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Salma Kuhlmann
Schwerpunkt Reelle Algebra und Geometrie,
Fachbereich Mathematik und Statistik,
Universität Konstanz,
78457 Konstanz, Germany

Email: salma.kuhlmann@uni-konstanz.de

The slides of this talk are available at:

<http://math.usask.ca/~skuhlman/slidesmaltsev2010.pdf>

Valued Differential Fields.

Joint work with M. Matusinski

I. Motivation

I.1 Ax - Kochen Ershov Principles for Valued Fields.

Let K be a field and (Γ, \preceq) a totally ordered abelian group (written multiplicatively). A surjective map

$$v : K^\times \rightarrow \Gamma$$

is a **field valuation** if for all $a, b \in K^\times$:

$$v(a.b) = v(a).v(b) \text{ (homomorphism)}$$

$$v(a + b) \preceq \max\{v(a), v(b)\} \text{ (ultrametric inequality).}$$

$K_v := \{a \in K \mid v(a) \preceq 1\}$ is the **valuation ring** of K

$I_v := \{a \in K \mid v(a) \prec 1\}$ the **maximal ideal** of K_v .

$v(K) := \Gamma$ is the **value group** (also: monomials group)

$K_v/I_v := \overline{K}$ is the residue field.

$v(K)$ and \overline{K} are important invariants of a valued field:

AKE Transfer Principle:

Let K and L be two valued fields

(*plus additional conditions*).

Assume that:

\overline{K} is elementarily equivalent to \overline{L}

$v(K)$ is elementarily equivalent to $v(L)$.

Then K is elementarily equivalent to L (?)

If in addition L is an extension of K , one can replace:
“elementarily equivalent” by “elementary substructure” or
“existentially closed” in the above query.

I.2. Kaplansky Embedding Theorem for Valued Fields.

Theorem: Let K be a valued field with $\text{char}(K) = \text{char}(\overline{K})$. Then K is analytically isomorphic to a subfield of a suitable generalized series field.

Let k be a (coefficients) field and (Γ, \preceq) a totally ordered abelian (monomials) group.

$K = k((\Gamma))$ denotes the **generalised series field**. It is the set of maps

$$\begin{aligned} a &: \Gamma \rightarrow k \\ \alpha &\mapsto a_\alpha \end{aligned}$$

such that $\text{Supp } a = \{\alpha \in \Gamma \mid a_\alpha \neq 0\}$ is anti-well-ordered in Γ .

We write these maps $a = \sum_{\alpha \in \text{Supp } a} a_\alpha \alpha$.

This set provided with component-wise sum and the following convolution product

$$\left(\sum_{\alpha \in \text{Supp } a} a_\alpha \alpha \right) \left(\sum_{\beta \in \text{Supp } b} b_\beta \beta \right) = \sum_{\gamma \in \Gamma} \left(\sum_{\alpha + \beta = \gamma} a_\alpha b_\beta \right) \gamma$$

is a field.

For any series $0 \neq a$, we define its **leading monomial**:

$$\text{LM}(a) := \max(\text{Supp } a) \in \Gamma.$$

The map

$$\text{LM} : K^\times \rightarrow \Gamma$$

is the canonical valuation on K .

E.g. $\Gamma = \{x^z ; z \in \mathbb{Z}\}$ (respectively $\Gamma = \{x^z ; z \in \mathbb{R}\}$) gives:

$\mathbb{R}((\Gamma))$ the Laurent series field (respectively the Levi-Civita series field).

- We have classification invariants and universal domains.
- What if the valued fields carry additional structure? Additional structure induced on the value group and residue field. AKE in this framework?
- In particular, generalised series fields are suitable domains for the study of real algebra.

Are they suitable domains for the study of real differential algebra ?

This work is the first step in this project:

Endow $K := \mathbb{R}((\Gamma))$ with derivations.

I.3. Hardy fields. The set of germs at infinity of real valued functions of a real variable forms a ring under point-wise addition and multiplication of germs.

A **Hardy field** is a subfield closed under differentiation of germs.

A Hardy field H carries a natural valuation:

$$H_v := \{f \in H ; \lim_{x \rightarrow \infty} f \in \mathbb{R}\}$$

.

Hardy fields are prime examples of valued differential fields.

II. Defining Derivations.

II.1. Hahn groups as monomial groups. Let (Φ, \preceq) be a totally ordered set, that we call the set of **fundamental monomials**.

Consider the set Γ of formal products $\gamma \in \Gamma$ of the form

$$\gamma = \prod_{\phi \in \Phi} \phi^{\gamma_\phi}$$

where $\gamma_\phi \in \mathbb{R}$, and the support of γ

$$\text{supp } \gamma := \{\phi \in \Phi \mid \gamma_\phi \neq 0\}$$

is an anti-well-ordered subset of Φ .

Multiplication of formal products is defined pointwise: for $\alpha, \beta \in \Gamma$

$$\alpha\beta = \prod_{\phi \in \Phi} \phi^{\alpha_\phi + \beta_\phi}$$

Γ is an abelian group with identity 1 (the product with empty support).

We endow Γ with the anti lexicographic ordering \preceq which extends \preceq of Φ :

$\gamma \succ 1$ if and only if $\gamma_\phi > 0$, for $\phi := \max(\text{supp } \gamma)$.

The **leading fundamental monomial** of $1 \neq \gamma \in \Gamma$ is $\text{LF}(\gamma) := \max(\text{supp } \gamma)$.

Γ is a totally ordered abelian group, the **Hahn group of generalised monic monomials**.

Hahn's Embedding Theorem: Hahn groups are universal domains.

II.2. Summable Families of Series.

We want to differentiate

$$a = \sum_{\alpha \in \Gamma} a_{\alpha} \alpha$$

term by term.

There are two problems:

(i) we first have to know how to differentiate a monomial $\alpha \in \Gamma$,

(ii) then we have to make sense of

$$a' = \sum_{\alpha \in \Gamma} a_{\alpha} \alpha'$$

a possibly infinite sum of field elements.

sometimes it is possible, but it can go wrong. Easy examples.

Let I be an infinite index set and $\mathcal{F} = \{a_i ; i \in I\}$ be a family of series in K . \mathcal{F} is said to be **summable** if:

(SF1) $\text{Supp } \mathcal{F} := \bigcup_{i \in I} \text{Supp } a_i$ (the support of the family) is an anti-well-ordered subset of Γ .

(SF2) For any $\alpha \in \text{Supp } \mathcal{F}$, the set

$$S_\alpha := \{i \in I \mid \alpha \in \text{Supp } a_i\} \subseteq I$$

is finite.

Write $a_i = \sum_{\alpha \in \Gamma} a_{i,\alpha} \alpha$, and assume that $\mathcal{F} = (a_i)_{i \in I}$ is summable.

Then

$$\sum_{i \in I} a_i := \sum_{\alpha \in \text{Supp } \mathcal{F}} \left(\sum_{i \in S_\alpha} a_{i,\alpha} \right) \alpha$$

is a well defined element of K that we call the **sum** of \mathcal{F} .

Series derivations.

Let

$$\begin{aligned} d_\Phi &: \Phi \rightarrow K \setminus \{0\} \\ \phi &\mapsto \phi' \end{aligned}$$

be a map.

We say d_Φ **extends to a series derivation on Γ** if the following property holds:

(SD1) For any anti-well-ordered subset $E \subset \Phi$,

the family $\left(\frac{\phi'}{\phi}\right)_{\phi \in E}$ is summable.

Then the **series derivation** d_Γ on Γ (extending d_Φ) is defined to be the map

$$d_\Gamma : \Gamma \rightarrow K$$

obtained through the following axioms:

(D0) $1' = 0$

(D1) Strong Leibniz rule:

$$\text{If } \alpha = \prod_{\phi \in \text{supp } \alpha} \phi^{\alpha_\phi} \text{ then } (\alpha)' = \alpha \sum_{\phi \in \text{supp } \alpha} \alpha_\phi \frac{\phi'}{\phi} .$$

We say that a series derivation d_Γ on Γ **extends to a series derivation on K** if the following property holds:

(SD2) For any anti-well-ordered subset $E \subset \Gamma$,
the family $(\alpha')_{\alpha \in E}$ is summable.

Then the **series derivation d** on K (extending d_Γ) is defined to be the map

$$d : K \rightarrow K$$

obtained through the following axiom:

(D2) Strong linearity:

$$\text{If } a = \sum_{\alpha \in \text{Supp } a} a_\alpha \alpha, \text{ then } a' = \sum_{\alpha \in \text{Supp } a} a_\alpha \alpha'.$$

We now study necessary and sufficient condition on the map d_Φ so that properties (SD1) and (SD2) hold.

II.3 Sequential Characterization Summability.

We use the following two key observations:

(i) \mathcal{F} is summable if and only if every countably infinite subfamily is summable.

(ii) (Infinite Ramsey.) Let Γ be a totally ordered set. Every sequence $(\alpha_n)_{n \in \mathbb{N}}$ in Γ has an infinite subsequence which is either constant, or strictly increasing, or strictly decreasing.

We isolate the following two crucial “bad” hypotheses:

(H1) There exists a strictly decreasing sequence $(\phi_n)_{n \in \mathbb{N}}$ in Φ and an increasing sequence $(\tau^{(n)})_{n \in \mathbb{N}}$ in Γ such that $\tau^{(n)} \in \text{Supp } \frac{\phi'_n}{\phi_n}$ for all $n \in \mathbb{N}$.

(H2) There exist strictly increasing sequences $(\phi_n)_{n \in \mathbb{N}}$ in Φ and $(\tau^{(n)})_{n \in \mathbb{N}}$ in Γ such that $\tau^{(n)} \in \text{Supp } \frac{\phi'_n}{\phi_n}$ and $\text{LF} \left(\frac{\tau^{(n+1)}}{\tau^{(n)}} \right) \succeq \phi_{n+1}$, for all $n \in \mathbb{N}$,

Theorem A: A map $d_\Phi : \Phi \rightarrow K \setminus \{0\}$ extends to a series derivation on K if and only if (H1) and (H2) fail.