Mal’tsev Meeting 2010
In honour of Y.L. Ershov, May 2-6 2010.

May 5, 2010

Salma Kuhlmann
Schwerpunkt Reelle Algebra und Geometrie,
Fachbereich Mathematik und Statistik,
Universität Konstanz,
78457 Konstanz, Germany

Email: salma.kuhlmann@uni-konstanz.de

The slides of this talk are available at:
Valued Differential Fields.
Joint work with M. Matusinski

I. Motivation

I.1 Ax - Kochen Ershov Principles for Valued Fields.

Let $K$ be a field and $(\Gamma, \preceq)$ a totally ordered abelian group (written multiplicatively). A surjective map
\[ v : K^\times \to \Gamma \]

is a field valuation if for all $a, b \in K^\times$:
\[ v(ab) = v(a).v(b) \] (homomorphism)
\[ v(a + b) \preceq \max\{v(a), v(b)\} \] (ultrametric inequality).

$K_v := \{a \in K \mid v(a) \preceq 1\}$ is the valuation ring of $K$
$I_v := \{a \in K \mid v(a) < 1\}$ the maximal ideal of $K_v$.
$v(K) := \Gamma$ is the value group (also: monomials group)
$K_v/I_v := \overline{K}$ is the residue field.
$v(K)$ and $\overline{K}$ are important invariants of a valued field:
AKE Transfer Principle:

Let $K$ and $L$ be two valued fields (plus additional conditions).
Assume that:

$K$ is elementarily equivalent to $L$
$v(K)$ is elementarily equivalent to $v(L)$.

Then $K$ is elementarily equivalent to $L$ (?)

If in addition $L$ is an extension of $K$, one can replace: “elementarily equivalent” by “elementary substructure” or “existentially closed” in the above query.

**Theorem:** Let $K$ be a valued field with $\text{char}(K) = \text{char}(\overline{K})$. Then $K$ is analytically isomorphic to a subfield of a suitable generalized series field.

Let $k$ be a (coefficients) field and $(\Gamma, \preceq)$ a totally ordered abelian (monomials) group.

$K = k((\Gamma))$ denotes the **generalised series field**. It is the set of maps

$$a : \Gamma \rightarrow k$$

$$\alpha \mapsto a_\alpha$$

such that $\text{Supp} a = \{\alpha \in \Gamma \mid a_\alpha \neq 0\}$ is anti-well-ordered in $\Gamma$.

We write these maps $a = \sum_{\alpha \in \text{Supp} \ a} a_\alpha \alpha$. 
This set provided with component-wise sum and the following convolution product
\[
( \sum_{\alpha \in \text{Supp } a} a_\alpha \alpha ) ( \sum_{\beta \in \text{Supp } b} b_\beta \beta ) = \sum_{\gamma \in \Gamma} ( \sum_{\alpha \beta = \gamma} a_\alpha b_\beta ) \gamma
\]
is a field.

For any series \( 0 \neq a \), we define its **leading monomial**:
\[
\text{LM } (a) := \max (\text{Supp } a) \in \Gamma.
\]
The map
\[
\text{LM } : K^\times \to \Gamma
\]
is the canonical valuation on \( K \).

E.g. \( \Gamma = \{ x^z ; z \in \mathbb{Z} \} \) (respectively \( \Gamma = \{ x^z ; z \in \mathbb{R} \} \))
gives:
\[
\mathbb{R}((\Gamma)) \text{ the Laurent series field (respectively the Levi-Civita series field).}
\]
• We have classification invariants and universal domains.

• What if the valued fields carry additional structure? Additional structure induced on the value group and residue field. AKE in this framework?

• In particular, generalised series fields are suitable domains for the study of real algebra.

*Are they suitable domains for the study of real differential algebra?*

This work is the first step in this project:

*Endow $K := \mathbb{R}((\Gamma))$ with derivations.*
I.3. **Hardy fields.** The set of germs at infinity of real valued functions of a real variable forms a ring under pointwise addition and multiplication of germs.

A **Hardy field** is a subfield closed under differentiation of germs.

A Hardy field $H$ carries a natural valuation:

$$H_v := \{ f \in H ; \lim_{x \to \infty} f \in \mathbb{R} \}$$

Hardy fields are prime examples of valued differential fields.
II. Defining Derivations.

II.1. Hahn groups as monomial groups. Let $(\Phi, \preceq)$ be a totally ordered set, that we call the set of fundamental monomials.

Consider the set $\Gamma$ of formal products $\gamma \in \Gamma$ of the form

$$\gamma = \prod_{\phi \in \Phi} \phi^{\gamma_{\phi}}$$

where $\gamma_{\phi} \in \mathbb{R}$, and the support of $\gamma$

$$\text{supp} \ \gamma := \{ \phi \in \Phi \mid \gamma_{\phi} \neq 0 \}$$

is an anti-well-ordered subset of $\Phi$.

Multiplication of formal products is defined pointwise: for $\alpha, \beta \in \Gamma$

$$\alpha \beta = \prod_{\phi \in \Phi} \phi^{\alpha_{\phi} + \beta_{\phi}}$$

$\Gamma$ is an abelian group with identity $1$ (the product with empty support).
We endow $\Gamma$ with the anti lexicographic ordering $\preceq$ which extends $\preceq$ of $\Phi$:

$$\gamma \succ 1 \text{ if and only if } \gamma_\phi > 0, \text{ for } \phi := \max(\text{supp } \gamma).$$

The **leading fundamental monomial** of $1 \neq \gamma \in \Gamma$ is $\text{LF}(\gamma) := \max(\text{supp } \gamma)$.

$\Gamma$ is a totally ordered abelian group, the **Hahn group of generalised monic monomials**.

Hahn’s Embedding Theorem: Hahn groups are universal domains.
II.2. Summable Families of Series.

We want to differentiate

\[ a = \sum_{\alpha \in \Gamma} a_\alpha \alpha \]

term by term.

There are two problems:

(i) we first have to know how to differentiate a monomial \( \alpha \in \Gamma \),

(ii) then we have to make sense of

\[ a' = \sum_{\alpha \in \Gamma} a_\alpha \alpha' \]

a possibly infinite sum of field elements.

*sometimes it is possible, but it can go wrong. Easy examples.*
Let $I$ be an infinite index set and $\mathcal{F} = \{a_i ; i \in I\}$ be a family of series in $K$. $\mathcal{F}$ is said to be **summable** if:

(SF1) $\text{Supp } \mathcal{F} := \bigcup_{i \in I} \text{Supp } a_i$ (the support of the family) is an anti-well-ordered subset of $\Gamma$.

(SF2) For any $\alpha \in \text{Supp } \mathcal{F}$, the set

$$S_{\alpha} := \{i \in I \mid \alpha \in \text{Supp } a_i\} \subseteq I$$

is finite.

Write $a_i = \sum_{\alpha \in \Gamma} a_{i,\alpha} \alpha$, and assume that $\mathcal{F} = (a_i)_{i \in I}$ is summable. Then

$$\sum_{i \in I} a_i := \sum_{\alpha \in \text{Supp } \mathcal{F}} \left( \sum_{i \in S_{\alpha}} a_{i,\alpha} \right) \alpha$$

is a well defined element of $K$ that we call the **sum** of $\mathcal{F}$. 

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Series derivations.

Let 

\[ d_\Phi : \Phi \to K \setminus \{0\} \]

\[ \phi \mapsto \phi' \]

be a map.

We say \( d_\Phi \) extends to a series derivation on \( \Gamma \) if the following property holds:

**(SD1)** For any anti-well-ordered subset \( E \subset \Phi \),

the family \( \left( \frac{\phi'}{\phi} \right)_{\phi \in E} \) is summable.

Then the series derivation \( d_\Gamma \) on \( \Gamma \) (extending \( d_\Phi \)) is defined to be the map

\[ d_\Gamma : \Gamma \to K \]

obtained through the following axioms:

**(D0)** \( 1' = 0 \)

**(D1)** Strong Leibniz rule:

If \( \alpha = \prod_{\phi \in \text{supp} \, \alpha} \phi^{\alpha_\phi} \) then

\[ (\alpha)' = \alpha \sum_{\phi \in \text{supp} \, \alpha} \alpha_\phi \frac{\phi'}{\phi}. \]
We say that a series derivation $d_{\Gamma}$ on $\Gamma$ \textbf{extends to a series derivation on} $K$ if the following property holds:

\textbf{(SD2)} For any anti-well-ordered subset $E \subset \Gamma$, the family $(\alpha')_{\alpha \in E}$ is summable.

Then the \textbf{series derivation} $d$ on $K$ (extending $d_{\Gamma}$) is defined to be the map

$$d : K \rightarrow K$$

obtained through the following axiom:

\textbf{(D2) Strong linearity:}

If $a = \sum_{\alpha \in \text{Supp} a} a_{\alpha} \alpha$, then $a' = \sum_{\alpha \in \text{Supp} a} a_{\alpha} \alpha'$.

\textit{We now study necessary and sufficient condition on the map $d_{\Phi}$ so that properties (SD1) and (SD2) hold.}
II.3 Sequential Characterization Summability.

We use the following two key observations:

(i) $\mathcal{F}$ is summable if and only if every countably infinite subfamily is summable.

(ii) (Infinite Ramsey.) Let $\Gamma$ be a totally ordered set. Every sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $\Gamma$ has an infinite subsequence which is either constant, or strictly increasing, or strictly decreasing.
We isolate the following two crucial “bad” hypotheses:

(H1) There exists a strictly decreasing sequence \((\phi_n)_{n \in \mathbb{N}}\) in \(\Phi\) and an increasing sequence \((\tau^{(n)})_{n \in \mathbb{N}}\) in \(\Gamma\) such that \(\tau^{(n)} \in \text{Supp} \frac{\phi'_n}{\phi_n}\) for all \(n \in \mathbb{N}\).

(H2) There exist strictly increasing sequences \((\phi_n)_{n \in \mathbb{N}}\) in \(\Phi\) and \((\tau^{(n)})_{n \in \mathbb{N}}\) in \(\Gamma\) such that \(\tau^{(n)} \in \text{Supp} \frac{\phi'_n}{\phi_n}\)
and \(\text{LF} \left( \frac{\tau^{(n+1)}}{\tau^{(n)}} \right) \geq \phi_{n+1}\), for all \(n \in \mathbb{N}\),

**Theorem A:** A map \(d_\Phi : \Phi \to K \setminus \{0\}\) extends to a series derivation on \(K\) if and only \((H1)\) and \((H2)\) fail.