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The slides of this talk are available at:  
Closures of Quadratic Modules in Locally Convex Topologies.

Positivity and Duality.

• Let $V := \mathbb{R}[X] := \mathbb{R}[X_1, \cdots, X_n]$ be the real vector space of polynomials in $n$ variables and real coefficients.

• Fix $\tau$ a locally convex topological vector space topology on $V$. Denote $V_\tau$ the corresponding topological space.

• Let $K \subset \mathbb{R}^n$. Consider polynomials positive semi-definite on $K$:

$$\text{Pos}(K) := \{ f \in V \mid f(x) \geq 0 \text{ for all } x \in K \}$$

• Let $C \subset V$. Define

$$K_C := \{ x \in \mathbb{R}^n \mid g(x) \geq 0 \ \forall \ g \in C \}$$
• Let $C \subset V$. Define the dual of $C$:

$$C^\vee := \{ L | L : V_\tau \to \mathbb{R}; \text{cts linear functional}; L(C) \geq 0 \}$$

and the double dual of $C$:

$$C^{\vee\vee} := \{ f \in V | L(f) \geq 0 \ \forall \ L \in C^\vee \}$$

Straightforward properties are:

(i) Contravariance

(ii) $C \subset C^{\vee\vee}$

(iii) $C^{\vee\vee\vee} = C^\vee$

etc...
The Moment Property and the Strong Moment Property.

Haviland’s representation theorem for multidimensional moment sequences:

Let $K \subset \mathbb{R}^n$ closed, and $L : V \to \mathbb{R}$ a linear functional $\neq 0$. The following are equivalent:

(i) $L(f) \geq 0$ for all $f \in \text{Pos}(K)$

(ii) $\exists$ a positive Borel measure $\mu$ on $K$ such that

$$L(f) = \int_K f \, d\mu , \forall f \in V$$

We use Haviland’s theorem and the properties of duality to deduce the following:
Theorem 1: Let $V_\tau$ as above, $C \subset V$, $K \subset \mathbb{R}^n$, $K$ closed. The following are equivalent:

1. $C^\vee \subset \text{Pos}(K)^\vee$
2. $C^{\vee\vee} \supset \text{Pos}(K)$
3. $\forall L \in C^\vee \exists \mu$ on $K$ such that:
   \[ L(f) = \int_K f d\mu, \forall f \in V \]

Definitions: $C$ satisfies (or solves) $K$ MP if any of the equivalent conditions of the above theorem hold. $C$ satisfies (or solves) the strong $K$ MP if $C$ satisfies $K$ MP with $K = K_C$.

Remark: These notions were introduced and studied by Schmüdgen for $\tau = \varphi :=$ the finest locally convex topology, thus studying representation of arbitrary linear functionals. Here we consider representation of $\tau$ continuous linear functionals. We shall return to this issue later.
Double duals and closures.

If $C \subset V$ is a (convex) cone (closed under addition and scalar multiplication by positive reals), then

$$C^\vee\vee = \overline{C}$$

in $V_\tau$ (Hahn–Banach). We obtain the following:

**Corollary 1** Let $C \subset V$ be a cone, $K \subset \mathbb{R}^n$ closed. The following are equivalent:

(2) $\overline{C} \supset \text{Pos}(K)$

(3) $\forall L \in C^\vee \exists \mu$ on $K$ such that:

$$L(f) = \int_K f d\mu , \forall f \in V$$

These results are particularly interesting in the special case when $C$ is a finitely generated quadratic module as we shall explain now.
Finite solvability of the $K$ Moment Problem for continuous linear functionals.

Let $S := \{g_1, \ldots, g_s\} \subset V$. We define the (finitely generated) quadratic module:

$$M_S := \{\sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s \mid \sigma_i \in \Sigma V^2\}.$$ 

For $S$, $M_S$, $K$ as above we obtain the following:

**Corollary 2:** The following are equivalent:

1. $M_S \supset \text{Pos}(K)$
2. If $L$ is a continuous linear functional s.t.
   $$L(h^2) \geq 0, L(h^2 g_1) \geq 0, \cdots, L(h^2 g_s) \geq 0$$
   (for all $h \in V$), then there exists $\mu$ on $K$ such that:
   $$L(f) = \int_K f d\mu, \forall f \in V.$$ 

Thus existence of representation via measures amounts to checking psd-ness of finitely many Hankel matrices.
**Definition:** The $K$ MP is finitely solvable if $S$ finite exists such that any of the equivalent conditions of Corollary 2 holds.

**Remark:** If $n \geq 2$ and $K$ contains a 2-dimensional affine cone, the $K$ MP is never finitely solvable for the finest topology $\varphi$ (K–Marshall). The hope with this approach is to get finite solvability for representation of linear functionals continuous in a coarser topology $\tau$. We discuss this now.
Closures of the cone of Sums of Squares

Theorem 2:

(1) \[ \sum V^2 = \sum V^2 \text{ in } V_\varphi. \]

(2) \[ \sum V^2 = \text{Pos } [-1, 1]^n \text{ in } V_p. \]

Here, for \( 1 \leq p \leq \infty \):

\( V_p : = V \) endowed with the \( \ell_p \)-norm topology (on the coefficients of polynomials).

Two aspects: 

a. Approximating nonnegative polynomials on the closed hypercube by sums of squares, and

b. applications to solvability of MP:
Corollary 2 and this theorem, say for $p = 1$, establish the following:

**Corollary 3:** Let $L$ be a continuous linear functional on $V_1$, i.e. $L$ is a linear functional on $V$ with a bounded sequence of moments $(L(x^\alpha))_{\alpha \in \mathbb{N}^n}$.

Assume that

$$L(h^2) \geq 0 \ \forall \ h \in V.$$ 

Then

$$\exists \mu \text{ on } [-1, 1]^n \text{ such that } L(f) = \int f \, d\mu \ \forall \ f \in V.$$
Closures in Weighted $\ell_p$ Topologies.

Let $1 \leq p < \infty$, $r = (r_1, \ldots, r_n)$ be a $n$-tuple of positive real numbers.

- Set
  \[ \ell_p(r, \mathbb{N}^n) = \{ s \in \mathbb{R}^{\mathbb{N}^n} : \sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^{pr^\alpha} < \infty \} \]
  endowed with the norm defined by
  \[ \|(s)\|_{p,r} = \left( \sum_{\alpha \in \mathbb{N}^n} |s(\alpha)|^{pr^\alpha} \right)^{\frac{1}{p}}. \]

- Denote by $V_{p,r}$ the topological vector space $V$ endowed with the $\| \cdot \|_{p,r}$ norm.

- We compute the closure of the cone of sums of squares in these norm topologies.
Let $f \in V$. Assume that 

$$f \geq 0 \text{ on } \prod_{i=1}^{n} [-r_i, r_i].$$

Then the polynomial $\tilde{f}(X) = f(r_1 X_1, \ldots, r_n X_n)$ is a non-negative polynomial on $[-1, 1]^n$.

Combining this observation with Berg’s result we get:

**Theorem 3**

$$\sum V^2 = \text{Pos} \left( \prod_{i=1}^{n} [-r_i, r_i] \right)$$

in $V_1, r$.

**Theorem 4** For $1 < p < \infty$,

$$\sum V^2 = \text{Pos} \left( \prod_{i=1}^{n} [-r_i^p, r_i^p] \right)$$

in $V_p, r$.

Here, $q$ is the conjugate of $p$.

**The End**