Session Surreal Numbers.
Joint Mathematics Meetings.

January 13, 2010

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The slides of this talk are available at:


Exponential - Logarithmic Series Fields.

Preliminaries.


Let \( G \neq 1 \) be an ordered abelian group.

\( \mathbb{R}((G)) \) will denote the field of \textbf{generalized series} with real coefficients, of which support is an anti well ordered and subset of \( G \).

\( f = \sum_{g \in G} f_g g \) with \( f_g \in \mathbb{R} \) and

\[
\text{supp}(f) := \{ g \in G ; f_g \text{ nonzero} \}
\]

is and anti-wellordered.

Pointwise addition, convolution formula for multiplication of series, anti-lexicographic order, natural valuation is given by “leading monomial”.
• let $G^{>1}$ be the semigroup of elements greater than 1.

• $\mathbb{R}((G^{>1}))$ consists of “purely infinite” series with support in $G^{>1}$.

• $\mathbb{R}((G^{\leq 1}))$ and $\mathbb{R}((G^{<1}))$ denote respectively the valuation ring of bounded elements, and the valuation ideal of infinitesimal elements of $\mathbb{R}((G))$.

We have the following direct sum (respectively, multiplicative direct sum) decompositions:

$$\mathbb{R}((G)) = \mathbb{R}((G^{>1})) \oplus \mathbb{R} \oplus \mathbb{R}((G^{<1})),$$  \hspace{1cm} (1)

$$\mathbb{R}((G))^{>0} = G \cdot \mathbb{R}^{>0} \cdot (1 + \mathbb{R}((G^{<1}))).$$  \hspace{1cm} (2)

Indeed given $f \in \mathbb{R}((G))$ write

• $f = f^{>1} + r + f^{<1}$ and

• for $f > 0$ and $g := \max \ \text{supp } f$, write

$$f = g \cdot c \cdot (1 + \epsilon)$$

with $c \in \mathbb{R}$, $c > 0$, $\epsilon \in \mathbb{R}((G^{<1}))$. 

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• If $G$ is divisible, $\mathbb{R}((G))$ is a (non-archimedean) **real closed field**, i.e. by Tarski’s Transfer Principle, $\mathbb{R}((G))$ is elementarily equivalent to the ordered field of real numbers $(\mathbb{R}, <)$.

• What about $(\mathbb{R}, <, \exp)$?

• *How to construct nonarchimedean logarithmic fields using fields of generalized series?*

• The additive and multiplicative decompositions will be exploited.
• Use Taylor expansion of the logarithm to define the logarithm of a generalized series?

**Summable families of series:** Given a family

\[ \{ s_i ; i \in I \} \subset \mathbb{R}((G)) \]

how to make sense of \( \sum_{i \in I} s_i \) as an element of \( \mathbb{R}((G)) \)?

• This is the case if (i) the support of the family, i.e. \( \bigcup_{i \in I} \text{support } s_i \) is anti wellordered, and (ii) for every \( \gamma \) in the support of the family, the set of \( i \in I \) for which \( \gamma \in \text{support } s_i \) is finite.

• **B.H.Neumann:** For \( \epsilon \in \mathbb{R}((G^{<1})) \),

\[
\sum_{i=1}^{+\infty} (-1)^{(i-1)} \frac{\epsilon^i}{i}
\]

makes sense.

• The condition on \( \epsilon \) is necessary!
Defining the logarithm.

• We have seen: the Taylor expansion defines a surjective logarithm from $\mathbb{R}^{>0} \cdot (1 + \mathbb{R}((G^{<1})))$ onto $\mathbb{R} \oplus \mathbb{R}((G^{<1}))$.

• A logarithmic section is an embedding of ordered groups

$$l : (G, \cdot, \prec) \rightarrow (\mathbb{R}((G^{>1})), +).$$

If we have a logarithmic section, we can define now a logarithm.
• Given $f \in \mathbb{R}((G))$, $f > 0$ and $g := \max \text{ supp } f$, write

$$f = g \cdot c \cdot (1 + \epsilon)$$

with $c \in \mathbb{R}$, $c > 0$, $\epsilon \in \mathbb{R}((G^{<1}))$.

• We extend $l$ as follows:

$$L(f) = l(g \cdot c \cdot (1 + \epsilon)) = l(g) + \log c + \sum_{i=1}^{+\infty} (-1)^{i-1} \frac{\epsilon^i}{i}$$

• $L : (\mathbb{R}((G))^{>0}, \cdot) \rightarrow (\mathbb{R}((G)), +)$ is an order preserving embedding of groups, extending the logarithmic section $l$ (the logarithm associated to the logarithmic section $l$).
Logarithmic sections from Hahn groups

Let us now consider a totally ordered set $\Gamma$,

- Consider the multiplicative “Hahn group” $H(\Gamma)$ which consists of formal products $g = \prod f^r$, $f \in \Gamma$, $r \in \mathbb{R}$, with support $g$ an anti well ordered subset of $\Gamma$. Multiplication is point wise, order is anti lexicographic, 1 is the product with empty support.

- Hahn Embedding’s Theorem states that every ordered abelian group $G$ is a subgroup of a Hahn group $H(\Gamma)$ (and $\Gamma$ is uniquely determined by $G$).
• We shall from now on assume that $G$ is a Hahn group $H(\Gamma)$, and explain how this data determines a logarithmic section:

• Consider $l : G \to \mathbb{R}((G^{>1}))$ defined by

$$l(\prod f_i^{\text{r}_i}) := \sum r_if_i,$$

defines indeed a logarithmic section on $\mathbb{R}((G))$. 

This logarithmic section has two defects:

(I) It violates the growth axiom.

(II) It does not map $G$ surjectively onto the ring of purely infinite series $\mathbb{R}((G^{\succ -1}))$ (so its associated logarithm will not be surjective).

To construct models we shall fix these two defects as follows:
(I) We assume that $\Gamma$ admits an order preserving automorphism which is a **leftward shift:**

\[ \sigma(f) \prec f \text{ for all } f \in \Gamma. \]

- The automorphism $\sigma$ induces the logarithmic section:

\[ l(\prod f_i^{r_i}) := \sum r_i \sigma(f_i). \]

This fixes (I) but is still not surjective. We shall now explain the core step in constructing exponentials of infinitely large elements to deal with (II): Since $l : G \rightarrow (\mathbb{R}((G^{>1})), +)$ is not surjective, there exists elements of $\mathbb{R}((G^{>1})) \setminus l(G)$ of which exponentials are not defined. We shall enlarge our group of monomials $G$ to a group extension $G^\#$ to include the missing exponentials.
Exponential Extension

We take $G^\#$ to be a multiplicative copy $e[\mathbb{R}((G^{\succ 1}))]$ of $\mathbb{R}((G^{\succ 1}))$ over $l(G)$.

- More precisely, we construct $G^\#$ formally as follows:
  \[
  G^\# := \{ e(\alpha); \alpha \in \mathbb{R}((G^{\succ 1})), \text{ where } e(\alpha) := g \text{ if } \exists g \in G \text{ s.t. } \alpha = l(g) \}
  \]
  By its definition, $G$ is a subset of $G^\#$.

- We define multiplication on $G^\#$ as follows:
  \[
  e(\alpha_1)e(\alpha_2) := e(\alpha_1 + \alpha_2).
  \]
  In particular, if $g_1 = e(\alpha_1), g_2 = e(\alpha_2) \in G$, then $e(\alpha_1)e(\alpha_2) = e(l(g_1) + l(g_2)) = e(l(g_1g_2)) = g_1g_2$, so $G$ is a subgroup of $G^\#$.

- We equip $G^\#$ with a total order:
  \[
  e(\alpha_1) < e(\alpha_2) \text{ if and only if } \alpha_1 < \alpha_2 \text{ in } \mathbb{R}((G^{\succ 1})).
  \]
  Again, if $g_1 = e(\alpha_1), g_2 = e(\alpha_2) \in G$, then $e(\alpha_1) < e(\alpha_2)$ if and only if $l(g_1) < l(g_2)$ in $\mathbb{R}((G^{\succ 1}))$ if and only if $g_1 < g_2$ in $G$, so $G$ is an ordered subgroup of $G^\#$.
Since $G \subseteq G^#$ as ordered abelian multiplicative groups, we view $\mathbb{R}((G))$ as an ordered subfield of $\mathbb{R}((G^#))$ (by identifying $\mathbb{R}((G))$ with the elements of $\mathbb{R}((G^#))$ having support in $G$).

- One verifies that the map
  \[
  l^# : (G^#, \cdot) \to \mathbb{R}((G^#)^1), +
  \]
  defined by:
  \[
  l^#(e(\alpha)) := \alpha
  \]
  for $\alpha \in \mathbb{R}((G^{>1}))$ is a prelogarithmic section with:
  \[
  l^#(G^#) = \mathbb{R}((G^{>1}))
  \]
  and $l^#$ extends $l$ on $G$.

- By construction of the logarithms $L$ and $L^#$ on $\mathbb{R}((G))^{>0}$ and $\mathbb{R}((G^#))^{>0}$ respectively, $L^#$ is an extension of $L$.

We define the \textbf{exponential extension} of $(\mathbb{R}((G)), L)$ to be $(\mathbb{R}((G^#)), L^#)$. 
The Exponential Closure

We now close under exponentiation by induction on $n$.

- If $n = 0$ set $(\mathbb{R}((G))^n, L^n) := (\mathbb{R}((G)), L)$.

For $n \in \mathbb{N}$, define inductively the $n$-th exponential extension of $(\mathbb{R}((G)), L)$:

$(\mathbb{R}((G))^n, L^n) :=$ the exponential extension of $(\mathbb{R}((G)^{n-1}), L^{n-1})$.

- Set $\mathbb{R}((G))^{EL} := \bigcup \mathbb{R}((G))^n$ and $\text{Log} := \bigcup L^n$.

We call $(\mathbb{R}((G))^{EL}, \text{Log})$ is EL-series field over $(\Gamma, \sigma)$. 
We see that pairwise distinct left shifts on \( \Gamma \) will induce pairwise distinct logarithms. We do more: we construct logarithms of pairwise distinct growth rates.

The **rank** of \((\Gamma, \sigma)\) is the order type of the quotient \( \Gamma / \sim_{\sigma} \), where \( a \sim_{\sigma} a' \) if and only if there exists \( n \in \mathbb{N} \) such that \( \sigma^{(n)}(a) \geq a' \) and \( \sigma^{(n)}(a') \geq a \).

Similarly the **logarithmic rank** of \((K^>^0, l)\) is defined via the equivalence relation: \( a, a' \in K^>^0 \) are log-equivalent if \( a \sim_l a' \), that is, if and only if there exists

\[ n \in \mathbb{N} \text{ such that } l^{(n)}(a) \leq a' \text{ and } l^{(n)}(a') \leq a . \]

**Proposition 0.1** The logarithmic rank of \((\mathbb{R}((G)), l_{\sigma})\) is equal to the rank of \((\Gamma, \sigma)\).
An asymptotic scale indexed by 
\[ \aleph_1 \times \mathbb{Z}^2. \]

We construct a totally ordered set of germs at infinity of real valued functions of a real variable, which admits \(2^{\aleph_1}\) left shifts.

- For \((p, q) \in \mathbb{Z}^2\), we denote by \(g_{p,q}\) the germ at \(+\infty\) of the infinitely large transmonomial

\[
x \mapsto \exp(x^q \exp(x^p)) .
\]

If we endow \(\mathbb{Z}^2\) with the lexicographic order, then \((p, q) < (p', q')\) implies \(g_{p,q} < g_{p',q'}\).
Now let \( \{ h_\alpha; \alpha \in \aleph_1 \} \) be a sequence of germs at \( +\infty \) of infinitely large transmonomials \( h_\alpha \), in such a way that \( \alpha < \beta \) implies \( h_\alpha \prec h_\beta \).

One can describe for example the first \( \epsilon_0 \) terms of such a sequence. Set \( h_0(x) := x \). We define \( h_\alpha \) by transfinite induction for \( \alpha < \epsilon_0 \). If the Cantor normal form of \( \alpha \) is \( \omega^{\beta_r}d_r + \cdots + \omega^{\beta_1}d_1 + d_0 \), with \( \beta_1 < \cdots < \beta_r < \alpha \) and \( d_0, \ldots, d_r \in \mathbb{N} \), set

\[
h_\alpha(x) := \exp (d_r h_{\beta_r}(x) + \cdots + d_1 h_{\beta_1}(x)) \exp(x)^{d_0}.
\]

We can set \( h_{\epsilon_0} := t(x) \) where \( t(x) \) is a germ of transexponential growth.
Finally: for all $(\alpha, p, q) \in \mathbb{N}_1 \times \mathbb{Z}^2$, we denote $f_{\alpha,p,q}$ the germ at $+\infty$ of the transmonomial $\exp_3(h_\alpha(x))g_{p,q}(x)$.

These germs are defined in such a way that if $(\alpha, p, q) < (\alpha', p', q')$ for the lexicographic order, then $f_{\alpha,p,q} < f_{\alpha',p',q'}$. This set of germs $\Gamma$ is thus totally ordered.
We construct $2^\aleph_1$ left-shifts of pairwise distinct ranks on $\Gamma$. To this end, we consider the two automorphisms defined on $\Gamma_1 = \{g_{p,q}, (p,q) \in \mathbb{Z}^2\}$ by:

$$\sigma (g_{p,q}) = g_{p-1,q}$$
$$\rho (g_{p,q}) = g_{p,q-1}$$

It follows easily from the definition of $g_{p,q}$ that the rank of $(\Gamma_1, \sigma)$ is 1 and the rank of $(\Gamma_1, \rho)$ is $\mathbb{Z}$. We define now, for every $S \subset \aleph_1$, the decreasing automorphism $\tau_S$ on $\Gamma$ by:

$$\tau_S (f_{\alpha,p,q}) = \begin{cases} f_{\alpha,p-1,q} = \exp_3 (h_\alpha) \sigma (g_{p,q}) & \text{si } \alpha \in S \\ f_{\alpha,p,q-1} = \exp_3 (h_\alpha) \rho (g_{p,q}) & \text{si } \alpha \not\in S \end{cases}$$

The End