F}} \setminus F_0$. We write $a_i = c_i + \varepsilon_i$ with $c_i \in F_0w = Fw$ and $w\varepsilon_i > 0$; let $c = (c_1,\ldots,c_k)$. By the Taylor expansion, the following assertions hold (they are elementary $\mathcal{L}_{\mathcal{F}}$-sentences and thus hold in the T_{an}-model M): for all $m \in \mathbb{N}$,

$$\left| f(a_1,\ldots,a_k) - \sum_{\nu=(0,\ldots,0)}^{(m,\ldots,m)} \frac{\partial^\nu f}{\partial X^\nu}(c_1,\ldots,c_k)\frac{\varepsilon^\nu}{\nu!} \right| \leq |\varepsilon_1 \cdot \ldots \cdot \varepsilon_k|^m$$

(for $\nu = (\nu_1,\ldots,\nu_k) \in \mathbb{N}^k$, $\frac{\partial^\nu f}{\partial X^\nu}$ stands for $\frac{\partial^{\nu_1}\ldots\partial^{\nu_k} f}{\partial X_1^{\nu_1}\ldots\partial X_k^{\nu_k}}$, and $\nu!$ stands for $\nu_1! \cdot \ldots \cdot \nu_k!$). By (3) it follows that for all $m \in \mathbb{N}$,

$$w\left(f(a_1,\ldots,a_k) - \sum_{\nu=(0,\ldots,0)}^{(m,\ldots,m)} \frac{\partial^\nu f}{\partial X^\nu}(c_1,\ldots,c_k)\frac{\varepsilon^\nu}{\nu!} \right) \geq m(w\varepsilon_1 + \ldots + w\varepsilon_k)\,.$$

Since wF_0 is archimedean and $w\varepsilon_i > 0$, the sequence $m(w\varepsilon_1 + \ldots + w\varepsilon_k)$, $m \in \mathbb{N}$, is cofinal in wF_0. This shows that the partial sums form a Cauchy sequence in (F_0,w), with limit $f(a)$. Note that since $\mathcal{F}$ is closed under partial derivatives and Fw is $\mathcal{F}$-closed, the coefficients $\frac{\partial^\nu f}{\partial X^\nu}(c_1,\ldots,c_k)$ lie in $Fw \subset F_0$. So the partial sums are indeed elements of F_0.

Suppose that the sequence has no limit in F_0. Then we can apply Lemma 2.8 to obtain that $F_0(f(a))$ is embeddable in the completion of (F_0,w) and hence also in the completion of (F,w). But this contradicts the maximality of F_0. Hence, there is some $b \in F_0$ which is also a limit of this sequence (observe that it is not necessarily a Cauchy sequence in (M,w)). Then by Lemma 2.5, $w(f(a)-b) > wF_0$. With $y := f(a) - b \neq 0$, we have found the desired element y which satisfies $wy > wF$. □

At this point, it might be helpful to give an example which shows that an element y as in the assertion of the above lemma can indeed exist. Take L to be any T_{an}-model with non-archimedean value group. Choose $y,t \in L$ such that $vy \gg vt > 0$. Then $vy > \mathbb{Q}vt = v\mathbb{R}(t)^{\mathrm{r}}$. It is well known that in general, $(\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} \neq \mathbb{R}(t)^{\mathrm{r}}$. Take any $a \in (\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} \setminus \mathbb{R}(t)^{\mathrm{r}}$. As $\mathbb{R}(t)^{\mathrm{r}} \subset (\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} \subset \mathbb{R}(t)^{\mathrm{r}\mathcal{F}}$, Lemma 3.2 below yields that $v(\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}} = v\mathbb{R}(t)^{\mathrm{r}}$. Further, $(\mathbb{R}(t)^{\mathrm{r}})^{\mathcal{F}}v = \mathbb{R} = \mathbb{R}(t)^{\mathrm{r}}v$. Hence also $v\mathbb{R}(t,a)^{\mathrm{r}} = v\mathbb{R}(t)^{\mathrm{r}}$ and $\mathbb{R}(t,a)^{\mathrm{r}}v = \mathbb{R}(t)^{\mathrm{r}}v$. That is, the extension $(\mathbb{R}(t,a)^{\mathrm{r}}|\mathbb{R}(t)^{\mathrm{r}},v)$ is immediate. Hence by Lemma 2.6, a is a limit of a pseudo Cauchy sequence without limit in $(\mathbb{R}(t)^{\mathrm{r}},v)$. Set $z := a + y$. Then $v(z-a) = vy > v\mathbb{R}(t)^{\mathrm{r}}$, and Lemma 2.5 shows that z is also a limit of this pseudo Cauchy sequence. Hence by Lemma 2.7, the extension $(\mathbb{R}(t)^{\mathrm{r}}(z)|\mathbb{R}(t)^{\mathrm{r}},v)$ is immediate. It follows from Lemma 2.10 that also $(\mathbb{R}(t,z)^{\mathrm{r}}|\mathbb{R}(t)^{\mathrm{r}},v)$ is immediate. On the other hand, $a \in (\mathbb{R}(t,z)^{\mathrm{r}})^{\mathcal{F}}$ and consequently, $y \in (\mathbb{R}(t,z)^{\mathrm{r}})^{\mathcal{F}}$ with $vy > v\mathbb{R}(t)^{\mathrm{r}} = v\mathbb{R}(t,z)^{\mathrm{r}}$. So $(F,w) = (\mathbb{R}(t,z)^{\mathrm{r}},v)$ is our desired example.

The following lemma is an easy consequence of the Valuation Property of restricted analytic functions (Corollary 3.7 of [D–M–M1]). Note that it only uses the case of the Valuation Property where the value group changes. The immediate case of the Valuation Property will only be needed in the proof of Lemma 5.2.

Lemma 3.2 *Assume that K is an $\mathrm{r}\mathcal{F}_{\mathrm{an}}$-closed subfield of M. Take $x_1,\ldots,x_n \in M$ such that the values $vx_1,\ldots,vx_n$ are rationally independent over vK. Then*

$$vK(x_1,\ldots,x_n)^{\mathrm{r}\mathcal{F}} = vK(x_1,\ldots,x_n)^{\mathrm{r}} = vK \oplus \bigoplus_{i=1}^{n} \mathbb{Q}vx_i \;. \tag{13}$$

In particular, if $x_1,\ldots,x_n \in M$ are such that $vx_1,\ldots,vx_n$ are rationally independent, then

$$v\mathbb{R}(x_1,\ldots,x_n)^{\mathrm{r}\mathcal{F}} = v\mathbb{R}(x_1,\ldots,x_n)^{\mathrm{r}} = \bigoplus_{i=1}^{n} \mathbb{Q}vx_i \;.$$

Proof: Suppose we have already shown (13) for $\mathcal{F} = \mathcal{F}_{\mathrm{an}}$ and some $j < n$ in the place of n. Since the vx_i are rationally independent over vK, $vx_{j+1} \notin vK \oplus \bigoplus_{i=1}^{j} \mathbb{Q}vx_i = vK(x_1,\ldots,x_j)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. In particular, $x_{j+1} \notin K(x_1,\ldots,x_j)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. Corollary 3.7 of [D–M–M1] shows that

$$\begin{aligned} vK(x_1,\ldots,x_{j+1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} &= v\left((K(x_1,\ldots,x_j)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}})(x_{j+1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}\right) \\ &= v\left((K(x_1,\ldots,x_j)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}})(x_{j+1})^{\mathrm{r}}\right) \\ &= vK(x_1,\ldots,x_j)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} \oplus \mathbb{Q}vx_{j+1} \;=\; vK \oplus \bigoplus_{i=1}^{j+1} \mathbb{Q}vx_i \;. \end{aligned}$$

By induction, we obtain equation (13) for $\mathcal{F} = \mathcal{F}_{\mathrm{an}}$ and every n. As

$$K(x_1,\ldots,x_n)^{\mathrm{r}} \subset K(x_1,\ldots,x_n)^{\mathrm{r}\mathcal{F}} \subset K(x_1,\ldots,x_n)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} \;,$$

we obtain (13) for arbitrary $\mathcal{F}$ and every n. □

Note that in the case of $\mathcal{F} \neq \mathcal{F}_{\mathrm{an}}$ it may not suffice to assume K real closed and $\mathcal{F}$-closed. Indeed, then it might not be closed under definable functions, in which case remark (3.9) of [D–M–M1] cannot be applied.

Lemma 3.3 *Let $x_i \in M$ such that the values vx_i, $i \in I$ are rationally independent. Further, let w be any convex valuation. Assume that there is a subset $I_w \subset I$ such that $wx_i = 0$ for all $i \in I_w$ and that the values wx_i, $i \in I \setminus I_w$ are rationally independent. Then*

$$w\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} = \bigoplus_{i \in I \setminus I_w} \mathbb{Q}wx_i \quad \text{and} \quad w\mathbb{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}} = \bigoplus_{i \in I \setminus I_w} \mathbb{Z}wx_i \;,$$

and

$$\begin{aligned} \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}w &= \mathbb{R}(x_i \mid i \in I_w)^{\mathrm{r}\mathcal{F}} \;=\; (\mathbb{R}(x_i \mid i \in I)w)^{\mathrm{r}\mathcal{F}} \\ \mathbb{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}}w &= \mathbb{R}(x_i \mid i \in I_w)^{\mathrm{h}\mathcal{F}} \;=\; (\mathbb{R}(x_i \mid i \in I)w)^{\mathrm{h}\mathcal{F}} \;. \end{aligned}$$

Proof: We set $L := \mathbb{R}(x_i \mid i \in I)$ and $K := \mathbb{R}(x_i \mid i \in I_w)$. By Corollary 2.4, $vL = \bigoplus_{i\in I} \mathbb{Z}vx_i$, $vK = \bigoplus_{i\in I_w} \mathbb{Z}vx_i$, $wL = \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i$ and $Lw = K$. From Lemma 3.2 we infer that $vL^{\mathrm{r}\mathcal{F}} = \bigoplus_{i\in I} \mathbb{Q}vx_i = \mathbb{Q}\otimes vL$ and that $vK^{\mathrm{r}\mathcal{F}} = \bigoplus_{i\in I_w} \mathbb{Q}vx_i = \mathbb{Q}\otimes vK$. The former implies that $wL^{\mathrm{r}\mathcal{F}} = \mathbb{Q}\otimes wL$, which is our assertion on the value groups for the r$\mathcal{F}$-closure.

We prove the assertions of our lemma for the h$\mathcal{F}$-closure. The proof for the residue field of the r$\mathcal{F}$-closure is analogous. If our assertions are not true, then there is some $b \in L^{\mathrm{h}\mathcal{F}}$ such that $wb \notin \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i$ or $bw \notin K^{\mathrm{h}\mathcal{F}}$. But b is already contained in some finitely generated subextension of $L^{\mathrm{h}\mathcal{F}}|\mathbb{R}$. This in turn is contained in some subfield $\mathbb{R}(x_1,\ldots,x_n)^{\mathrm{h}\mathcal{F}} \subset L^{\mathrm{h}\mathcal{F}}$, where $x_1,\ldots,x_n$ are suitably chosen from the x_i's. So we see that it suffices to prove our lemma in the case of a finite set $I = \{1,\ldots,n\}$.

Since vK is contained in the convex subgroup H_w associated with w, we find that also $vK^{\mathrm{r}\mathcal{F}} = \mathbb{Q}\otimes vK \subset H_w$. That is, w is trivial on $K^{\mathrm{r}\mathcal{F}}$ and thus also on $K^{\mathrm{h}\mathcal{F}}$. Therefore, $K^{\mathrm{h}\mathcal{F}} \subset L^{\mathrm{h}\mathcal{F}}w$. We will show that equality holds.

First assume that wL is archimedean. Then $wL^{\mathrm{r}\mathcal{F}} = \mathbb{Q}\otimes wL$ is archimedean, and so is $wL^{\mathrm{h}\mathcal{F}} \subset wL^{\mathrm{r}\mathcal{F}}$. Set $F := K^{\mathrm{h}\mathcal{F}}(x_i \mid i \in I\setminus I_w)$. Then $L^{\mathrm{h}\mathcal{F}} = K^{\mathrm{h}\mathcal{F}}(x_i \mid i \in I\setminus I_w)^{\mathrm{h}\mathcal{F}} = F^{\mathrm{h}\mathcal{F}}$, and by Lemma 2.3, $Fw = K^{\mathrm{h}\mathcal{F}}$ and $wF = wL$. By Zorn's Lemma, we find a maximal subfield F_0 of $F^{\mathrm{h}\mathcal{F}}$ containing F and embeddable in the completion of (F,w). Since $wF_0 = wF$ is archimedean and $F_0w = Fw$ is $\mathcal{F}$-closed, we can apply Lemma 3.1 to see that F_0 is $\mathcal{F}$-closed. From Lemma 2.9 we infer that F_0 must be equal to its henselization, i.e., it is henselian. Therefore, $F_0 = F^{\mathrm{h}\mathcal{F}} = L^{\mathrm{h}\mathcal{F}}$, showing that $w\mathbb{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}} = wL^{\mathrm{h}\mathcal{F}} = wF = \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i$ and $\mathbb{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}}w = L^{\mathrm{h}\mathcal{F}}w = Fw = K^{\mathrm{h}\mathcal{F}}$. (For the r$\mathcal{F}$-closure, one takes F_0 to be a maximal subfield of $F^{\mathrm{r}\mathcal{F}}$ containing F^{r} and embeddable in the completion of (F^{r},w), and uses Lemma 2.10 in the place of Lemma 2.9.)

Now let wL be non-archimedean. Since it is finitely generated, it has finite rank. So we can proceed by induction on the rank. Let H be the largest proper convex subgroup of wL. Since H is finitely generated, we can choose $y_1,\ldots,y_\ell \in L$ such that the values $wy_1,\ldots,wy_\ell$ form a set of rationally independent generators of H. We take w' to be a convex valuation on M whose restriction to L is the valuation associated with H. Since wL is finitely generated, we can choose $y_{\ell+1},\ldots,y_m \in L$ such that the values $w'y_i = wy_i + H$, $\ell < i \le m$, form a set of rationally independent generators of wL/H. Then $m = \ell + (m-\ell) = \dim_{\mathbb{Q}} \mathbb{Q}\otimes (wL/H) + \dim_{\mathbb{Q}} \mathbb{Q}\otimes H = \dim_{\mathbb{Q}} \mathbb{Q}\otimes wL = |I\setminus I_w| = \operatorname{trdeg} L|K$. Since the values $wy_1,\ldots,wy_m$ are rationally independent over $wK = \{0\}$, the elements $y_1,\ldots,y_m$ are algebraically independent over K. Consequently, L is algebraic over $K(y_1,\ldots,y_m)$. By our choice of the y_i, $wL = wK(y_1,\ldots,y_m)$. By Lemma 2.3, $Lw = K = K(y_1,\ldots,y_m)w$. Thus, the extension is immediate. Now Lemma 2.7 shows that $K(y_1,\ldots,y_m)^{\mathrm{h}} = L^{\mathrm{h}}$. This implies that $L^{\mathrm{h}\mathcal{F}} = K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}$, $K(y_1,\ldots,y_m)^{\mathrm{r}} = L^{\mathrm{r}}$, and $K(y_1,\ldots,y_m)^{\mathrm{r}\mathcal{F}} = L^{\mathrm{r}\mathcal{F}}$.

The rank of $wK(y_1,\ldots,y_\ell) = H$ is smaller than that of wL. Hence by induction hypothesis,

$$wK(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}} = \bigoplus_{1\le i\le \ell} \mathbb{Z}wy_i \quad \text{and} \quad K(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}}w = K^{\mathrm{h}\mathcal{F}}\,.$$

On the other hand, the value group $w'K(y_1,\ldots,y_m) = wL/H$ is archimedean since H was chosen to be the largest convex subgroup of wL. By our choice of the elements y_i,

$w'y_i = 0$ for $1 \le i \le \ell$, and the values $w'y_{\ell+1}, \ldots, w'y_m$ are rationally independent. Thus, we can replace w by w' and apply the assertion of our lemma, which is already proved in the archimedean case, to deduce that

$$w'K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{\ell<i\le m} \mathbb{Z}w'y_i \;\text{ and }\; K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w' \;=\; K(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}}\;.$$

Since the values $w'y_i = wy_i + H$, $\ell < i \le m$, are rationally independent, the values wy_i, $\ell < i \le m$, are rationally independent over $H = \bigoplus_{1\le i\le\ell} \mathbb{Z}wy_i = wK(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}}$. It follows that

$$\begin{aligned} w\mathbb{R}(x_1,\ldots,x_n)^{\mathrm{h}\mathcal{F}} &= wK(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}} \;=\; wK(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}} \oplus \bigoplus_{\ell<i\le m} \mathbb{Z}wy_i \\ &= \bigoplus_{1\le i\le\ell} \mathbb{Z}wy_i \oplus \bigoplus_{\ell<i\le m} \mathbb{Z}wy_i \;=\; \bigoplus_{1\le i\le m} \mathbb{Z}wy_i \;=\; \bigoplus_{i\in I\setminus I_w} \mathbb{Z}wx_i \end{aligned}$$

and that

$$\begin{aligned} \mathbb{R}(x_1,\ldots,x_n)^{\mathrm{h}\mathcal{F}}w &= K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w \;=\; (K(y_1,\ldots,y_m)^{\mathrm{h}\mathcal{F}}w')w \\ &= K(y_1,\ldots,y_\ell)^{\mathrm{h}\mathcal{F}}w \;=\; K^{\mathrm{h}\mathcal{F}} \;=\; \mathbb{R}(x_i \mid i \in I_w)^{\mathrm{h}\mathcal{F}}\;. \end{aligned}$$

□

Let us note that the result of this lemma remains true if the henselization with respect to v is replaced by the henselization with respect to any convex valuation. — The lemma shows in particular that if the values vx_i, $i \in I$, are rationally independent, then

$$v\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} \;=\; \bigoplus_{i\in I} \mathbb{Q}vx_i \quad\text{and}\quad v\mathbb{R}(x_i \mid i \in I)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{i\in I} \mathbb{Z}vx_i\;. \tag{14}$$

Corollary 3.4 *Let $x_i \in M$ such that the values vx_i, $i \in I$ are rationally independent. Further, let w be any convex valuation. Then there exist some index set J_w and algebraically independent elements $y_j \in \mathbb{R}(x_i \mid i \in I)$, $j \in J_w$, such that*

$$\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}w \;=\; \mathbb{R}(y_j \mid j \in J_w)^{\mathrm{r}\mathcal{F}}\;.$$

Proof: By Zorn's Lemma, choose a maximal subset $I'_w \subset I$ such that the values wx_i, $i \in I'_w$, are rationally independent. We set $J_w := I \setminus I'_w$. Then for every $j \in J_w$, there are $i_1, \ldots, i_\ell \in I'_w$ and $n, n_1, \ldots, n_\ell \in \mathbb{Z}$ such that $wy_j = 0$ for $y_j := x_j^n \cdot x_{i_1}^{n_1} \cdot \ldots \cdot x_{i_\ell}^{n_\ell}$. Then $x_j^n \in \mathbb{R}(x_i, y_j \mid i \in I'_w, j \in J_w) =: F$ and thus, $x_j \in F^{\mathrm{r}}$. Therefore, $F^{\mathrm{r}\mathcal{F}} = \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$. By Lemma 3.3, $F^{\mathrm{r}\mathcal{F}}w = \mathbb{R}(y_j \mid j \in J_w)^{\mathrm{r}\mathcal{F}}$. □

For use in Sections 5 and 6, we add the following lemma:

Lemma 3.5 *Let $x_i \in M$ such that $x_i > 0$ and the values vx_i, $i \in I$ are rationally independent. Further, let $x_i^{1/k}$ denote the unique positive real k-th root of x_i. Then*

$$\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} \;=\; \bigcup_{I_0\subset I \text{ finite}} \;\bigcup_{k\in\mathbb{N}} \mathbb{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}} \tag{15}$$

with

$$v\mathbb{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}} \;=\; \bigoplus_{i\in I_0} \mathbb{Z}\frac{vx_i}{k} \;\;,$$

a finitely generated group.

Proof: The assertion for the value group follows from (14). Let U denote the union on the right hand side of (15). Every field in the union is henselian, so U is henselian. The value group vU is divisible and the residue field $Uv = \mathbb{R}$ is real closed. Hence by Lemma 2.10, U is real closed. By construction, U is also $\mathcal{F}$-closed. Since all $x_i^{1/k}$ are in the real closed field $\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$, we find that U is contained in $\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$. Since this field is the smallest real closed and $\mathcal{F}$-closed field containing $\mathbb{R}(x_i \mid i \in I)$, it follows that $U = \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$. □

4 Closures of $\mathbb{R}(x)$ under $\mathcal{F}$, log and exp

From now on, let M always be a model of $T_{\mathrm{an,exp}}$. We take $\mathcal{F}$ as before, but always assume in addition that $\mathcal{F}_{LE} \subset \mathcal{F}$. Hence, if F is $\mathcal{F}$-closed, then $\exp\varepsilon \in F$ and $\log(1+\varepsilon) \in F$ for every infinitesimal ε in F.

Lemma 4.1 *Let K be a log- and $\mathrm{r}\mathcal{F}$-closed subfield of M, containing $\mathbb{R}$. Let w be a convex valuation of M. Assume that the residue field Kw is a subfield of $\mathcal{O}_w \cap K$, relatively exp-closed in $\mathcal{O}_w^\times$. Take any $a \in K$ such that $\exp a \notin K$. Then $w\exp a$ is rationally independent over wK.*

Proof: Suppose that $w\exp a$ is not rationally independent over wK. Since wK is divisible by Lemma 2.10, it follows that $w\exp a = wb \in wK$ for some positive $b \in K$. Then $w\frac{\exp a}{b} = 0$ and by Lemma 2.13, $w(a-\log b) = w\log(\frac{\exp a}{b}) \geq 0$. Since K is log-closed, $\log b \in K$. Hence, there is $c \in Kw$ such that $w(a-\log b - c) > 0$. By Lemma 2.12, this shows that $w\,\frac{\exp a}{b\exp c} = w\exp(a-\log b-c) = 0$. In particular, we find that $w\exp c = w\frac{\exp a}{b} = 0$, that is, $\exp c \in \mathcal{O}_w^\times$. By assumption on Kw, $\exp c \in Kw \subset K$.

By Lemma 2.2, $w(a-\log b-c) > 0$ yields that $v(a-\log b-c) > 0$. Therefore, $\exp(a-\log b-c) \in K^{\mathrm{r}\mathcal{F}} = K$, showing that $\exp a = \exp(a-\log b-c)\cdot b\cdot \exp c \in K$. We conclude: if $\exp a \notin K$, then $w\exp a$ is rationally independent over wK. □

Lemma 4.2 *Assume that $K = \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} \subset M$ such that*
1) the values vx_i , $i \in I$, are rationally independent,
2) for all $i \in I$, $x_i > 0$ and $\log x_i \in K$.
Then K is log-closed.

Proof: Take $b \in K$. There is a finite subset $I_0 \subset I$ and rational numbers $q_i \in \mathbb{Q}$ such that $vb = \sum_{i\in I_0} q_i vx_i$. So we can write $b = \prod_{i\in I_0} x_i^{q_i}\cdot r\cdot(1+\varepsilon)$ with $r \in \mathbb{R}$ and $\varepsilon \in K$ such that $v\varepsilon > 0$. We have that $\log(1+\varepsilon) \in K$ since K is $\mathcal{F}$-closed. Moreover, $\log r \in \mathbb{R} \subset K$. Therefore,

$$\log b \;=\; \sum_{i\in I_0} q_i \log x_i \;+\; \log r \;+\; \log(1+\varepsilon) \;\in\; K\;.$$

$\square$

Lemma 4.3 *Assume that* K *is of the form*

$$\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} \text{ log-closed, with } x_i > 0 \text{ and } vx_i,\ i \in I, \text{ rationally independent.} \tag{16}$$

Take any $a \in K$ *such that* $\exp a \notin K$. *Then* $v \exp a$ *is rationally independent over* vK, *and*

$$vK(\exp a)^{\mathrm{r}\mathcal{F}} \;=\; vK \oplus \mathbb{Q}\, v \exp a\;. \tag{17}$$

Moreover, $K(\exp a)^{r\mathcal{F}}$ *is again log-closed, and therefore of the form (16). It contains* $\exp b$ *whenever* $b \in K(\exp a)^{r\mathcal{F}}$ *and* $v \exp b$ *is rationally dependent over* $vK(\exp a)^{r\mathcal{F}}$.

Proof: Applying Lemma 4.1 with $w = v$ and $Kw = \mathbb{R}$, we obtain that $v \exp a$ is rationally independent over vK and that $\exp b \in K(\exp a)^{r\mathcal{F}}$ whenever $b \in K(\exp a)^{r\mathcal{F}}$ and $v \exp b$ is rationally dependent over $vK(\exp a)^{r\mathcal{F}}$. Equation (17) follows from Lemma 3.3 with $w = v$ and $I_w = \emptyset$. We infer from Lemma 4.2 that $K(\exp a)^{r\mathcal{F}}$ is log-closed. $\square$

Next, we show how to construct such log-closed fields K. **From now on, we always assume that $x \in M$ is a positive infinite element, i.e., $x > 0$ and $vx < 0$.**

Lemma 4.4 *The field*

$$\mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$$

is log-closed. The convex hull of its value group in vM *is equal to the smallest convex subgroup containing* vx. *If* w *is a convex valuation such that* $wx = 0$, *then the field* $\mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$ *lies in* $\mathcal{O}_w$.

Proof: From Lemma 2.13 we know that

$$vx \ll v \log x \ll \ldots \ll v \log^m x \ll \ldots < 0\;. \tag{18}$$

Hence, the values $v \log^m x$ are rationally independent. So it follows from Lemma 4.2 that $\mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$ is log-closed.

From Lemma 3.3 we infer that $v\mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}} = \bigoplus_{m\geq 0} \mathbb{Q}\, v \log^m x$. Now (18) yields that this group is contained in the smallest convex subgroup H of vM which contains vx. If w is a convex valuation such that $wx = 0$, then H is contained in the convex subgroup H_w associated with w. Thus, w is trivial on $\mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$, that is, this field lies in $\mathcal{O}_w$. $\square$

Next, we build up $LE_{\mathcal{F}}(x)$ and its residue fields. Let w be a convex valuation on M and H_w its associated convex subgroup of vM. Further, let $K_0^w \subset \mathcal{O}_w$ be any field of the form (16). For example, if $wx = 0$, then we can take $K_0^w = \mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$.

Now we construct K_1^w as follows. Assume that $a \in K_0^w$ such that $\exp a \notin K_0^w$, but $v \exp a \in H_w$. Then by Lemma 4.3, $K_0^w(\exp a)^{\mathrm{r}\mathcal{F}}$ is again of the form (16), with $vK_0^w(\exp a)^{\mathrm{r}\mathcal{F}} = vK \oplus \mathbb{Q}\, v \exp a \subset H_w$. The latter shows that it is again a subfield of $\mathcal{O}_w$. We repeat this procedure until we arrive at a field $K_1^w \subset \mathcal{O}_w$ of the form (16), which contains $\exp a$ for every $a \in K_0^w$ such that $\exp a \in \mathcal{O}_w^\times$. Then we construct K_2^w from K_1^w in the same way as we constructed K_1^w from K_0^w. We iterate to obtain fields $K_n^w \subset \mathcal{O}_w$, of the form (16). Their union

$$K_\infty^w := \bigcup_{n\in\mathbb{N}} K_n^w \subset \mathcal{O}_w$$

is r$\mathcal{F}$-closed and of the form (16). It is also relatively exp-closed in $\mathcal{O}_w^\times$. To see this, let $a \in K_\infty^w$. Then $a \in K_n^w$ for some n. If $\exp a \in \mathcal{O}_w^\times$, then by construction, $\exp a \in K_{n+1}^w$. On the other hand, every other log- and r$\mathcal{F}$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing K_0^w, must also contain K_∞^w. This proves:

Lemma 4.5 *K_∞^w is the uniquely determined smallest log- and r$\mathcal{F}$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing K_0^w. It is of the form (16).*

We derive some further information from our construction. Take $n \in \mathbb{N}$.

Lemma 4.6 *If $a \in K_n^w$ with $va < 0$, $a > 0$, then $v \log a \in vK_{n-1}^w$ and $v \log^n a \in vK_0^w$.*

Proof: By the construction of K_n^w from K_{n-1}^w, there are elements $a_j \in K_{n-1}^w$, $j \in J$, such that $vK_n^w = vK_{n-1}^w \oplus \bigoplus_{j\in J} \mathbb{Q} v \exp(a_j)$. Hence, $a \in K_n^w$ can be written as

$$a = \prod_{j\in J_0} \exp(a_j)^{q_j} \cdot c \cdot r \cdot (1+\varepsilon)$$

with J_0 a finite subset of J, $q_j \in \mathbb{Q}$, $c \in K_{n-1}^w$, $r \in \mathbb{R}$ and $\varepsilon \in K_n^w$ with $v\varepsilon > 0$. Then $\log a = \sum_{j\in J_0} q_j a_j + \log c + \log r + \log(1+\varepsilon)$. Since $v \log a < 0$ by Lemma 2.13, but $v \log(1+\varepsilon) > 0$, we find that $v \log a = v(\sum_{j\in J_0} q_j a_j + \log c + \log r) \in vK_{n-1}^w$. By induction it follows that $v \log^n a \in vK_0^w$. □

If $wx = 0$ and we start our construction from $K_0^w = \mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$, then K_∞^w will be the uniquely determined smallest log- and r$\mathcal{F}$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}_w^\times$ and containing $\mathbb{R}(x)$. We denote it by $LE_\mathcal{F}^w(x)$.

If we choose u to be the trivial valuation on M, then $\mathcal{O}_u = M$ and $H_u = vM$. In this case, $LE_\mathcal{F}^u(x)$ is exp-closed and contains x. Therefore,

$$LE_\mathcal{F}^u(x) = LE_\mathcal{F}(x) .$$

This proves:

Theorem 4.7 *$LE_\mathcal{F}(x)$ is of the form (16). The elements x_i can be chosen so as to include x and $\log^m x$ for all $m \in \mathbb{N}$.*

Assume that $wx = 0$. By our construction, $LE^w_{\mathcal{F}}(x) \subset LE_{\mathcal{F}}(x)$. So we can rerun our construction of $LE^u_{\mathcal{F}}(x) = LE_{\mathcal{F}}(x)$ starting with $K^u_0 = LE^w_{\mathcal{F}}(x)$. We note that K^u_0 is of the form $\mathbb{R}(x_i \mid i \in I_w)^{\mathrm{r}\mathcal{F}}$, where the x_i , $i \in I_w$, are obtained from the above construction (and thus, their values $vx_i \in H_w$ are rationally independent). Since $K^u_0 \subset \mathcal{O}_w$, we have that $K^u_0 w = K^u_0$ and that $wx_i = 0$ for $i \in I_w$. Suppose that while building up $LE^u_{\mathcal{F}}(x)$ from this field by the above construction, we have reached a field K of the form $\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}}$ with $Kw = K^u_0$, $I_w \subset I$ and such that the values wx_i , $i \in I \setminus I_w$, are rationally independent. If $a \in K$, but $\exp a \notin K$, then Lemma 4.1 shows that $w \exp a$ is rationally independent over wK. Therefore, the values $w \exp a, wx_i$, $i \in I \setminus I_w$, are rationally independent, and Lemma 3.3 shows that

$$K(\exp a)^{\mathrm{r}\mathcal{F}} w \;=\; \mathbb{R}(\exp a, x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} w \;=\; \mathbb{R}(x_i \mid i \in I_w)^{\mathrm{r}\mathcal{F}} \;=\; K^u_0 \;.$$

Hence, $K(\exp a)^{\mathrm{r}\mathcal{F}}$ is again of the same form as K. By induction, it follows that

$$LE_{\mathcal{F}}(x)w \;=\; LE^w_{\mathcal{F}}(x) \quad \text{if } wx = 0.$$

In our above considerations, the only assumption on x was that it is a positive infinite element. So we can well replace it by $\log^{m_0} x$, for arbitrary $m_0 \in \mathbb{N}$. Note that $LE_{\mathcal{F}}(x) = LE_{\mathcal{F}}(\log^{m_0} x)$. If $w \log^{m_0} x = 0$, then we find that

$$LE_{\mathcal{F}}(x)w \;=\; LE^w_{\mathcal{F}}(\log^{m_0} x) \;. \tag{19}$$

Theorem 4.8 *Let w be an arbitrary convex valuation of $LE_{\mathcal{F}}(x)$, different from the natural valuation. Then there is an integer $m_0 \geq 0$ such that $w \log^{m_0} x = 0$. With every such m_0 , equation (19) holds. If $wx = 0$, then we can choose $m_0 = 0$.*

Proof: Starting our above construction from $K^u_0 = \mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$, we can write $LE_{\mathcal{F}}(x) = \bigcup_{n\in\mathbb{N}} K^u_n$. Take any convex valuation w which does not coincide with the natural valuation on $LE_{\mathcal{F}}(x)$. Further, take any negative element α in its associated convex subgroup H_w . Then there is some $n \in \mathbb{N}$ and a positive $a \in K^u_n$ such that $\alpha = va$. By Lemma 4.6, $v \log^n a \in K^u_0$. Lemma 2.13 tells us that $va < v \log^n a < 0$. On the other hand, the values $v \log^m x$ are not bounded away from 0 in $v\mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$. So there is some m_0 such that $v \log^n a < v \log^{m_0} x < 0$. Thus, $\alpha < v \log^{m_0} x < 0$, which yields that $v \log^{m_0} x \in H_w$. That is, $w \log^{m_0} x = 0$, and equation (19) holds. □

From this theorem together with the uniqueness of $LE^w_{\mathcal{F}}(x)$ (which also works with $\log^{m_0} x$ in the place of x), we obtain Theorem 1.1. Further, if $\mathcal{F}_1 \subset \mathcal{F}_2$, then the smallest log- and $\mathrm{r}\mathcal{F}_1$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}^\times_w$ and containing $\mathbb{R}(x)$, is contained in the smallest log- and $\mathrm{r}\mathcal{F}_2$-closed subfield of $\mathcal{O}_w$, relatively exp-closed in $\mathcal{O}^\times_w$ and containing $\mathbb{R}(x)$. Hence:

Corollary 4.9 *Suppose that $\mathcal{F}_1 \subset \mathcal{F}_2$ are sets of convergent power series, closed under partial derivations and containing $\mathcal{F}_{LE}$. Then for every convex valuation w such that $wx = 0$,*

$$LE^w_{\mathcal{F}_1}(x) \;\subset\; LE^w_{\mathcal{F}_2}(x) \;.$$

Finally, let us apply Lemma 4.6 to show that $LE_{\mathcal{F}}(x)$ has exponential rank 1, in the sense of [K–K2]:

Lemma 4.10 *The sequence* $\exp^m x$, $m \geq 0$, *is cofinal in* $LE_{\mathcal{F}}(x)$.

Proof: Let $a \in LE_{\mathcal{F}}(x)$ be positive infinite. From Lemma 4.6 we infer that $v \log^n a \in v\mathbb{R}(\log^m x \mid m \geq 0)^{\mathrm{r}\mathcal{F}}$ for some $n \in \mathbb{N}$. By Lemma 4.4, every element $\alpha < 0$ in this value group is either archimedean equivalent to vx, or satisfies $vx \ll \alpha < 0$. Since $v \log^n a \ll v \log^{n+1} a < 0$ by Lemma 2.13, it follows that $vx \ll v \log^{n+1} a < 0$. Hence by (1), $x > \log^{n+1} a$ and therefore, $\exp^{n+1} x > a$. □

5 Applications

In this section we show how our approach can be used to deduce the applications which van den Dries, Macintyre and Marker give in their paper [D–M–M2].

We take $M = H(\mathbb{R}_{\mathrm{an,exp}})$ and x to be the germ of the identity function. Recall that this choice yields that $H(\mathbb{R}_{\mathrm{an,exp}}) = LE_{\mathcal{F}_{\mathrm{an}}}(x)$ and $LE = LE_{\mathcal{F}_{LE}}(x)$.

We deduce Corollary 2.10 of [D–M–M2]:

Corollary 5.1 *If* $f : \mathbb{R} \to \mathbb{R}$ *is definable in* $\mathbb{R}_{\mathrm{an,exp}}$, *then there are* $c \in \mathbb{R}$ *and* $n \in \mathbb{N}$ *such that* $f(z) < \exp^n(z)$ *for all* $z > c$.

Proof: Let $f \in H(\mathbb{R}_{\mathrm{an,exp}})$ denote the germ of the function $f(z)$. By Lemma 4.10, there is some $n \in \mathbb{N}$ such that $f < \exp^n x$ (as elements in the ordered field $H(\mathbb{R}_{\mathrm{an,exp}})$). Since this says that the germ of $\exp^n z$ is bigger than that of $f(z)$, it follows that $f(z) < \exp^n(z)$ for all large enough $z \in \mathbb{R}$. □

From now on, we will not any more distinguish the variable x from the germ x of the identity function. Note that if f is definable in $\mathbb{R}_{\mathrm{an,exp}}$ and $g \in H(\mathbb{R}_{\mathrm{an,exp}})$ is the germ of the function $g(x)$, then the element $f(g) \in H(\mathbb{R}_{\mathrm{an,exp}})$ is defined to be the germ of the function $f(g(x))$; in this way, f is made into a function on $H(\mathbb{R}_{\mathrm{an,exp}})$. In particular, the element $f(x) \in H(\mathbb{R}_{\mathrm{an,exp}})$ is the germ of the function $f(x)$.

5.1 The Hardy problem

Take two functions $f, g : \mathbb{R} \to \mathbb{R}$, definable in $\mathbb{R}_{\mathrm{an,exp}}$. Assume that $\exp f(x)$ is asymptotic to $g(x)$, that is, $\lim_{x\to\infty} \frac{\exp f(x)}{g(x)} = 1$. This is equivalent to $\lim_{x\to\infty} f(x) - h(x) = 0$, where $h : (r, \infty) \to \mathbb{R}$ for suitable $r \in \mathbb{R}$ is the function $\log g(x)$, which again is definable in $\mathbb{R}_{\mathrm{an,exp}}$. This means that the function $f(x) - h(x)$ is ultimately smaller than every nonzero constant function. Equivalently, its germ $f - h$ in $H(\mathbb{R}_{\mathrm{an,exp}})$ is infinitesimal, or in other words, $v(f - h) > 0$.

As in [D–M–M2], let the function $i(x)$ denote the compositional inverse of the function $x \log x$. Identifying $i(x)$ with its germ, we have that $i(x) \in H(\mathbb{R}_{\mathrm{an,exp}})$. But by an argument about Liouville extensions of the Hardy field $\mathbb{R}(x)$, Corollary 4.6 of [D–M–M2] shows that $i(x) \notin LE$. Assume that $\exp i(x)$ were asymptotic to a function $g(x)$ which is a composition of semialgebraic functions, exp and log. Through identification with its germ, the latter means that $g(x) \in LE$. Then also $h(x) := \log g(x) \in LE$, and $v(i(x) - h(x)) > 0$. Further, one shows as in [D–M–M2] that there is a convergent power series $f(X, Y)$ such that

$$i(x) \;=\; \frac{x}{\log x}\left(1 + f\left(\frac{\log\log x}{\log x}\,,\,\frac{1}{\log x}\right)\right)\;.$$

Now let w be the convex valuation corresponding to the largest convex subgroup not containing vx. It contains $v\log x$. We have that $wx^{-1} = -wx > 0$. Further,

$$v\left(\frac{i(x)}{x} - \frac{h(x)}{x}\right) \;>\; vx^{-1}\;.$$

By Lemma 2.2 it follows that

$$w\left(\frac{i(x)}{x} - \frac{h(x)}{x}\right) \;>\; wx^{-1} \;>\; 0\;. \tag{20}$$

By Corollary 4.9, $LE^w_{\mathcal{F}_{LE}}(\log x) \subset LE^w_{\mathcal{F}_{\rm an}}(\log x)$. Hence, (20) yields that the w-residues of $\frac{i(x)}{x}$ and $\frac{h(x)}{x}$ in $LE^w_{\mathcal{F}_{\rm an}}(\log x)$ are equal. On the other hand,

$$\frac{i(x)}{x} \;=\; \frac{1}{\log x}\left(1 + f\left(\frac{\log\log x}{\log x}\,,\,\frac{1}{\log x}\right)\right) \in \mathbb{R}(\log x, \log\log x)^{r\mathcal{F}_{\rm an}} \;\subset\; LE^w_{\mathcal{F}_{\rm an}}(\log x)\;,$$

showing that the w-residue of $\frac{i(x)}{x}$ in $LE^w_{\mathcal{F}_{\rm an}}(\log x)$ is $\frac{i(x)}{x}$. Since it is also the w-residue of $\frac{h(x)}{x}$, we see that

$$\frac{i(x)}{x} \;\in\; LE^w_{\mathcal{F}_{LE}}(\log x) \;\subset\; LE\;.$$

But then also $i(x) \in LE$, a contradiction. This proves that $\exp i(x)$ is not asymptotic to any function with germ in LE.

5.2 Undefinable functions

We choose a representation $H(\mathbb{R}_{\rm an,exp}) = \mathbb{R}(x_i \mid i \in I)^{\rm r\mathcal{F}_{\rm an}}$ with vx_i, $i \in I$, rationally independent, which exists by Theorem 4.7. For the applications, we will assume in addition that x is among the x_i.

Lemma 5.2 *Take any positive infinitesimal element t in $H(\mathbb{R}_{\rm an,exp})$. Suppose that the element $h \in H(\mathbb{R}_{\rm an,exp})$ satisfies*

$$\left| h \;-\; \sum_{n=0}^{m} r_n t^n \right| \;<\; r'_m t^m \quad \textit{for all } m \in \mathbb{N},$$

where $r_n, r'_n \in \mathbb{R}$, $r'_n > 0$. Then $\sum_{n=0}^{\infty} r_n X^n$ converges in $\mathbb{R}$ near 0.

Proof: By Lemma 2.11, h is a limit of the pseudo Cauchy sequence formed by the partial sums $S_m = \sum_{n=0}^m r_n t^n$. For simplicity, we assume that $r_n \neq 0$ for all n. Otherwise, we have to define k_n to be the n-th index k for which $r_k \neq 0$; then we apply Lemma 2.11 with $z_n = t^{k_n}$ to obtain that the partial sums $S_m = \sum_{n=0}^m r_{k_n} t^{k_n}$ form a pseudo Cauchy sequence with limit h. Note that in all cases, $vh = vS_0 = vt^{n_0} = n_0 vt$ for some $n_0 \in \mathbb{N}$.

Let H be the convex subgroup of $vH(\mathbb{R}_{\mathrm{an,exp}})$ generated by vt, and w the convex valuation associated with H. Then $wt = 0$ and $wh = 0$. By Theorem 1.1, we can choose an element $h_w \in LE^w_{\mathcal{F}_{\mathrm{an}}}(t^{-1})$ such that $w(h - h_w) > 0$. That is, $v(h - h_w) > vt^{m+1} = v(S_{m+1} - S_m)$ for all m. So Lemma 2.5 shows that h_w is a limit of $(S_m)_{m\geq 0}$, too. Since $v\mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{Q}vt$ (cf. Lemma 3.2), this is a Cauchy sequence in $(\mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}, v)$. Since $\mathbb{Q}vt$ is cofinal in $vLE^w_{\mathcal{F}_{\mathrm{an}}}(t^{-1})$ by our choice of H, it is also a Cauchy sequence in $(LE^w_{\mathcal{F}_{\mathrm{an}}}(t^{-1}), v)$. Hence, h_w is the only limit that the sequence admits in this field. If $h_w \in \mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$, then trivially, $v\mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}(h_w) = \mathbb{Q}vt$. Otherwise, this follows by Lemma 2.7. Thus by Corollary 3.7 of [D–M–M1], $v\mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{Q}vt$.

In view of (14), we can write $vt = \sum_{i\in I} q_i vx_i$ with $q_i \in \mathbb{Q}$, only finitely many of them nonzero. Take $i_0 \in I$ with $q_{i_0} \neq 0$. Then by the rational independence of the values vx_i ,

$$vt \notin \sum_{i\in I\setminus\{i_0\}} \mathbb{Q}vx_i \;=\; v\mathbb{R}(x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} \;.$$

So $t \notin \mathbb{R}(x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. An application of the Exchange Lemma for o-minimal theories ([P–S]) to this model of T_{an} shows that $x_{i_0} \in \mathbb{R}(t, x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. Hence, $H(\mathbb{R}_{\mathrm{an,exp}}) = \mathbb{R}(t, x_i \mid i \in I \setminus \{i_0\})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. Moreover, the values vt, vx_i , $i \in I \setminus \{i_0\}$, are rationally independent. Now choose $\{x_1, \ldots, x_\ell\} \subset \{x_i \mid i \in I \setminus \{i_0\}\}$ with ℓ minimal such that $h_w \in \mathbb{R}(t, x_1, \ldots, x_\ell)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. Suppose that $\ell > 0$. Because of the minimality of ℓ, it follows from the Exchange Lemma that $x_\ell \in \mathbb{R}(h_w, t, x_1, \ldots, x_{\ell-1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}(x_1, \ldots, x_{\ell-1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. By Lemma 3.2 and what we have shown for $\mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$, we know that $v\mathbb{R}(h_w, t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}(x_1, \ldots, x_{\ell-1})^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} = \mathbb{Q}vt \oplus \mathbb{Q}vx_1 \oplus \ldots \oplus \mathbb{Q}vx_{\ell-1}$. But this group does not contain vx_ℓ . This contradiction shows that $\ell = 0$, i.e., $h_w \in \mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$.

Now let $\mathbb{R}\langle t\rangle$ denote the set of convergent Puiseux series in t, that is, the subset of the completion of $\mathbb{R}(t)^r$ consisting of all series $\sum_{n=n_0}^\infty r_n t^{n/k}$, where $n_0 \in \mathbb{Z}$, $k \in \mathbb{N}$, $r_n \in \mathbb{R}$, and $\sum_{n=0}^\infty r_n X^n$ converges near 0. Then $\mathbb{R}\langle t\rangle$ is a real closed field such that it $f(X_1, \ldots, X_m)$ is a power series over $\mathbb{R}$ converging near 0 and $\varepsilon_1, \ldots, \varepsilon_m$ are infinitesimals in $\mathbb{R}\langle t\rangle$, then $f(\varepsilon_1, \ldots, \varepsilon_m) \in \mathbb{R}\langle t\rangle$. This shows that $\mathbb{R}\langle t\rangle$ is r$\mathcal{F}_{\mathrm{an}}$-closed. By its definition it is clear that the r$\mathcal{F}_{\mathrm{an}}$-closure of $\mathbb{R}(t)$ in $\mathbb{R}\langle t\rangle$ must be equal to $\mathbb{R}\langle t\rangle$. By induction along the lines of the proof of Lemma 3.1, one shows that there is a unique isomorphism $\mathbb{R}(t)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}} \simeq \mathbb{R}\langle t\rangle$ (of valued fields) which is the identity on $\mathbb{R}(t)$. Since h_w is the limit of the Cauchy sequence $(S_m)_{m\geq 0}$, this isomorphism sends h_w to the unique limit $\sum_{n=0}^\infty r_n t^n$, which consequently must lie in $\mathbb{R}\langle t\rangle$. By definition of $\mathbb{R}\langle t\rangle$, $\sum_{n=0}^\infty r_n X^n$ must be convergent near 0. □

If a definable function $f : (r, +\infty) \to \mathbb{R}$ has an asymptotic expansion $f(x) \sim \sum r_n f_n(x)$ in the sense of [D–M–M2], then for some $C \in \mathbb{R}$, $C > 0$,

$$\left| f(x) \;-\; \sum_{n=0}^m r_n f_n(x) \right| \;<\; C f_m(x)$$

holds in $H(\mathbb{R}_{\rm an,exp})$ for all $m\in\mathbb{N}$. Hence if $f(x)\sim\sum r_nx^{-n}$, then with $t:=x^{-1}$, it follows from the foregoing lemma that $\sum_{n=0}^m r_nX^n$ is a convergent series. Using the asymptotic expansions as given in [D–M–M2], it follows that the Gamma-function and the functions

$$\int_0^x e^{-t^2}dt\,,\;\;\int_x^\infty \frac{e^{-t}}{t}\,dt\,,\;\;\int_0^\infty \frac{e^{-t}}{t+x}\,dt\,,\;\;\int_0^\infty \frac{e^{-t}}{1+xt}\,dt\,,\;\;\int_0^x e^{t^2}dt\,,\;\;\int_0^x e^{e^t}dt$$

on $(0,+\infty)$ are not definable in $\mathbb{R}_{\rm an,exp}$.

Lemma 5.3 *Suppose that the element $h\in H(\mathbb{R}_{\rm an,exp})$ satisfies*

$$\left|h-\sum_{n=0}^m d_n\right| < r'_m d_m \quad \textit{for all } m\in\mathbb{N}, \tag{21}$$

where $0<r'_n\in\mathbb{R}$, and the d_n are positive monomials such that the values vd_n are strictly increasing. Then these values are contained in a finitely generated subgroup of $vH(\mathbb{R}_{\rm an,exp})$.

Proof: From Lemma 3.5 we infer that $h\in\mathbb{R}(x_i^{1/k}\mid i\in I_0)^{\rm h\mathcal{F}_{an}}=:K$ for some $k\in\mathbb{N}$ and some finite subset $I_0\subset I$, and that the value group of this field is the finitely generated subgroup $vK=\bigoplus_{i\in I_0}\mathbb{Z}\frac{vx_i}{k}$ of $vH(\mathbb{R}_{\rm an,exp})$. From the rational independence of the values vx_i it follows for every monomial d that $vd\in vK$ if and only if $d\in K$.

Suppose that $vd_n\notin vK$ for some $n\in\mathbb{N}$, and take n to be the smallest integer with this property. Then $d_j\in K$ for $1\le j<n$. Consequently, $h-S_{n-1}\in K$. But by Lemma 2.11, $v(h-S_{n-1})=vd_n\notin vK$, a contradiction. □

For the application to the Riemann zeta function, we run our construction of $LE_{\mathcal{F}_{\rm an}}(x)$ with a slight refinement. We choose a $\mathbb{Q}$-basis $\mathcal{B}$ of $\mathbb{R}$ containing the $\mathbb{Q}$-linearly independent elements $\log p$, where p runs through all primes. Starting our construction from $K_0^u=\mathbb{R}(\log^m x\mid m\ge m_0)^{\rm r\mathcal{F}_{an}}$, we may first adjoin all elements $\exp(rx)$ as new x_i's. Indeed, Lemma 2.14 shows that the values $v\exp(rx)$, $r\in\mathcal{B}$, are rationally independent over vK_0^u; therefore for all $s\in\mathcal{B}$, $\exp(sx)\notin K_0^u(\exp(rx)\mid r\in\mathcal{B}\setminus\{s\})$. So we can assume the elements $\exp(x\log p)$ to be among the x_i .

The restriction ζ of the zeta function to $(1,+\infty)$ has the asymptotic expansion $\zeta(x)\sim\sum\exp(-x\log n)$. Writing $n=\prod_{p\,\rm prime}p^{\nu_p}$ with integers $\nu_p\ge 0$, we obtain that

$$\exp(-x\log n)=\exp(-x\log\prod_{p\,\rm prime}p^{\nu_p})=\exp(-x\sum_{p\,\rm prime}\nu_p\log p)=\prod_{p\,\rm prime}(\exp(x\log p))^{-\nu_p}\,,$$

which is a monomial. If ζ were definable in $\mathbb{R}_{\rm an,exp}$, it would thus follow from the foregoing lemma that the values $v\exp(-x\log n)$, and in particular the values $v\exp(-x\log p)$, p prime, lie in a finitely generated group. But this is not the case since the latter values are rationally independent. This proves that the restriction of the zeta function to $(1,+\infty)$ is not definable in $\mathbb{R}_{\rm an,exp}$.

6 How small is $LE_{\mathcal{F}}(x)$?

Throughout this section, we work with a representation

$$LE_{\mathcal{F}}(x) \;=\; \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} \quad \text{with } \; vx_i,\, i \in I, \text{ rationally independent,} \tag{22}$$

which exists by Theorem 4.7. Note that it follows from our construction that for any two convex valuations w, w',

$$LE_{\mathcal{F}}(x)w \;=\; LE_{\mathcal{F}}^{w}(x) \;\subset\; LE_{\mathcal{F}}^{w'}(x) \;=\; LE_{\mathcal{F}}(x)w' \quad \text{if } \; w \text{ is finer than } w'. \tag{23}$$

We note in advance that Lemmas 6.1 and 6.2 as well as Theorem 6.4 do not require any further assumptions on the x_i (not even that x is among them). Similarly, we only need (23), without any further information on the structure induced on the residue fields. So the results will hold for every field of the form (22), since for every real closed field, one can construct embeddings of the residue fields such that (23) holds. This can be done by transfinite induction on an arbitrary enumeration of the elements of the field. However, it should not be overlooked that Corollaries 6.5 and 6.6 and the application to the Hardy problem need additional assumptions on the x_i and the residue fields.

Let (K, w) be a valued field and $a \in K$. We write $a =_w \sum_{n=0}^{\infty} a_n$ if all $a_n \in K$ and
– either there is some m_0 such that $a = \sum_{n=0}^{m_0} a_n$, and $a_n = 0$ for all $n > m_0$,
– or the values wa_n form a strictly monotone cofinal sequence in wK and a is the (unique) limit of the Cauchy sequence $(\sum_{n=0}^{m} a_n)_{m\geq 0}$ in (K, w).

Lemma 6.1 *Let (K, w) be a valued field and assume that $Kw \subset K$. Further, let wK be archimedean. Suppose that $K = Kw(z_j \mid j \in J)$, where the values wz_j , $j \in J$, are rationally independent. Let*

$$\mathcal{D} := \{ \prod_{j\in J_0} z_j^{n_j} \mid \; J_0 \subset J \textit{ finite and } n_j \in \mathbb{Z} \textit{ for every } j \in J_0 \}.$$

Take any $a \in K$. Then there are uniquely determined elements $c_n \in Kw$ and $d_n \in \mathcal{D}$ such that

$$a \;=_w\; \sum_{n=0}^{\infty} c_n d_n \;.$$

The same holds for every element a in the henselization or the completion of (K, w).

Proof: Let R denote the subring of K consisting of all finite sums $c_1d_1 + \ldots + c_md_m$ with $c_i \in Kw$ and $d_i \in \mathcal{D}$. We show that R is w-dense in K, that is, for every $a \in K$ and every $\alpha \in wK$ there is $a' \in R$ such that $w(a - a') > \alpha$. From the rational independence of the values wz_j it follows that every two distinct elements $d, d' \in \mathcal{D}$ have distinct values. On the other hand, every $a \in K$ can be written as a quotient of two polynomials in finitely many of the z_j , and therefore also as a quotient $\frac{b_1d_1+\ldots+b_md_m}{b'_1d'_1+\ldots+b'_\ell d'_\ell}$ where $d_1, \ldots, d_m \in \mathcal{D}$ are distinct and $d'_1, \ldots, d'_\ell \in \mathcal{D}$ are distinct, and $b_i, b'_i \in Kw$. We may assume that $b'_1d'_1$ is the summand of least value in the denominator. We write

$$b'_1d'_1 + \ldots + b'_\ell d'_\ell \;=\; b'_1d'_1(1 + d') \quad \text{with } \; d' := \frac{b'_2d'_2}{b'_1d'_1} + \ldots + \frac{b'_\ell d'_\ell}{b'_1d'_1} \;.$$

Note that $\frac{d'_2}{d'_1}, \ldots, \frac{d'_\ell}{d'_1}$ are elements of $\mathcal{D}$ of positive value. Hence, also $wd' > 0$. By the geometric expansion,

$$w\left(\frac{1}{1+d'} - \sum_{i=0}^{k}(-d')^i\right) \;=\; (k+1)wd'$$

for every integer $k \geq 1$. Take $\alpha \in wK$. Since wK is archimedean, we can choose k as big as to obtain that $(k+1)wd' > \alpha - w(b_1d_1 + \ldots + b_md_m)(b'_1d'_1)^{-1}$. For

$$a' := (b_1d_1 + \ldots + b_md_m)(b'_1d'_1)^{-1}\sum_{i=0}^{k}(-d')^i \in R\;,$$

this yields that $w(a - a') > \alpha$.

Every valued field (K, w) is w-dense in its completion (by definition). Since wK is archimedean, then the henselization of (K, w) lies in the completion and thus, (K, w) is also w-dense in its henselization. Since density is transitive, we find that R is also w-dense in the henselization and in the completion of (K, w).

Every element of the ring R can be written as a sum $c_1d_1 + \ldots + c_md_m$ with distinct $d_i \in \mathcal{D}$, and such that $wd_1 < wd_2 < \ldots < wd_m$. Its value is equal to wd_1. Therefore, such a sum can only be equal to 0 if it is trivial. Consequently, the representation of every element as a sum of this form is uniquely determined.

Now we choose $\alpha \in wK$, $\alpha > 0$. Then the sequence $k\alpha$, $k > 0$, is cofinal in the archimedean group wK. For every k, we choose $a_k \in R$ such that $w(a - a_k) > k\alpha$. For $k' > k > 0$, $w(a_{k'} - a_k) \geq \min\{w(a - a_{k'}), w(a - a_k)\} > k\alpha$. Thus, the summands of value $\leq k\alpha$ in the representations of a_k and $a_{k'}$ have to be the same. So we take c_nd_n to be the uniquely determined n-th summand appearing in the representation of all a_m, for m large enough. Since distinct elements of $\mathcal{D}$ have distinct values, d_n and thus also c_n is uniquely determined from the element c_nd_n. □

Lemma 6.2 *Take $h \in LE_{\mathcal{F}}(x)$. Then there are convex valuations w, w' such that:*
a) the value group of $(LE_{\mathcal{F}}(x)w', w)$ is archimedean,
b) $h \in LE_{\mathcal{F}}(x)w' \setminus LE_{\mathcal{F}}(x)w$,
c) there are monomials $d_n \in LE_{\mathcal{F}}(x)w'$ and elements $c_n \in LE_{\mathcal{F}}(x)w$ such that in $(LE_{\mathcal{F}}(x)w', w)$,

$$h \;=_w\; \sum_{n=0}^{\infty} c_nd_n\;, \tag{24}$$

d) the summands c_nd_n are uniquely determined,
e) the values vc_nd_n lie in a finitely generated subgroup of $vLE_{\mathcal{F}}(x)$.

Proof: From Lemma 3.5 we infer that $h \in \mathbb{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}} =: K$ for some $k \in \mathbb{N}$ and some finite subset $I_0 \subset I$. Since vK is finitely generated, it has finite rank. That is, there are only finitely many distinct convex valuations on K. Therefore, there are convex valuations w'_0, w_0 on $LE_{\mathcal{F}}(x)$ such that the value group $w_0(Kw'_0)$ is archimedean and $h \in LE_{\mathcal{F}}(x)w'_0 \setminus LE_{\mathcal{F}}(x)w_0$.

Every element in vK is the value of a monomial built up from the elements x_i, $i \in I_0$. Hence, we can choose monomials $z_1, \ldots, z_\ell \in K$ such that:

• $vz_1, \ldots, vz_{\ell_1}$ form a set of rationally independent generators of $v(Kw_0)$,
• $w_0 z_{\ell_1+1} = vz_{\ell_1+1} + v(Kw_0), \ldots, w_0 z_{\ell_2} = vz_{\ell_2} + v(Kw_0)$ form a set of rationally independent generators of $w_0(Kw_0')$, and
• $w_0' z_{\ell_2+1} = vz_{\ell_2+1} + v(Kw_0'), \ldots, w_0' z_\ell = vz_\ell + v(Kw_0')$ form a set of rationally independent generators of $w_0' K$.

Similarly as in the proof of Lemma 3.3, one finds that $\ell = |I_0|$ and that $\mathbb{R}(z_1, \ldots, z_\ell)^{\mathrm{h}\mathcal{F}} = \mathbb{R}(x_i^{1/k} \mid i \in I_0)^{\mathrm{h}\mathcal{F}}$. From Lemma 3.3 it follows that $Kw_0 = \mathbb{R}(z_1, \ldots, z_{\ell_1})^{\mathrm{h}\mathcal{F}}$ and $Kw_0' = \mathbb{R}(z_1, \ldots, z_{\ell_2})^{\mathrm{h}\mathcal{F}}$. Now we have that $Kw_0 \subset Kw_0' \subset K$. Since $Kw_0 = K \cap LE_{\mathcal{F}}(x)w_0$ and $Kw_0' = K \cap LE_{\mathcal{F}}(x)w_0'$, we obtain that $h \in Kw_0' \setminus Kw_0$. Since $w_0(Kw_0') = \bigoplus_{\ell_1 < j \leq \ell_2} \mathbb{Z} w_0 z_j$ is archimedean, we can apply Lemma 6.1, where we set $J = \{\ell_1 + 1, \ldots, \ell_2\}$, to obtain the unique representation (24). Here, the d_n are monomials built up from $z_{\ell_1+1}, \ldots, z_{\ell_2}$. Thus, they are also monomials built up from x_i, $i \in I_0$. Note that the d_n depend on our choice of the elements z_j, $j = \ell_1 + 1, \ldots, \ell_2$. These in turn are uniquely determined only up to multiplication with monomials with trivial w-value. Thus, the d_n are in general not uniquely determined. However, the uniqueness of the summands $c_n d_n$ can be shown as in the proof of the foregoing lemma. The values $vc_n d_n$ lie in the value group vK, which is finitely generated, according to Lemma 3.5.

It remains to find appropriate valuations w, w' on $LE_{\mathcal{F}}(x)$. Since $h \notin Kw_0$, there is at least one summand $c_n d_n$ such that $w_0 c_n d_n \neq 0$. We take w to be the valuation associated with the smallest convex subgroup H of $vLE_{\mathcal{F}}(x)$ containing $vc_n d_n$. Then w is the finest convex valuation on $LE_{\mathcal{F}}(x)$ which coincides with w_0 on K. Similarly, the valuation w' associated with the largest convex subgroup H' of $vLE_{\mathcal{F}}(x)$ not containing $vc_n d_n$ is the coarsest convex valuation on $LE_{\mathcal{F}}(x)$ coinciding with w_0' on K. Finally, $w(LE_{\mathcal{F}}(x)w') = H/H'$ is archimedean. □

For each monomial $d \in LE_{\mathcal{F}}(x)$ we define w_d to be the convex valuation associated with the largest convex subgroup not containing vd. Then w_d is the coarsest convex valuation such that $w_d d \neq 0$. The residue field $LE_{\mathcal{F}}(x)w_d$ can be thought of as the largest residue field "below d"; it is the largest residue field in which the residue of either d or d^{-1} is 0. Note that if $w_d d < w_d d' < 0$, then the largest convex subgroup not containing vd must be equal to that not containing vd' and therefore, $w_d = w_{d'}$. The following theorem is the intrinsic version of "truncation at 0".

Theorem 6.3 *Take $h \in LE_{\mathcal{F}}(x)$ such that $vh < 0$. Then there exist $m \in \mathbb{N}$, monomials $d_n \in LE_{\mathcal{F}}(x)$, elements $c_n \in LE_{\mathcal{F}}(x)w_{d_n}$, $1 \leq n \leq m$, some $r_h \in \mathbb{R}$, and $h^+ \in LE_{\mathcal{F}}(x)$ of value $vh^+ > 0$, such that*

$$h \;=\; c_0 d_0 + \ldots + c_m d_m + r_h + h^+ \quad \textit{with} \;\; vc_0 d_0 < \ldots < vc_m d_m < 0\,, \tag{25}$$

and such that $w_{d_n} d_n < w_{d_n} d_{n+1}$ for all $n < m$. The summands $c_n d_n$ and the elements r_h and h^+ are uniquely determined.

Proof: In the representation (24) of h given by Lemma 6.2, there are only finitely many summands $c_0 d_0, \ldots, c_{m_1} d_{m_1}$ of value < 0. Note that the valuation w' of that lemma coincides with w_{d_0}. Therefore, $c_0, \ldots, c_{m_1} \in LE_{\mathcal{F}}(x)w_{d_0}$. Assume that $w_{d_0} c_{m_1} d_{m_1} < 0$. Then also $w_{d_0} c_n d_n < 0$ and thus, $w_{d_0} = w_{d_n}$ for $0 \leq n \leq m_1$. It follows that $c_n \in$

$LE_{\mathcal{F}}(x)w_{d_0} = LE_{\mathcal{F}}(x)w_{d_n}$ for $0 \leq n \leq m_1$ and $w_{d_n}d_n < w_{d_n}d_{n+1}$ for $0 \leq n < m_1$. We have that $v(h - c_1d_1 - \ldots - c_{m_1}d_{m_1}) \geq 0$. Consequently, there is a unique $r_h \in \mathbb{R}$ such that $h^+ := h - c_1d_1 - \ldots - c_{m_1}d_{m_1} - r_h$ has value $vh^+ > 0$, and we are done.

Now assume that $w_{d_0}c_{m_1}d_{m_1} = 0$ (note that $w_{d_0}c_nd_n < 0$ for $n < m_1$ by condition c) of Lemma 6.2). It follows that $c_{m_1}d_{m_1} \in LE_{\mathcal{F}}(x)w_{d_0}$, and we apply the lemma again to this element in the place of h. We repeat this procedure, thereby descending through the convex valuations of $LE_{\mathcal{F}}(x)$. But we are actually working with elements inside of r$\mathcal{F}$-closure of the field K which we used in the proof of Lemma 6.2. Since the value group of K has finite rank, there are only finitely many distinct convex valuations on K. Therefore, after a finite repetition of our procedure, we reach a convex valuation which coincides with v on K (if the procedure doesn't stop before). If $c_\ell d_\ell, \ldots, c_md_m$ are the summands obtained from Lemma 6.2 at this step, then by their choice we have that $w_{d_\ell}c_nd_n = vc_nd_n < 0$ for $\ell \leq n \leq m$, and our procedure will stop here. Now necessarily $v(h - c_1d_1 - \ldots - c_md_m) \geq 0$ since otherwise, we would have to obtain a further summand from our procedure. — The uniqueness of the summands c_nd_n follows from the uniqueness assertion of Lemma 6.2. □

Given the representation (25) of an element h according to this theorem, the uniquely determined finite sum

$$\mathrm{pp}(h) := c_1d_1 + \ldots + c_md_m$$

will be called the **principal part of** h; we set $\mathrm{pp}(h) := 0$ if $vh \geq 0$. The principal part is uniquely determined, once the set of monomials in $LE_{\mathcal{F}}(x)$ is fixed. Note that $v(h - \mathrm{pp}(h) - r_h) > 0$ with $r_h \in \mathbb{R}$.

Theorem 6.4 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions, definable in $\mathbb{R}_{\mathrm{an,exp}}$ and ultimately positive. Then f is asymptotic to rg for some positive $r \in \mathbb{R}$ if and only if the germs $\log f$ and $\log g$ in $H(\mathbb{R}_{\mathrm{an,exp}})$ have the same principal part.*

Proof: We know already that f is asymptotic to rg if and only if $v(\log f - \log rg) > 0$. This in turn is equivalent to $v(\log f - \log g) \geq 0$, since if the latter holds, then there is some $r_0 \in \mathbb{R}$ such that $v(\log f - \log g - r_0) > 0$, and we set $r = \exp r_0$. By the uniqueness of the principal part, $v(\log f - \log g) \geq 0$ if and only if $\mathrm{pp}(\log f) = \mathrm{pp}(\log g)$. □

To apply this theorem in the spirit of the Hardy problem, we take $\mathcal{F}$ to be any set of restricted analytic functions, closed under partial derivations. Then by running our construction of Section 4 simultaneously for $\mathcal{F}$ and $\mathcal{F}_{\mathrm{an}}$, we find index sets $I_{\mathcal{F}} \subset I$ and elements x_i such that $LE_{\mathcal{F}}(x) = \mathbb{R}(x_i \mid i \in I_{\mathcal{F}})^{\mathrm{r}\mathcal{F}}$ and $LE_{\mathcal{F}_{\mathrm{an}}}(x) = \mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}_{\mathrm{an}}}$. So the monomials of $LE_{\mathcal{F}}(x)$ will also be monomials of $LE_{\mathcal{F}_{\mathrm{an}}}(x)$. Moreover, we can take

$$LE_{\mathcal{F}}(x)w = LE^w_{\mathcal{F}}(\log^{m_0} x) \subset LE^w_{\mathcal{F}_{\mathrm{an}}}(\log^{m_0} x) = LE_{\mathcal{F}_{\mathrm{an}}}(x)w$$

for each convex valuation w and suitable m_0, according to Theorem 4.8 and Corollary 4.9. Using principal parts determined by this choice of the x_i and the residue fields, we get:

Corollary 6.5 *Assume that $h : \mathbb{R} \to \mathbb{R}$ is definable in $\mathbb{R}_{\mathrm{an,exp}}$. Then $\exp h$ is asymptotic to a composition of semialgebraic functions, exp, log and restricted analytic functions in $\mathcal{F}$, if and only if $\mathrm{pp}(h) \in LE_{\mathcal{F}}(x)$.*

As an example, let us reconsider the Hardy problem. Here we assume in addition that the x_i include x (cf. Theorem 4.7). We choose w as in Section 5.1. The representation of $i(x)$ is just $i(x) = cx$, where $c = \frac{1}{\log x}(1+\tilde{f}) \in H(\mathbb{R}_{\mathrm{an,exp}})w$. Thus, $\mathrm{pp}(i(x)) = i(x) \notin LE$. Hence by our corollary, $\exp i(x)$ is not asymptotic to any element of LE.

Let us give a further application of Theorem 6.3. Denote by $\mathcal{L}_{\mathcal{F}}$ the language of ordered rings, enriched by symbols for the functions from $\mathcal{F}$. Recall that every generalized power series field $\mathbb{R}((G))$ has a canonical cross-section, sending $\alpha \in G$ to the element $1_\alpha \in \mathbb{R}((G))$ which has a 1 at α and zeros everywhere else. (1_α is the characteristic function of the singleton $\{\alpha\}$.)

Corollary 6.6 *Take any $\mathcal{L}_{\mathcal{F}}$-embedding of $LE_{\mathcal{F}}(x)$ in some generalized power series field $\mathbb{R}((G))$, and denote by L its image in $\mathbb{R}((G))$. Assume that the restriction of the canonical cross-section of $\mathbb{R}((G))$ to vL is a cross-section π of (L,v), and that $L = \mathbb{R}(\pi vL)^{\mathrm{r}\mathcal{F}}$. Then the nonzero elements of the support of each element in L are bounded away from 0.*

Proof: For every convex valuation w with associated convex subgroup $H_w \subset G$, we have that $\mathbb{R}((G))w = \mathbb{R}((H_w))$. These residue fields satisfy (23).

Let $I \subset vL$ be a maximal set of rationally independent values. Set $x_i := 1_\alpha$ for $i = \alpha \in I$. Then $\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}} = \mathbb{R}(\pi vL)^{\mathrm{r}}$ and hence, $\mathbb{R}(x_i \mid i \in I)^{\mathrm{r}\mathcal{F}} = \mathbb{R}(\pi vL)^{\mathrm{r}\mathcal{F}} = L$ by hypothesis. The monomials obtained from the x_i are precisely the elements of the form $r \cdot 1_\alpha$ with $r \in \mathbb{R}$ and $\alpha \in vL$. Note that if $\alpha < H_w$, then for every $c \in \mathbb{R}((H_w))$, the support of $c\,r1_\alpha$ is bounded away from 0 by every element β which satisfies $\alpha + H_w < \beta < 0$. For example, $\beta = \alpha/2$ is a good choice.

Take $h \in L$ and consider the representation (25) with respect to the monomials x_i and the residue fields $\mathbb{R}((H_w))$. Now $\mathrm{support}(h) \setminus \{0\}$ is the union of the support of $c_1 d_1 + \ldots + c_m d_m$ and the support of h^+. The latter is bounded away from 0 by vh^+. The support of $c_1 d_1 + \ldots + c_m d_m$ is the union of the supports of $c_1 d_1, \ldots, c_m d_m$. This union is bounded away from 0 by $\frac{1}{2} v d_m$. □

Note that the embeddings of $H(\mathbb{R}_{\mathrm{an,exp}})$ and of LE in the logarithmic power series field $\mathbb{R}((t))^{LE}$ given in [D–M–M2] satisfy the conditions of the corollary. (Recall that $\mathbb{R}((t))^{LE}$ can be viewed as a subfield of a suitable power series field.)

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