ALGEBRAIC GEOMETRIC TECHNIQUES FOR TRUNCATED MOMENT PROBLEMS

(JOINT WORK WITH L. FIALKOW AND M. MÖLLER)

Raúl E. Curto

Banff 2006

ABSTRACT

- For a degree $2n$ real d-dimensional multisequence $\beta \equiv \beta^{(2n)} = \{\beta_i\}_{i \in \mathbb{Z}_+^d, |i| \leq 2n}$ to have a *representing measure* μ, it is necessary for the associated moment matrix $\mathcal{M}(n)(\beta)$ to be positive semidefinite, and for the algebraic variety associated to β, $\mathcal{V} \equiv \mathcal{V}_\beta$, to satisfy $\operatorname{rank} \mathcal{M}(n) \leq \operatorname{card} \mathcal{V}$ as well as the following *consistency* condition: if a polynomial $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$ vanishes on $\mathcal{V}$, then $p(\beta) := \sum_{|i| \leq 2n} a_i \beta_i = 0$.
- In joint work with Lawrence Fialkow and Michael Möller, we prove that for the *extremal* case ($\operatorname{rank} \mathcal{M}(n) = \operatorname{card} \mathcal{V}$), positivity of $\mathcal{M}(n)$ and consistency are sufficient for the existence of a (unique, $\operatorname{rank} \mathcal{M}(n)$-atomic) representing measure.

ABSTRACT

- The new results build on our operator-theoretic approach to truncated moment problems, based on matrix positivity and extension, which, via a “functional calculus” for the columns of the associated moment matrix, allows us to obtain existence theorems in case the columns satisfy one of several natural constraints.
- The extremal case is inherent in TMP:
  C. Bayer and J. Teichmann (2005) (extending a classical theorem of V. Tchakaloff and its successive generalizations by I.P. Mysovskikh, M. Putinar and RC-LF) recently proved that if $\beta^{(2n)}$ has a representing measure, then it has a **finitely atomic** representing measure.

- RC-LF showed that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $M(n)$ admits an extension to a positive moment matrix $M(n+k)$ (for some $k \geq 0$, which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $M(n+k+1)$.
- In many instances, $M(n+k+1)$ is an extremal moment matrix for which there is a computable rank $M(n+k)$-atomic representing measure μ. Clearly, μ is also a finitely atomic representing measure for $\beta^{(2n)}$, and every finitely atomic representing measure for $\beta^{(2n)}$ arises in this way.

rcurto@math.uiowa.edu; http://www.math.uiowa.edu/˜rcurto

THE TRUNCATED COMPLEX MOMENT PROBLEM

- Given $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \dots, \gamma_{0,2n}, \dots, \gamma_{2n,0}$, with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, the **TCMP** entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

  $$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \le i + j \le 2n);$$

  μ is called a **rep. meas.** for γ.
- In earlier joint work with L. Fialkow,
- We have introduced an approach based on matrix positivity and extension, combined with a new "functional calculus" for the columns of the associated **moment matrix**.

- We have shown that when the TCMP is of **flat data type**, a solution always exists; this is compatible with our previous results for

$$\begin{array}{ll} \text{supp } \mu \subseteq \mathbb{R} & \text{(Hamburger TMP)} \\ \text{supp } \mu \subseteq [0, \infty) & \text{(Stieltjes TMP)} \\ \text{supp } \mu \subseteq [a, b] & \text{(Hausdorff TMP)} \\ \text{supp } \mu \subseteq \mathbb{T} & \text{(Toeplitz TMP)} \end{array}$$

- Along the way we have developed new machinery for analyzing TMP's in **one or several real or complex variables**. For simplicity, in this talk we focus on **one complex variable or two real variables**, although several results have multivariable versions.

- Our techniques also give concrete algorithms to provide finitely-atomic rep. meas. whose atoms and densities can be explicitly computed.
- We have fully resolved, among others, the cases

$$\bar{Z} = \alpha 1 + \beta Z$$

and

$$Z^k = p_{k-1}(Z, \bar{Z}) \ \ (1 \leq k \leq [\frac{n}{2}] + 1; \deg p_{k-1} \leq k-1).$$

- We obtain applications to quadrature problems in numerical analysis.
- We have obtained a duality proof of a generalized form of the Tchakaloff-Putinar Theorem on the existence of quadrature rules for positive Borel measures on $\mathbb{R}^d$.

- More recently, we have begun to use our methods to solve FULL moment problems, by first solving truncated MP's, and then applying J. Stochel's limiting argument.
- Our matrix extension approach works equally well to **localize the support** of a rep. meas.
- In the specific case of $K := \text{supp } \mu$, a semi-algebraic set determined by a finite collection of complex polynomials $\mathcal{P} = \{p_i(z, \bar{z})\}_{i=1}^{m}$, i.e.,

$$K = K_P := \{z \in \mathbb{C} : p_i(z, \bar{z}) \geq 0, 1 \leq i \leq m\},$$

we obtain an existence criterion expressed in terms of positivity and extension properties of the moment matrix $M(n)(\gamma)$ associated to γ and of the localizing matrix M_{p_i} corresponding to each p_i.

POSITIVITY OF BLOCK MATRICES

THEOREM

(Smul'jan, 1959)

$$\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Leftrightarrow \begin{cases} A \geq 0 \\ B = AW \\ C \geq W^*AW \end{cases} .$$

Moreover, rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ =rank $A \Leftrightarrow C = W^*AW$.

COROLLARY

Let $A \geq 0$ *and assume* rank $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ =rank A. *Then* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0$.

BASIC POSITIVITY CONDITION

$\mathcal{P}_n$: polynomials p in z and $\bar{z}$, $\deg p \leq n$

- Given $p \in \mathcal{P}_n$, $p(z, \bar{z}) \equiv \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$,

$$0 \leq \int \mid p(z, \bar{z}) \mid^2 d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \int \bar{z}^{i+\ell} z^{j+k} d\mu(z, \bar{z})$$

$$= \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell, j+k}.$$

- To understand this **"matricial" positivity**, we introduce the following lexicographic order on the rows and columns of $M(n)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots$$

Define $M[i,j]$ as in

$$M[3,2] := \begin{pmatrix} \gamma_{32} & \gamma_{41} & \gamma_{50} \\ \gamma_{23} & \gamma_{32} & \gamma_{41} \\ \gamma_{14} & \gamma_{23} & \gamma_{32} \\ \gamma_{05} & \gamma_{14} & \gamma_{23} \end{pmatrix}$$

Then

$$(\text{\textbf{“matricial” positivity}}) \quad \sum_{ijk\ell} a_{ij} \bar{a}_{k\ell} \gamma_{i+\ell,j+k} \geq 0$$

$$\Leftrightarrow M(n) \equiv M(n)(\gamma) := \begin{pmatrix} M[0,0] & M[0,1] & \ldots & M[0,n] \\ M[1,0] & M[1,1] & \ldots & M[1,n] \\ \ldots & \ldots & \ldots & \ldots \\ M[n,0] & M[n,1] & \ldots & M[n,n] \end{pmatrix} \geq 0.$$

For example,

- $$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

- $$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{12} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In general,

$$M(n+1) = \begin{pmatrix} M(n) & B \\ B^* & C \end{pmatrix}$$

Similarly, one can build $M(\infty)$.

In the real case, $\mathcal{M}(n)_{ij} := \gamma_{i+j}, \ i,j \in \mathbb{Z}_+^2$.

Positivity Condition is not sufficient:

By modifying an example of K. Schmüdgen, we have built a family $\gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{06}, \ldots, \gamma_{60}$ with positive invertible moment matrix $M(3)$ but **no** rep. meas. But this can also be done for $n = 2$.

MOMENT PROBLEMS AND NONNEGATIVE POLYNOMIALS (FULL MP CASE)

- $\mathcal{M} := \{\gamma \equiv \gamma^{(\infty)} : \gamma$ admits a rep. meas. $\mu\}$
- $\mathcal{B}_+ := \{\gamma \equiv \gamma^{(\infty)} : M(\infty)(\gamma) \geq 0\}$

  Clearly, $\mathcal{M} \subseteq \mathcal{B}_+$

- (Berg, Christensen and Ressel) $\gamma \in \mathcal{B}_+$, γ bounded $\Rightarrow \gamma \in \mathcal{M}$
- (Berg and Maserick) $\gamma \in \mathcal{B}_+$, γ exponentially bounded $\Rightarrow \gamma \in \mathcal{M}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ finite rank $\Rightarrow \gamma \in \mathcal{M}$
- (RC and L. Fialkow) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ flat $\Rightarrow \gamma \in \mathcal{M}$

- $\mathcal{P}_+$: nonnegative poly's

  Σ^2 : sums of squares of poly's

  Clearly, $\Sigma^2 \subseteq \mathcal{P}_+$

Duality

For C a cone in $\mathbb{R}^{\mathbb{Z}_+^2}$, we let

$$C^* := \{\xi \in \mathbb{R}^{\mathbb{Z}_+^2} : \text{supp}(\xi) \text{ is finite and } \langle p, \xi \rangle \geq 0 \text{ for all } p \in C\}.$$

- (Riesz-Haviland) $\mathcal{P}_+^* = \mathcal{M}$

  For, consider the Riesz functional $\Lambda_\gamma(p) := p(\gamma) \equiv \langle p, \gamma \rangle$, which induces a map $\mathcal{M} \to \mathcal{P}_+^*$ $(\gamma \mapsto \Lambda_\gamma)$; Haviland's Theorem says that this maps is onto, that is, there exists μ r.m. for γ if and only if $\Lambda_\gamma \geq 0$ on $\mathcal{P}_+$.

  At present, we are attempting to formulate and prove a version of this result for TMP.

$\mathcal{P}_+ = \mathcal{M}^*$ (straightforward once we have a r.m.)

$\mathcal{B}_+ = (\Sigma^2)^*$ (straightforward)

(Berg, Christensen and Jensen) $(\mathcal{B}_+)^* = \Sigma^2$

$(n = 1)$ $\mathcal{P}_+ = \Sigma^2 \Rightarrow \mathcal{P}_+^* = (\Sigma^2)^* \Rightarrow \mathcal{M} = \mathcal{B}_+$ (Hamburger)

(Hilbert) Description of pairs (n, d) for which every poly of degree d in n indeterminates which is nonnegative on $\mathbb{R}^n$ is a sum of squares (cf. Reznick 2000)

Generally, SOS implies the existence of a representing measure.

FUNCTIONAL CALCULUS

For $p \in \mathcal{P}_n,\ p(z,\bar{z}) \equiv \sum_{0 \le i+j \le n} a_{ij} \bar{z}^i z^j$ define

$$p(Z,\bar{Z}) := \sum a_{ij} \bar{Z}^i Z^j.$$

If there exists a rep. meas. μ, then

$$p(Z,\bar{Z}) = 0 \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p).$$

The following is our analogue of recursiveness for the TCMP

(RG) **If** $p, q, pq \in \mathcal{P}_n$, **and** $p(Z,\bar{Z}) = 0,$

then $(pq)(Z,\bar{Z}) = 0.$

SINGULAR TMP; REAL CASE

- Given a finite family of moments, build moment matrix
- Identify all column relations
- Build algebraic variety $\mathcal{V}$
- Consider the ideal $\mathcal{I} \subseteq \mathcal{P} \equiv \mathbb{R}[x, y]$ generated by poly's arising from column relations
- When the moment matrix is flat, the ideal $\mathcal{I}$ is always radical, i.e., $\mathcal{I} = \sqrt{\mathcal{I}} := \left\{ f \in \mathcal{P} : f^k \in \mathcal{I} \text{ for some } k \geq 1 \right\}$
- If $\mathcal{V}$ is finite, then $\mathcal{I}$ is zero-dimensional, i.e., $V(\mathcal{I})$ is finite, where $V(\mathcal{I}) := \left\{ x \in \mathbb{C}^2 : f(x) = 0 \text{ for all } f \in \mathcal{I} \right\}$

SINGULAR TMP

- (Parrilo, 2002) Assume an ideal $\mathcal{I}$ is radical and zero-dimensional, and let f be a poly with $f \geq 0$ on $V(\mathcal{I})$. Then f is SOS modulo $\mathcal{I}$.
- (Lasserre, 2004) Assume $p \geq 0$ on $\mathbb{R}^2$, and let $\epsilon > 0$. Then there exists $t \in \mathbb{Z}_+$ such that
$$p + \varepsilon\left(\sum_{0}^{t} \frac{x^{2k} + y^{2k}}{k!}\right) \tag{5.1}$$
is SOS.

SINGULAR TMP

- Always true: $r \leq \text{card supp } \mu \leq v$, so if variety is finite there's a natural candidate for supp μ
- Finite rank case
- Flat case
- Extremal case
- Recursively generated relations
- Build positive extension, and check for extremality; repeat, and eventually flatten
- General case.

FIRST EXISTENCE CRITERION

THEOREM

(RC-LF, 1998) Let γ be a truncated moment sequence. TFAE:

- *(i) γ has a rep. meas.;*
- *(ii) γ has a rep. meas. with moments of all orders;*
- *(iii) γ has a compactly supported rep. meas.;*
- *(iv) γ has a finitely atomic rep. meas. (with at most $(n+2)(2n+3)$ atoms);*
- *(v) $M(n) \geq 0$ and for some $k \geq 0$ $M(n)$ admits a positive extension $M(n+k)$, which in turn admits a flat (i.e., rank-preserving) extension $M(n+k+1)$ (here $k \leq 2n^2+6n+6$)).*

CASE OF FLAT DATA

Recall: If μ is a rep. meas. for $M(n)$, then rank $M(n) \leq$ card supp μ.

γ is flat if $M(n) = \begin{pmatrix} M(n-1) & M(n-1)W \\ W^*M(n-1) & W^*M(n-1)W \end{pmatrix}$.

THEOREM

(RC-LF, 1996) If γ is flat and $M(n) \geq 0$, then $M(n)$ admits a unique flat extension of the form $M(n+1)$.

THEOREM

(RC-LF, 1996) The truncated moment sequence γ has a rank *$M(n)$-atomic rep. meas. if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$.*

To find μ concretely, let $r :=$rank $M(n)$ and look for the relation

$$Z^r = c_0 1 + c_1 Z + \ldots + c_{r-1} Z^{r-1}.$$

We then define

$$p(z) := z^r - (c_0 + \ldots + c_{r-1} z^{r-1})$$

and solve the Vandermonde equation

$$\begin{pmatrix} 1 & \cdots & 1 \\ z_0 & \cdots & z_{r-1} \\ \cdots & \cdots & \cdots \\ z_0^{r-1} & \cdots & z_{r-1}^{r-1} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \rho_1 \\ \cdots \\ \rho_{r-1} \end{pmatrix} = \begin{pmatrix} \gamma_{00} \\ \gamma_{01} \\ \cdots \\ \gamma_{0r-1} \end{pmatrix}.$$

Then

$$\mu = \sum_{j=0}^{r-1} \rho_j \delta_{z_j}.$$

LOCALIZING MATRICES

Consider the **full** MP

$$\int \bar{z}^i z^j \, d\mu = \gamma_{ij} \quad (i, j \geq 0),$$

where supp $\mu \subseteq K$, for K a closed subset of $\mathbb{C}$.

- The **Riesz functional** is given by

$$\Lambda_\gamma(\bar{z}^i z^j) := \gamma_{ij} \quad (i, j \geq 0).$$

- Riesz-Haviland:

  There exists μ with supp $\mu \subseteq K \Leftrightarrow \Lambda_\gamma(p) \geq 0$ for all p such that $p|_K \geq 0$.

If q is a polynomial in z and $\bar{z}$, and

$$K \equiv K_q := \{z \in \mathbb{C} : q(z, \bar{z}) \geq 0\},$$

then $L_q(p) := L(qp)$ must satisfy $L_q(p\bar{p}) \geq 0$ for μ to exist. For,

$$L_q(p\bar{p}) = \int_{K_q} qp\bar{p}\, d\mu \geq 0 \ \text{ (all } p).$$

- K. Schmüdgen (1991): If K_q is compact, $\Lambda_\gamma(p\bar{p}) \geq 0$ and $L_q(p\bar{p}) \geq 0$ for all p, then there exists μ with supp $\mu \subseteq K_q$.
- We shall establish a version of this result for truncated MP's.

First, recall that $p(Z, \bar{Z}) = 0$ implies supp $\mu \subseteq \mathcal{Z}(p)$. We define the algebraic variety of γ as

$$\mathcal{V}(\gamma) := \bigcap_{\substack{p \in \mathcal{P}_n \\ p(Z,\bar{Z})=0}} \mathcal{Z}(p),$$

and observe that rank $M(n) \leq$ card supp $\mu \leq$ card $V(\gamma)$, from which it follows that

$$\text{card } \mathcal{V}(\gamma) < \text{rank } M(n) \Rightarrow \text{there is } \textbf{no} \text{ rep. meas. } \mu.$$

LOCALIZATION OF SUPPORT: MAIN THEOREM

THEOREM

(RC-LF, 2000) Let $M(n) \geq 0$ *and suppose* $\deg(q) = 2k$ *or* $2k - 1$ *for some* $k \leq n$*. Then* $\exists\ \mu$ *with* $\operatorname{rank} M(n)$ *atoms and* $\operatorname{supp}\ \mu \subseteq K_q$ *if and only if* $\exists$ *a flat extension* $M(n+1)$ *for which* $M_q(n+k) \geq 0$*. In this case,* $\exists\ \mu$ *with exactly* $\operatorname{rank} M(n) - \operatorname{rank}\ M_q(n+k)$ *atoms in* $\mathcal{Z}(q)$*.*

REMARK

M. Laurent (2005) has recently found an alternative proof, using ideas from real algebraic geometry.

THE ALGEBRAIC VARIETY OF A TMP

Recall that if $\gamma^{(2n)}$ admits a rep. meas., then

$$\begin{aligned} M(n) &\equiv M(n)(\gamma) \geq 0 \\ & M(n) \text{ is } RG \\ \operatorname{rank} M(n) &\leq \operatorname{card} \mathcal{V}(\gamma). \end{aligned} \tag{10.1}$$

QUESTION

Assume $M(n)$ satisfies (10.1), and $M(n)$ is singular. Does γ admit a rep. meas.?

QUESTION

For which $p \in \mathcal{P}_n$ *do (10.1) and* $p(Z, \bar{Z}) = 0$ *imply that* γ *has a rep. meas.?*

(Fialkow) Consider $p(z, \bar{z}) \equiv z^k - q(z)$, with $\deg q < k$. If k is minimal, if γ satisfies (10.1) and if $p(Z, \bar{Z}) = 0$, then $B := \{1, Z, Z^2, ..., Z^{k-1}\}$ is lin. indep. Moreover, $k \geq \text{card } \mathcal{V}(\gamma) \geq \operatorname{rank} M(n) \geq k$, so B is indeed a basis for $\mathcal{C}_{M(n)}$.

It follows that $M(n)$ is flat, and it therefore admits a k-atomic rep. meas.

THE QUARTIC MOMENT PROBLEM

Recall the lexicographic order on the rows and columns of $M(2)$:

$$1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2$$

- $Z = A\,1$ (Dirac measure)
- $\bar{Z} = A\,1 + B\,Z$ (supp $\mu \subseteq$ line)
- $Z^2 = A\,1 + B\,Z + C\,\bar{Z}$ (flat extensions always exist)
- $\bar{Z}Z = A\,1 + B\,Z + C\,\bar{Z} + D\,Z^2$

$$\begin{aligned} D &= 0 \Rightarrow \bar{Z}Z = A\,1 + B\,Z + \bar{B}\,\bar{Z} \text{ and } C = \bar{B} \\ &\Rightarrow (\bar{Z} - B)(Z - \bar{B}) = A + |B|^2 \\ &\Rightarrow \bar{W}W = 1 \text{ (circle), for } W := \frac{Z - \bar{B}}{\sqrt{A + |B|^2}}. \end{aligned}$$

The functional calculus we have constructed is such that $p(Z, \bar{Z}) = 0$ implies supp $\mu \subseteq \mathcal{Z}(p)$.

When $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ is a basis for $\mathcal{C}_{M(2)}$, the associated algebraic variety is the zero set of a real quadratic equation in $x := \text{Re}[z]$ and $y := \text{Im}[z]$.

Using the flat data result, one can reduce the study to cases corresponding to the following four real conics:

(a)	$\bar{W}^2 = -2iW + 2i\bar{W} - W^2 - 2\bar{W}W$	parabola; $y = x^2$
(b)	$\bar{W}^2 = -4i1 + W^2$	hyperbola; $yx = 1$
(c)	$\bar{W}^2 = W^2$	pair of intersect. lines; $yx = 0$
(d)	$\bar{W}W = 1$	unit circle; $x^2 + y^2 = 1$.

THEOREM QUARTIC

(RC-LF, 2005) Let $\gamma^{(4)}$ *be given, and assume* $M(2) \geq 0$ *and* $\{1, Z, \bar{Z}, Z^2, \bar{Z}Z\}$ *is a basis for* $\mathcal{C}_{M(2)}$. *Then* $\gamma^{(4)}$ *admits a rep. meas.* μ. *Moreover, it is possible to find* μ *with* card supp $\mu = \text{rank } M(2)$, *except in some cases when* $\mathcal{V}(\gamma^{(4)})$ *is a pair of intersecting lines, in which cases there exist* μ *with* card supp $\mu \leq 6$.

Consider now the following property for a polynomial $P \in \mathbb{R}_n[x, y]$:

$$\beta \equiv \beta^{(2n)} \text{ has a rep. meas. supported in } \mathcal{Z}(P) \text{ if and only if} \qquad (A'_n)$$

$$\mathcal{M}(n)(\beta) \text{ is positive semi-definite, RG },$$

$$P(X, Y) = 0 \text{ in } \mathcal{C}_{\mathcal{M}(n)}, \text{ and } \text{ rank } \mathcal{M}(n) \leq \text{card } \mathcal{V}(\mathcal{M}(n)).$$

Polynomials which satisfy (A'_n) form an attractive class, because if P satisfies (A'_n), then the degree-$2n$ moment problem on $P(x, y) = 0$ can be solved by concrete tests involving only elementary linear algebra and the calculation of roots of polynomials.

THEOREM

(RC-LF, 2005) If $\deg P \leq 2$*, then* P *satisfies* (A'_n) *for every* $n \geq \deg P$*.*

This Theorem is motivated in part by results of J. Stochel, who solved the **full** moment problem on planar curves of degree at most 2. Paraphrasing Stochel's work (i.e., translating from the language of *moment sequences* into the language of moment matrices), we consider the following property of a polynomial P:

$$\beta^{(\infty)} \text{ has a rep. meas. supported in } P(x,y)=0 \tag{A}$$

$$\text{if and only if } \mathcal{M}(\infty)(\beta) \geq 0 \text{ and } P(X,Y)=0 \text{ in } \mathcal{C}_{\mathcal{M}(\infty)}.$$

THEOREM

(Stochel, 1992) If $\deg P \leq 2$*, then* P *satisfies* (A)*.*

- Stochel also proved that there exist polynomials of degree 3 that do not satisfy (A).
- The link between TMP and FMP is provided by another result of Stochel (2001):

THEOREM

$\beta^{(\infty)}$ has a rep. meas. supported in a closed set $K \subseteq \mathbb{R}^2$ if and only if, for each n, $\beta^{(2n)}$ has a rep. meas. supported in K.

Recall

- Riesz-Haviland: There exists μ with supp $\mu \subseteq K \Leftrightarrow \Lambda_\gamma(p) \geq 0$ for all p such that $p|_K \geq 0$.

- For TMP, the natural analogue won't work; for example, if

  $d = 1,\ K = \mathbb{R}$, and

$$\mathcal{M}(2) := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \geq 0,$$

  then Λ_β is $\mathbb{R}$-positive, but no r.m. exists.

- **Possible Analogue**: $\beta^{(2n)}$ admits a K-r.m. if and only if Λ_β admits a K-positive extension $\Lambda : \mathcal{P}_{2n+2} \to \mathbb{R}$.

- Another useful and relevant notion is that of $\mathcal{V}$-positivity.

EXTREMAL MP; $r = v$

$$\beta_i = \int_{\mathbb{R}^d} x^i \; d\mu, \;\; |i| \leq 2n; \tag{12.1}$$

$\mathcal{P} \equiv \mathbb{R}^d[x] = \mathbb{R}[x_1, ..., x_d]$: space of real valued d-variable polynomials

$\mathcal{P}_k \equiv \mathbb{R}^d_k[x]$: the subspace of $\mathcal{P}$ consisting of polynomials p with $\deg p \leq k \;\; (k \geq 1)$.

$\Lambda \equiv \Lambda_\beta : \mathcal{P}_{2n} \to \mathbb{R}$ (Riesz functional): if $p(x) \equiv \sum_{|i| \leq 2n} a_i x^i$, then $\Lambda(p) := \sum_{|i| \leq 2n} a_i \beta_i$

- In the presence of a representing measure μ, we have $\Lambda(p) = \int p \; d\mu$.

$\hat{p}$: coefficient vector (a_i) of p.

$\mathcal{M}(n) \equiv \mathcal{M}(n)(\beta)$: moment matrix, with rows and columns X^i indexed by the monomials of $\mathcal{P}_n$ in degree-lexicographic order

$d = n = 2$: the columns of $\mathcal{M}(2)$ are denoted as $1, X_1, X_2, X_1^2, X_2X_1, X_2^2$

- $\mathcal{M}(n)$ is a real symmetric matrix characterized by

$$\langle \mathcal{M}(n)\hat{p}, \hat{q} \rangle = \Lambda(pq) \ (p, q \in \mathcal{P}_n). \tag{12.2}$$

- If μ is a representing measure for β, then $\langle \mathcal{M}(n)\hat{p}, \hat{p} \rangle = \Lambda(p^2) = \int p^2 d\mu \geq 0$; it follows that $\mathcal{M}(n) \geq 0$).

The *algebraic variety* of β is

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, \hat{p} \in \ker \mathcal{M}(n)} \mathcal{Z}_p,$$

where $\mathcal{Z}_p = \{x \in \mathbb{R}^d : p(x) = 0\}$.

- If β admits a representing measure μ, then

$$p \in \mathcal{P}_n \text{ satisfies } \hat{p} \in \ker \mathcal{M}(n) \Leftrightarrow \text{supp } \mu \subseteq \mathcal{Z}_p$$

Thus supp $\mu \subseteq \mathcal{V}$, so $r :=$ rank $\mathcal{M}(n)$ and $v :=$ card $\mathcal{V}$ satisfy

$$r \le \text{card supp } \mu \le v.$$

If $p \in \mathcal{P}_{2n}$ and $p|_{\mathcal{V}} \equiv 0$, then $\Lambda(p) = \int p \, d\mu = 0$.

BASIC NECESSARY CONDITIONS FOR THE EXISTENCE OF A REPRESENTING MEASURE

- $$\text{(Positivity)}\ \mathcal{M}(n) \geq 0 \tag{12.3}$$

- $$\text{(Consistency)}\ p \in \mathcal{P}_{2n},\ p|_{\mathcal{V}} \equiv 0 \Longrightarrow \Lambda(p) = 0 \tag{12.4}$$

- $$\text{(Variety Condition)}\ r \leq v, \text{ i.e., } \operatorname{rank}\ \mathcal{M}(n) \leq \operatorname{card}\ \mathcal{V}. \tag{12.5}$$

- Consistency implies

$$\text{(Recursiveness)}\ p, q, pq \in \mathcal{P}_n,\ \hat{p} \in \ker \mathcal{M}(n) \Longrightarrow \widehat{pq} \in \ker \mathcal{M}(n). \tag{12.6}$$

Previous results:

- For $d = 1$ (the T *Hamburger* MP for $\mathbb{R}$), positivity and recursiveness are sufficient
- For $d = 2$, there exists $\mathcal{M}(3) > 0$ for which β has no representing measure
- In general, *Positivity*, *Consistency* and the *Variety Condition* are **not** sufficient.

QUESTION C

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, β is consistent, and $r \leq v$, does β admit a representing measure?

More generally, the following question remained unsolved until very recently.

QUESTION RG

Suppose $\mathcal{M}(n)(\beta)$ is singular. If $\mathcal{M}(n)$ is positive, recursively generated, and $r \leq v$, does β admit a representing measure?

- RC-LF: If $d = 2$ and $\mathcal{M}(n)\hat{p} = 0$ for some p with deg $p \leq 2$, then Question RG has an affirmative answer. (Theorem Quartic)
- RC-LF-MM: If $d = 2$, $\widehat{y - x^3} \in \ker \mathcal{M}(n)$ and $r = v \leq 7$, then Question RG has an affirmative answer.
- RC-LF-MM: If $d = 2$, $\widehat{y - x^3} \in \ker \mathcal{M}(n)$ and $r = v = 8$, **then Question RG has a negative answer.**

The next result gives an affirmative answer to Question C in the *extremal* case, i.e., $r = v$.

THEOREM EXT

(RC-LF and M. Möller, 2005) For $\beta \equiv \beta^{(2n)}$ **extremal**, *i.e.,* $r = v$, *the following are equivalent:*

(i) β *has a representing measure;*

(ii) β *has a unique representing measure, which is* rank $\mathcal{M}(n)$*-atomic (minimal);*

(iii) $\mathcal{M}(n) \geq 0$ *and* β *is consistent.*

- In many cases, the conditions of Theorem EXT provide a concrete solution to the extremal case of TMP. Indeed, only elementary linear algebra is required to verify that $\mathcal{M}(n) \geq 0$, to compute rank $\mathcal{M}(n)$, and to identify the column relations which define $\mathcal{V}$.

- If the points of $\mathcal{V}$ can be computed exactly, then only elementary linear algebra is required to verify that β is consistent.

- Question RG is significant because recursiveness is generally a simpler condition to work with than consistency. For example, we often have $M(n) \geq 0$ and $M(n-1) > 0$ (positive definite), in which case $M(n)$ is obviously recursively generated, but we do not know whether $r \leq v$ implies that $M(n)$ is consistent in this case.

- The extremal case is inherent in TMP: C. Bayer and J. Teichmann (2006) (extending a classical theorem of V. Tchakaloff and its successive generalizations by I.P. Mysovskikh, M. Putinar and RC-LF) recently proved that if $\beta^{(2n)}$ has a representing measure, then it has a **finitely atomic** representing measure;

- RC-LF showed that $\beta^{(2n)}$ has a finitely atomic representing measure if and only if $M(n)$ admits an extension to a positive moment matrix $M(n+k)$ (for some $k \geq 0$, which in turn admits a rank-preserving (i.e., *flat*) moment matrix extension $M(n+k+1)$;

- In many instances, $M(n+k+1)$ is an extremal moment matrix for which there is a computable rank $M(n+k)$-atomic representing measure μ. Clearly, μ is also a finitely atomic representing measure for $\beta^{(2n)}$, and every finitely atomic representing measure for $\beta^{(2n)}$ arises in this way.

- there exist extremal TCMP of arbitrarily large degree

- Moment theory can sometimes be used to estimate the number and location of the zeros of a prescribed polynomial; for example, we can use TMP tecniques to show that the polynomial

$$\text{p(z)} \equiv z^{2n} + az^{2n-1} - az - 1 \quad (0 < a < 1)$$

has $2n$ distinct zeros, all in the unit circle.

REAL IDEALS AND NECESSARY CONDITIONS

- If $\beta^{(2n)}$ has a representing measure, then the Riesz functional

  $$\Lambda : \mathcal{P}_{2n} \to \mathbb{R}, \quad \Lambda(x^i) := \beta_i \equiv \int_{\mathbb{R}^d} x^i \, d\mu \ \ (|i| \leq 2n),$$

  is square positive, that is,

  $$p \in \mathcal{P}_n \Rightarrow \Lambda(p^2) \geq 0.$$

- If we assume that for a representing measure μ all moments

  $$\int_{\mathbb{R}^d} x^i d\mu, \quad i \in \mathbb{Z}_+^d$$

  are convergent, then we can extend Λ to $\mathcal{P}$ by letting

  $$\Lambda(x^i) := \int_{\mathbb{R}^d} x^i d\mu, \quad i \in \mathbb{Z}_+^d,$$

  thus obtaining a square positive functional over $\mathcal{P}$.

Under this assumption the set

$$\mathcal{I} := \{p \in \mathcal{P} : \Lambda(p^2) = 0\}$$

is a *real ideal*, i.e., it is an ideal ($p_1, p_2 \in \mathcal{I} \Rightarrow p_1 + p_2 \in \mathcal{I}$ and $p \in \mathcal{I},\ q \in \mathcal{P} \Rightarrow pq \in \mathcal{I}$) and satisfies one of the two equivalent conditions:

(i) For $s \in \mathbb{Z}_+, p_1, \ldots, p_s \in \mathcal{P} : \sum_{i=1}^{s} p_i^2 \in \mathcal{I} \Rightarrow \{p_1, \ldots, p_s\} \subseteq \mathcal{I}$
(ii) There exists $G \subseteq \mathbb{R}^d$such that for all $p \in \mathcal{P} : p|_G \equiv 0 \Rightarrow p \in \mathcal{I}$.

If $\mathcal{I}$ is a real ideal, then one may take for $G \subseteq \mathbb{R}^d$ the real variety

$$V_{\mathbb{R}}(\mathcal{I}) := \{w \in \mathbb{R}^d : f(w) = 0 \quad (\text{all } f \in \mathcal{I})\}.$$

If $\mathcal{I}$ is an ideal, its subset $\mathcal{I}_k := \mathcal{I} \cap \mathcal{P}_k$ is an $\mathbb{R}$-vector subspace of $\mathcal{P}_k$. One can then introduce the *Hilbert function* of $\mathcal{I}$ by

$$H_{\mathcal{I}}(k) := \dim \mathcal{P}_k - \dim \mathcal{I}_k, \quad k \in \mathbb{Z}_+.$$

- Both $k \mapsto \dim \mathcal{I}_k$ and $k \mapsto H_{\mathcal{I}}(k)$ are nondecreasing functions.
- For sufficiently large k, say $k \geq k_0$, $H_{\mathcal{I}}(k)$ becomes a polynomial in k, the so called *Hilbert polynomial of $\mathcal{I}$*, whose degree equals the dimension of $\mathcal{I}$.

Assume now that $\beta^{(2n)}$ admits a representing measure μ. Then, irrespective of whether the Riesz functional $\Lambda : \mathcal{P}_n \longrightarrow \mathbb{R}$ can be extended to a square positive functional $\Lambda : \mathcal{P} \longrightarrow \mathbb{R}$, we can define the ideal

$$\mathcal{I}(\mu) := \{p \in \mathcal{P} : p|_{\text{supp } \mu} \equiv 0\}. \tag{13.1}$$

Since supp $\mu \subseteq \mathbb{R}^d$, $\mathcal{I}(\mu)$ is a real ideal, which we will call the *real ideal of* $\beta^{(2n)}$.

LEMMA

Assume $\beta^{(2n)}$ has a representing measure, and let $\mathcal{I}(\mu)$ be its real ideal. Then

$$\{p \in \mathcal{P}_n : \mathcal{M}(n)\hat{p} = 0\} = \mathcal{I}(\mu) \bigcap \mathcal{P}_n. \tag{13.2}$$

If $t_1, \ldots, t_N$ denote the monomials $x^i \in \mathcal{P}_n$ in degree-lexicographic order, then the row vectors of $\mathcal{M}(n)$ and the row vectors of $W_n := \{(t_1(w), \ldots, t_N(w)) : w \in \operatorname{supp}\ \mu\}$ span the same subspace of $\mathbb{R}^N$; in particular,

$$\operatorname{rank}\ \mathcal{M}(n) = H_{\mathcal{I}(\mu)}(n). \tag{13.3}$$

MOMENT MATRICES AND CONSISTENCY

- Recall that

$$\mathcal{V} \equiv \mathcal{V}_\beta := \bigcap_{p \in \mathcal{P}_n, p(X)=0} \mathcal{Z}_p.$$

Let $\mathcal{P}_n|_{\mathcal{V}}$ denote the restriction to $\mathcal{V}$ of the polynomials in $\mathcal{P}_n$, and consider the mapping $\varphi_\beta : \mathcal{C}_{M(n)} \to \mathcal{P}_n|_{\mathcal{V}}$ given by $p(X) \mapsto p|_{\mathcal{V}}$. The map φ_β is well-defined, and if β has a representing measure μ then φ_β is 1-1.

PROPOSITION

Let β, φ_β and $M(n)(\beta)$ be as before. Then

$$\beta \text{ consistent} \Longrightarrow \varphi_\beta \text{ 1-1} \Longrightarrow M(n)(\beta) \text{ recursively generated.}$$

PROPOSITION

For $d = 2$ *(the plane), if* $M(n)(\beta)$ *is recursively generated and* $\mathcal{V}_\beta$ *is a proper, infinite irreducible curve, then* β *is consistent.*

PROBLEM

Study the solubility of TMP on the irreducible algebraic set $y^2 - x^3 = 0$.

The FMP associated to this curve has been studied by Powers-Scheiderer and by Schmüdgen.

AN EXAMPLE WITH $r < v < +\infty$

EXAMPLE

Consider

$$\mathcal{M}(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 2 & 5 & 14 & 42 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 1 & 0 & 0 & 2 & 5 & 14 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 5 & 14 & 42 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 14 & 42 & 132 & 0 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 & 5 & 14 & 42 & 132 \\ 0 & 5 & 14 & 0 & 0 & 0 & 14 & 42 & 132 & 429 \\ 0 & 14 & 42 & 0 & 0 & 0 & 42 & 132 & 429 & 2000 \\ 0 & 42 & 132 & 0 & 0 & 0 & 132 & 429 & 2000 & 338881 \end{pmatrix}.$$

We have $\mathcal{M}(3) \geq 0$, $\mathcal{M}(2) > 0$, $r = 8$,

$$Y = X^3, \tag{15.1}$$

and

$$Y^3 = q(X, Y). \tag{15.2}$$

where $q(x, y) := -2285x + 5720y - 34441yx^2 + 578y^2x$. Here $v = 9$. Thus, $\mathcal{M}(3)$ is positive, recursively generated, $r < v$, and the minimal representing measure for $\beta^{(6)}$ is v-atomic.

(Previously known only when $\mathcal{V} = \mathbb{R}^2$, but here $\mathcal{V}$ is finite.

A VERSION OF RIESZ-HAVILAND FOR TMP

Recall:

$\mathcal{M} := \{\gamma \equiv \gamma^{(\infty)} : \gamma$ admits a representing measure $\mu\}$

$\mathcal{B}_+ := \{\gamma \equiv \gamma^{(\infty)} : M(\infty)(\gamma) \geq 0\}$

Clearly, $\mathcal{M} \subseteq \mathcal{B}_+$.

(i) $\gamma \in \mathcal{B}_+$, γ bounded $\Rightarrow \gamma \in \mathcal{M}$ (BCR)

(ii) $\gamma \in \mathcal{B}_+$, γ exponentially bounded $\Rightarrow \gamma \in \mathcal{M}$ (BeMa)

(iii) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ finite rank $\Rightarrow \gamma \in \mathcal{M}$ (CF)

(iv) $\gamma \in \mathcal{B}_+$, $M(\gamma)$ flat $\Rightarrow \gamma \in \mathcal{M}$ (CF)

- $\mathcal{P}_+$: nonnegative poly's

  Σ^2 : sums of squares of poly's

  Clearly, $\Sigma^2 \subseteq \mathcal{P}_+$

Duality

For C a cone in $\mathbb{R}^{\mathbb{Z}^2_+}$, we let

$$C^* := \{\xi \in \mathbb{R}^{\mathbb{Z}^2_+} : \text{supp}(\xi) \text{ is finite and } \langle p, \xi \rangle \geq 0 \text{ for all } p \in C\}.$$

- (Riesz-Haviland) $\mathcal{P}_+^* = \mathcal{M}$

  For, consider the Riesz functional $\Lambda_\gamma(p) := p(\gamma) \equiv \langle p, \gamma \rangle$, which induces a map $\mathcal{M} \to \mathcal{P}_+^*$ $(\gamma \mapsto \Lambda_\gamma)$; Haviland's Theorem says that this maps is onto, that is, there exists μ r.m. for γ if and only if $\Lambda_\gamma \geq 0$ on $\mathcal{P}_+$.

$\mathcal{P}_+ = \mathcal{M}^*$ (straightforward once we have a r.m.)

$\mathcal{B}_+ = (\Sigma^2)^*$ (straightforward)

(Berg, Christensen and Jensen) $(\mathcal{B}_+)^* = \Sigma^2$

$(n = 1)$ $\mathcal{P}_+ = \Sigma^2 \Rightarrow \mathcal{P}_+^* = (\Sigma^2)^* \Rightarrow \mathcal{M} = \mathcal{B}_+$ (Hamburger)

We say that the Riesz functional L is *K-positive* if $p \in \mathcal{P}$ and $p|K \geq 0$ $\Rightarrow$L(p)≥ 0; when $K = \mathbb{R}^2$, we say simply that L is *positive*.

In TMP, K-positivity is a necessary (but not sufficient) condition for a K-representing measure μ

THEOREM

$\beta \equiv \beta^{(2n)}$ *admits a K-representing measure if and only if* L_β *admits a K-positive linear extension* $L : \mathcal{P}_{2n+2} \longmapsto \mathbb{R}$.

This Theorem implies Riesz-Haviland, via Stochel's Theorem.

- **Main tool**. Let $\beta \equiv \beta^{(2n)}$ and let $K \subseteq \mathbb{R}^d$ be closed. Assume L_β is K-positive. Then $\beta \equiv \beta^{(2n-1)}$ has a K-representing measure.
- Related notion: $\mathcal{V}$-positivity, where $\mathcal{V} \equiv V(\mathcal{M}(n))$ is the algebraic variety associated to $\beta^{(2n)}$.
- We have been able to prove that $\mathcal{V}$-positivity for $L_{\beta^{(2n)}}$ implies the existence of representing measures for $\beta^{(2n)}$ when $d = 1$ or when $\mathcal{V}$ is compact.

PROBLEM

Let $\beta \equiv \beta^{(2n)}$ be given, and assume that L_β is $\mathcal{V}$-positive. Is TMP soluble?

$\mathcal{Q} \equiv \{q_1, \ldots, q_m\} \subseteq \mathcal{P}$

$K_Q := \{x \in \mathbb{R}^d : q_i(x) \geq 0 \ (1 \leq i \leq m)\}$.

$\langle M_p \hat{f}, \hat{g} \rangle = L(fgp) \ (f, g \in \mathcal{P})$.

If μ is a representing measure supported in $K_{\mathcal{Q}}$ and $r := q_{i_1} \cdots q_{i_s}$ (a prod. of dist. q_i's), then

$\langle M_r \hat{f}, \hat{f} \rangle = L(rf^2) = \int q_{i_1} \cdots q_{i_s} f^2 \ d\mu \geq 0$

(since $q_{i_j}|$ supp $\mu \geq 0$), whence $M_r \geq 0$.

- Consider the following property for $K_{\mathcal{Q}}$:
  (**S**) $\beta \equiv \beta^{(\infty)}$ has a r.m. supp. in $K_{\mathcal{Q}} \Leftrightarrow M \geq 0$ and $M_r \geq 0$ for each polynomial r that is a prod. of dist. q_i's.

- When $K_{\mathcal{Q}}$ is compact, Schmüdgen established property (**S**). He then used property (**S**) to establish a structure theorem for polynomials that are strictly positive on $K_{\mathcal{Q}}$.

- Consider the convex cone in $\mathcal{P}$ defined by

$$\Sigma_{\mathcal{Q}} := \{p \in \mathcal{P} : p = \sum f_j^2 + \sum g_k^2 r_k : \ f_j, g_k \in \mathcal{P}, \ r_k \text{ is prod. of dist. } q_i\text{'s}\}. \tag{15.3}$$

THEOREM

*A semialgebraic set $K_{\mathcal{Q}}$ satisfies (**S**) if and only if each polynomial that is strictly positive on $K_{\mathcal{Q}}$ belongs to $\Sigma_{\mathcal{Q}}$.*

Several authors have used techniques from real algebra to develop structure theorems for positive polynomials on certain noncompact $K_{\mathcal{Q}}$ satisfying (**S**): Kuhlmann-Marshall, Powers-Reznick, Powers-Scheiderer, Prestel, Putinar, Scheiderer, Schmüdgen.

They then derived moment theorems for measures supported on $K_{\mathcal{Q}}$.

- The analogue of condition (**S**) will be given by

  (**S**$_n$) $\beta^{(2n)}$ has a $K_{\mathcal{Q}}$-r.m. $\Leftrightarrow \mathcal{M}(n) \geq 0$ and $\mathcal{M}_r(n) \geq 0$ for every polynomial r that is prod. of dist. q_i's.

- We can now consider the following convex cone in $\mathcal{P}_{2n}$:

$$\begin{aligned} \Sigma_{\mathcal{Q}}^n \quad &: \quad = \{p \in \mathcal{P}_{2n} : p = \sum f_i^2 + \sum g_j^2 s_j, \\ f_i \quad &\in \quad \mathcal{P}_n, \ s_j \text{ is prod. of dist. } q_k\text{'s, and } g_j^2 s_j \in \mathcal{P}_{2n}\}. \end{aligned}$$

CONJECTURE

Suppose $K_{\mathcal{Q}}$ is compact. Then $K_{\mathcal{Q}}$ satisfies $(\mathbf{S}_n)$ *if and only if* $\Sigma_{\mathcal{Q}}^n$ *contains each polynomial in* $\mathcal{P}_{2n}$ *that is strictly positive on* $K_{\mathcal{Q}}$*.*

CONJECTURE

Let $d \geq 1$ and suppose $\mathcal{V}$ is compact. If L_β is $\mathcal{V}$-positive, then β admits a $\mathcal{V}$-representing measure μ with $\text{card supp } \mu \leq \dim \mathcal{P}_{2n}$*.*

SUMMARY

- Given a finite family of moments, build moment matrix
- Identify all column relations, and build algebraic variety $\mathcal{V}$
- Consider the ideal generated by poly's arising from column relations
- Always true: $r \leq \text{card supp } \mu \leq v$
- Finite rank case and flat case were completed earlier
- Extremal case is now done
- General singular case still open
- Riesz-Haviland for TMP is within reach
- Invertible case remains a big mystery...