Closed-form solutions to certain Moment Problems with Applications to Business

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Outline

1. Introduction
2. Extensions of Lo’s bound
3. Arbitrage bounds
4. Robust portfolio allocation
5. Final Remarks
General Moment Problem

\[ \sup / \inf \quad \mathbb{E}_\pi(f(s)) \equiv \int_{\mathbb{R}^n} f(s) d\pi(s) \]

s.t. \[ \mathbb{E}_\pi(1) = 1 \]
\[ \mathbb{E}_\pi(f^\alpha(s)) = \sigma_\alpha, \ \forall \ \alpha \in \mathcal{I} \]
\[ \pi \text{ is a probability distribution (p.d.) in } \mathcal{D}. \]

\textbf{Generalized Tchebycheff Inequalities} in Probability Theory

Example (One-Sided Tchebycheff Inequality)

Sharp upper bound on \[ \mathbb{P}(S \leq \mu - a) : \]

\[ \sup \quad \mathbb{E}_\pi(\mathbb{I}_{(-\infty,\mu-a]}) = \frac{\sigma^2}{\sigma^2 + a^2} \]

s.t. \[ \mathbb{E}_\pi(1) = 1 \]
\[ \mathbb{E}_\pi(s) = \mu \]
\[ \mathbb{E}_\pi(s^2) = \sigma^2 + \mu^2 \]
\[ \pi \text{ is a p.d. in } \mathbb{R} \]
General Moment Problem

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\sup \text{ / } \inf \quad \mathbb{E}_\pi(f(s)) \equiv \int_{\mathbb{R}^n} f(s) \, d\pi(s)
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s.t.

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\mathbb{E}_\pi(1) = 1 \quad \mathbb{E}_\pi(f^\alpha(s)) = \sigma_\alpha, \quad \forall \alpha \in \mathcal{I}
\]

\(\pi\) is a probability distribution (p.d.) in \(\mathcal{D}\).

- **Generalized Tchebycheff Inequalities** in Probability Theory

Example (One-Sided Tchebycheff Inequality)

Sharp upper bound on \(\mathbb{P}(S \leq \mu - a)\):

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\sup \quad \mathbb{E}_\pi(\mathbb{1}_{(-\infty, \mu-a]}) = \frac{\sigma^2}{\sigma^2 + a^2}
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s.t.

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\mathbb{E}_\pi(1) = 1 \quad \mathbb{E}_\pi(s) = \mu \quad \mathbb{E}_\pi(s^2) = \sigma^2 + \mu^2
\]

\(\pi\) is a p.d. in \(\mathbb{R}\)
Applications

- Lots of applications in:
  - Stochastic Programming
  - Inventory
  - Finance
  - Probability
  - Actuarial Science

- More so, thanks to the recent advances in numerical and theoretical techniques to solve moment problems

- Here, we are interested in closed-form solutions:
  - Practical & theoretical significance
  - Easy computation
  - Easy sensitivity analysis
  - Optimization over parameters
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A Call Option

- An option, is a financial instrument whose price depends on the price of another asset; this asset is called the underlying asset.
- A call option gives the holder the right to buy the underlying stock at an agreed price (the strike price), at an agreed future date (maturity time).
- Therefore, the payoff of a call option is:

\[
\max\{0, S - K\} := (S - K)^+
\]

where \(S\) represents the underlying asset price at maturity, and \(K\) is the strike price of the option.

- European: can only be exercised at maturity date
- American: can be exercised at any time before maturity.
A Call Option (cont.)

- **Strike price**: The price at which the holder of the call option can buy the underlying asset.
- **Asset price**: The price of the underlying asset.
- **Profit**: The difference between the strike price and the asset price at maturity.
- **Exercise**: The decision to buy the underlying asset at the strike price.
- **No exercise**: The decision not to buy the underlying asset at the strike price.

**Graph**: The diagram shows the relationship between the asset price and strike price over time, with the decision points for exercising or not exercising the call option indicated by dots at maturity (T).
Applications in Finance

Option Pricing:

- **Black-Scholes pricing**: full knowledge of underlying asset distribution is used to get option price
  \[
  \text{Option Price} = \mathbb{E}_{\pi^*} (\text{Option Payoff})
  \]

- **Semiparametric bounds**: partial knowledge of underlying asset distribution is used to get bounds on option price
  \[
  \text{Option Price} \in \left[ \sup / \inf (\mathbb{E}_{\pi} (\text{Option Payoff}) : \pi \in \mathcal{M}) \right]
  \]

Robust Portfolio Optimization:

- Instead of full knowledge of underlying returns distribution to calculate risk, partial information is used to calculate **worst-case risk** (El Ghaoui & Oks & Oustry)
Option Pricing:

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- Option Pricing:
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- Robust Portfolio Optimization:
  - Instead of full knowledge of underlying returns distribution to calculate risk, partial information is used to calculate **worst-case risk** (El Ghaoui & Oks & Oustry)
Applications to Finance (cont.)

Example

For a European call option: (payoff $\left(\text{payoff} := (s - 1)^+ := \max\{s - 1, 0\}\right)$, $s$: asset price at maturity, $1$: strike w.l.o.g.)

- **Black-Scholes formula**: Lognormal (underlying) asset dist.
- **Lo’s upper price bound**: Mean and variance of asset dist.

Theorem (Lo)

Let $\mu \geq 0$, $\gamma \geq 1$. Then

$$\sup_{\pi \text{ a p.d. in } \mathbb{R}_+} \mathbb{E}_{\pi}((s - 1)^+) \quad \text{s.t.} \quad \begin{align*}
\mathbb{E}_{\pi}(1) &= 1 \\
\mathbb{E}_{\pi}(s) &= \mu \\
\mathbb{E}_{\pi}(s^2) &= \gamma \mu^2
\end{align*}$$

$$= \begin{cases} 
\frac{\mu - 1}{\gamma} & \mu > \frac{2}{\gamma}, \\
\frac{1}{2} \left( \frac{\mu - 1 + \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2}}{\sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2}} \right) & \mu \leq \frac{2}{\gamma}.
\end{cases}$$
Applications to Finance (cont.)

**Example**

For a European call option: (payoff := \((s - 1)^+ := \max\{s - 1, 0\}\), \(s\): asset price at maturity, \(1\): strike w.l.o.g.)

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**Theorem (Lo)**

*Let \(\mu \geq 0, \gamma \geq 1\). Then*

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\begin{align*}
\sup_{\pi \text{ a p.d. in } \mathbb{R}_+} & \quad \mathbb{E}_\pi ((s - 1)^+) \\
\text{s.t.} & \quad \mathbb{E}_\pi (1) = 1 \\
& \quad \mathbb{E}_\pi (s) = \mu \\
& \quad \mathbb{E}_\pi (s^2) = \gamma \mu^2
\end{align*}
\]

\[
= \begin{cases} 
\mu - \frac{1}{\gamma} & \mu > \frac{2}{\gamma}, \\
\frac{1}{2} \left( \mu - 1 + \frac{\sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2}}{(\gamma - 1)\mu^2 + (\mu - 1)^2} \right) & \mu \leq \frac{2}{\gamma}.
\end{cases}
\]
Applications to Finance (cont.)

Example

For a European call option: (payoff \( = (s - 1)^+ \) \( = \max\{s - 1, 0\} \),
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\pi & \text{ a p.d. in } \mathbb{R}_+, \\
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\end{cases}
\]
Inf version of Lo’s bound

**Theorem**

For $\mu \geq 0$, $\gamma \geq 1$, let

$$z = \inf \quad \mathbb{E}_\pi ((s - 1)^+)$$

s.t.

- $\mathbb{E}_\pi (1) = 1$
- $\mathbb{E}_\pi (s) = \mu$
- $\mathbb{E}_\pi (s^2) = \gamma \mu^2$

$\pi$ a p.d. in $\mathbb{R}_+$, then $z = (\mu - 1)^+$

**Proof.**

- Clearly, $z \geq (\mu - 1)^+$

Consider for

- $0 < \epsilon < 1$, $n > 0$, $\mu(1 - \frac{1}{n}\sqrt{\gamma - 1/\epsilon}) \geq 0$

$$\pi_{\epsilon,n} = \begin{cases} 
\frac{1}{1+n^2}\epsilon & s = \mu(1 + n\sqrt{\gamma - 1/\epsilon}), \\
1 - \epsilon & s = \mu, \\
\frac{n^2}{1+n^2}\epsilon & s = \mu(1 - \frac{1}{n}\sqrt{\gamma - 1/\epsilon}). 
\end{cases}$$

$\pi_{\epsilon,n}$ is feasible and as $\epsilon \to 0$ gives bound
Inf version of Lo’s bound

**Theorem**

For $\mu \geq 0$, $\gamma \geq 1$, let

$$z = \inf_{\pi} \mathbb{E}_{\pi}((s - 1)^+)$$

subject to:

- $\mathbb{E}_{\pi}(1) = 1$
- $\mathbb{E}_{\pi}(s) = \mu$
- $\mathbb{E}_{\pi}(s^2) = \gamma \mu^2$

where $\pi$ is a p.d. in $\mathbb{R}_+$, then $z = (\mu - 1)^+$

**Proof.**

- Clearly, $z \geq (\mu - 1)^+$

Consider for

- $0 < \epsilon < 1$, $n > 0$, $\frac{\epsilon}{1+n^2} \leq \frac{1}{1-\epsilon}$, $n \geq 0$, $\mu(1 - \frac{1}{n} \sqrt{\gamma - 1/\epsilon}) \geq 0$

$$\pi_{\epsilon,n} = \begin{cases} \frac{1}{1+n^2} \epsilon & s = \mu(1 + n\sqrt{\gamma - 1/\epsilon}) , \\ 1 - \epsilon & s = \mu, \\ \frac{n^2}{1+n^2} \epsilon & s = \mu(1 - \frac{1}{n} \sqrt{\gamma - 1/\epsilon}) . \end{cases}$$

$\pi_{\epsilon,n}$ is feasible and as $\epsilon \to 0$ gives bound.
“Lo’s” 2\textsuperscript{nd} moment upper–lower bds. \((\gamma = 1.5, \beta = 3)\)
Consider adding third moment info to Lo’s bound

\[
\sup / \inf \quad \mathbb{E}_{\pi} ((s - 1)^+) \\
\text{s.t.} \quad \mathbb{E}_{\pi} (1) = 1 \\
\mathbb{E}_{\pi} (s) = \mu \\
\mathbb{E}_{\pi} (s^2) = \gamma \mu^2 \\
\mathbb{E}_{\pi} (s^3) = \beta \mu^3 \\
\pi \quad \text{a p.d. in } \mathbb{R}_+. 
\]

From classical moment theory, this problem is feasible iff:

(i) \( \gamma \geq 1, \beta = \gamma^2 \) and \( \mu \geq 0 \) or

(ii) \( \gamma > 1, \beta > \gamma^2 \) and \( \mu > 0 \).

Let’s consider first the case when (i): \( \beta = \gamma^2 \) holds...
Consider adding third moment info to Lo’s bound

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\begin{align*}
\sup / \inf \quad & \mathbb{E}_\pi ((s - 1)^+) \\
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Let’s consider first the case when (i): \( \beta = \gamma^2 \) holds...
Lo’s bound + third moment info (cont.)

**Theorem (Peña, Z.)**

For $\gamma \geq 1$, $\mu \geq 0$ let

\[
\begin{align*}
    z_{\beta=\gamma^2}^{\sup/\inf} &= \sup / \inf \\
    \text{s.t.} \\
    \mathbb{E}_\pi((s - 1)^+) \\
    \mathbb{E}_\pi(1) &= 1 \\
    \mathbb{E}_\pi(s) &= \mu \\
    \mathbb{E}_\pi(s^2) &= \gamma \mu^2 \\
    \mathbb{E}_\pi(s^3) &= \gamma^2 \mu^3 \\
    \pi &\text{ a p.d. in } \mathbb{R}_+,
\end{align*}
\]

then

\[
\begin{align*}
    z_{\beta=\gamma^2}^{\inf} &= \gamma^2 \\
    z_{\beta=\gamma^2}^{\sup} &= \gamma^2 \\
    = \left(\mu - \frac{1}{\gamma}\right)^+.
\end{align*}
\]

**Proof.**

\[
\begin{align*}
    \pi &= \begin{cases} \\
        \frac{1}{\gamma} & s = \gamma \mu, \\
        1 - \frac{1}{\gamma} & s = 0.
    \end{cases} \\
    \Rightarrow z_{\beta=\gamma^2}^{\inf} \leq \mathbb{E}_\pi(s - 1)^+ = \left(\mu - \frac{1}{\gamma}\right)^+ \leq z_{\beta=\gamma^2}^{\sup}
\end{align*}
\]

Suitable *dual* solutions give the equalities.
Lo’s bound + third moment info (cont.)

**Theorem (Peña, Z.)**

For $\gamma \geq 1$, $\mu \geq 0$ let

$$z_{\beta=\gamma^2}^{\sup/\inf} = \sup / \inf \ E_\pi ((s - 1)^+)$$

s.t.

- $E_\pi (1) = 1$
- $E_\pi (s) = \mu$
- $E_\pi (s^2) = \gamma \mu^2$
- $E_\pi (s^3) = \gamma^2 \mu^3$

$\pi$ a p.d. in $\mathbb{R}_+$,

Then

$$z_{\beta=\gamma^2}^{\inf} = z_{\beta=\gamma^2}^{\sup} = (\mu - \frac{1}{\gamma})^+$$

Expected payoff gets defined with up to $3^{rd}$ moment info.

**Proof.**

- $\pi = \begin{cases} 
  \frac{1}{\gamma} & s = \gamma \mu, \\
  1 - \frac{1}{\gamma} & s = 0. 
\end{cases}$

$\Rightarrow z_{\beta=\gamma^2}^{\inf} \leq E_\pi (s - 1)^+ = (\mu - \frac{1}{\gamma})^+ \leq z_{\beta=\gamma^2}^{\sup}$

Suitable dual solutions give the equalities.

□
Theorem (Peña, Z.)

For $\gamma \geq 1$, $\mu \geq 0$ let

\[ z^\sup_{\beta=\gamma^2} = \sup \, z^\inf_{\beta=\gamma^2} = \inf \]

subject to

\[ \mathbb{E}_\pi((s - 1)^+) \]
\[ \mathbb{E}_\pi(1) = 1 \]
\[ \mathbb{E}_\pi(s) = \mu \]
\[ \mathbb{E}_\pi(s^2) = \gamma \mu^2 \]
\[ \mathbb{E}_\pi(s^3) = \gamma^2 \mu^3 \]

$\pi$ a p.d. in $\mathbb{R}_+$,

Proof.

$\pi = \begin{cases} 
\frac{1}{\gamma} & s = \gamma \mu, \\
\frac{1}{1 - \frac{1}{\gamma}} & s = 0.
\end{cases}$

$\Rightarrow z^\inf_{\beta=\gamma^2} \leq \mathbb{E}_\pi(s - 1)^+ = (\mu - \frac{1}{\gamma})^+ \leq z^\sup_{\beta=\gamma^2}$

Suitable dual solutions give the equalities.
Now let's consider case (ii): $\beta > \gamma^2$. First Lower bound:

**Theorem (Du, Peña, Z.)**

For $\gamma > 1$, $\mu > 0$ let

$$z_{\beta > \gamma^2} = \inf_{\pi} \left\{ \begin{array}{ll}
E_{\pi}((s - 1)^+) & \\
E_{\pi}(1) & = 1 \\
E_{\pi}(s) & = \mu \\
E_{\pi}(s^2) & = \gamma \mu^2 \\
E_{\pi}(s^3) & = \beta \mu^3 \\
\pi & \text{a p.d. in } \mathbb{R}_+,
\end{array} \right. \text{ s.t.}

\begin{align*}
E_{\pi}(1) &= 1 \\
E_{\pi}(s) &= \mu \\
E_{\pi}(s^2) &= \gamma \mu^2 \\
E_{\pi}(s^3) &= \beta \mu^3 \\
\pi & \text{ a p.d. in } \mathbb{R}_+,
\end{align*}

then

$$z_{\beta > \gamma^2} = \left\{ \begin{array}{ll}
\frac{\mu - 1}{\beta \mu - \gamma} & \mu > \tilde{\mu} \\
\frac{1}{\gamma} < \mu \leq \tilde{\mu} & \\
0 & \mu \leq \frac{1}{\gamma}
\end{array} \right.$$

$$\tilde{\mu} = \frac{2(\gamma - 1)}{(\beta - \gamma) - \sqrt{(\beta - 3\gamma + 2)^2 + 4(\gamma - 1)^3}}$$

**Proof.**

Suitable primal & dual solutions + “Lo’s” 2nd moment bds.
Now let’s consider case (ii): $\beta > \gamma^2$. First Lower bound:

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For $\gamma > 1$, $\mu > 0$ let

$$z_{\beta > \gamma^2}^{\text{inf}} = \inf \quad \mathbb{E}_{\pi}((s - 1)^+),$$

s.t.

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$\pi$ a p.d. in $\mathbb{R}_+$,

then

$$z_{\beta > \gamma^2}^{\text{inf}} = \begin{cases} 
\mu - 1 & \mu > \tilde{\mu} \\
\frac{(\gamma \mu - 1)^2}{\beta \mu - \gamma} & \frac{1}{\gamma} < \mu \leq \tilde{\mu} \\
0 & \mu \leq \frac{1}{\gamma}
\end{cases},$$

$$\tilde{\mu} = \frac{2(\gamma - 1)}{(\beta - \gamma) - \sqrt{(\beta - 3\gamma + 2)^2 + 4(\gamma - 1)^3}}.$$

**Proof.**

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For $\gamma > 1$, $\mu > 0$ let

$$ z_{\beta > \gamma^2}^{\inf} = \inf_{\pi} \mathbb{E}_{\pi}((s - 1)^+) $$

s.t.

- $\mathbb{E}_{\pi}(1) = 1$
- $\mathbb{E}_{\pi}(s) = \mu$
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$\pi$ a p.d. in $\mathbb{R}_+$,

then

$$ z_{\beta > \gamma^2}^{\inf} = \begin{cases} 
\mu - 1 & \mu > \tilde{\mu} \\
\frac{(\gamma \mu - 1)^2}{\beta \mu - \gamma} & \frac{1}{\gamma} < \mu \leq \tilde{\mu} \\
0 & \mu \leq \frac{1}{\gamma} 
\end{cases} $$

$$ \tilde{\mu} = \frac{2(\gamma - 1)}{(\beta - \gamma) - \sqrt{(\beta - 3 \gamma + 2)^2 + 4(\gamma - 1)^3}} $$

**Proof.**

Suitable primal & dual solutions + “Lo’s” 2nd moment bds.
Now let's consider Upper bound in case (ii): $\beta > \gamma^2$:

**Theorem (Du, Peña, Z.)**

For $\gamma > 1$, $\mu > 0$ let

$$
\sup_{\beta > \gamma^2} \pi = \sup_{\pi \text{ a p.d. in } \mathbb{R}_+} \min\{\text{obj(dual sol.)}, \text{Lo's bd.}\} \vee \sup_{\beta > \gamma^2} \max\{\text{obj(primal sols.), } \beta = \gamma^2 \text{ bd.}\}
$$

$s.t.$

$$
\mathbb{E}_\pi((s - 1)^+) = 1 \\
\mathbb{E}_\pi(s) = \mu \\
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$$
Now let's consider Upper bound in case (ii): $\beta > \gamma^2$:

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*For $\gamma > 1$, $\mu > 0$ let*

$$z^{\text{sup}}_{\beta > \gamma^2} = \sup \text{ s.t. } \begin{align*}
\mathbb{E}_{\pi}((s - 1)^+) &= 1 \\
\mathbb{E}_{\pi}(1) &= 1 \\
\mathbb{E}_{\pi}(s) &= \mu \\
\mathbb{E}_{\pi}(s^2) &= \gamma \mu^2 \\
\mathbb{E}_{\pi}(s^3) &= \beta \mu^3 \\
\pi &\text{ a p.d. in } \mathbb{R}_+,
\end{align*}$$

*then*

$$\min \{ \text{obj(dual sol.), Lo's bd. } \} \bigg\| \bigg\| z^{\text{sup}}_{\beta > \gamma^2} \bigg\| \bigg\| \max \{ \text{obj(primal sols.), } \beta = \gamma^2 \text{ bd.} \}$$

$$z^{\text{sup}}_{\beta > \gamma^2} = \mu - \frac{1}{\gamma} \quad \mu \geq \frac{2}{\gamma}$$
“Lo’s” 3rd moment upper bound “sandwich”

\[ \gamma = 1.5, \ \beta = 3 \]

\[ E((s-1)^+) \]
“Lo’s” 3rd moment upper bound “sandwich”

\[ \gamma = 1.2, \beta = 1.5 \]
“Lo’s” 2\textsuperscript{nd}– 3\textsuperscript{rd} moment upper–lower bds. \((\gamma = 1.5, \beta = 3)\)

\[ \mathbb{E}[\max(s-1,0)] \quad \beta \gg \gamma^2 \]
“Lo’s” 2\textsuperscript{nd}– 3\textsuperscript{rd} moment upper–lower bds. \((\gamma = 1.5, \beta = 3)\)

\[ \mathbb{E}[\max(s-1, 0)] \quad \beta \gg \gamma^2 \]
“Lo’s” 2nd–3rd moment upper–lower bds. ($\gamma = 1.2, \beta = 1.5$)

\[ E[\max(s-1, 0)] \quad \beta \approx \gamma^2 \]
"Lo's" 2\textsuperscript{nd}–3\textsuperscript{rd} moment upper–lower bds. \( (\gamma = 1.2, \beta = 1.5) \)
Bounds vs. Black-Scholes \((\rho = 0, \sigma = 0.5, T = 1)\)

\[\mathbb{E}[\max(s-1, 0)] \quad \text{Lognormal: } \beta = \gamma^3 \gg \gamma^2\]
Application in Inventory

Example

Let \( x \): inventory of a single product, \( c \): product’s unit cost, and \( r \): product’s unit price. What is the minimum expected profit over demand distributions with given mean and variance?

Theorem (Scarf)

\[
\inf_{\pi} \mathbb{E}_\pi (r \min\{x, s\} - cx) \quad \text{s.t.} \quad \begin{align*}
\mathbb{E}_\pi (1) &= 1 \\
\mathbb{E}_\pi (s) &= \mu \\
\mathbb{E}_\pi (s^2) &= \gamma \mu^2 \\
\pi &\text{ a p.d. in } \mathbb{R}_+, \\
x &\leq \frac{\gamma \mu}{2}, \\
x &\geq \frac{\gamma \mu}{2}.
\end{align*}
\]

Scarf finds \( x^* \) that maximizes the min. expected profit.

\((s - x)^+ = s - \min(x, s) \Rightarrow \text{extension to third moment info!}\).
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\mathbb{E}_{\pi}(1) = 1 \\
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\pi \quad \text{a p.d. in } \mathbb{R}_+, \\
\begin{align*}
\mathbb{E}_{\pi}(s) &= \mu \\
\mathbb{E}_{\pi}(s^2) &= \gamma \mu^2
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\]

Scarf finds \( x^* \) that maximizes the min. expected profit.

\( (s - x)^+ = s - \min(x, s) \Rightarrow \text{extension to third moment info!} \).
Application in Inventory

Example

Let $x$: inventory of a single product, $c$: product’s unit cost, and $r$: product’s unit price. What is the minimum expected profit over demand distributions with given mean and variance?

Theorem (Scarf)

$$\inf \pi \left( r \min \{ x, s \} - cx \right)$$

s.t.

$$\begin{align*}
E_{\pi} (1) &= 1 \\
E_{\pi} (s) &= \mu \\
E_{\pi} (s^2) &= \gamma \mu^2 \\
\pi & \text{ a p.d. in } \mathbb{R}_+,
\end{align*}$$

$$\begin{cases}
x \left( \frac{r}{\gamma} - c \right) & x \leq \frac{\gamma \mu}{2}, \\
x \left( \frac{r}{2} \left( \mu + x - \frac{\sqrt{\gamma \mu^2 - 2 \mu x + x^2}}{\gamma \mu^2 - 2 \mu x + x^2} \right) - cx \right) & x \geq \frac{\gamma \mu}{2}.
\end{cases}$$

- Scarf finds $x^*$ that maximizes the min. expected profit.
- $(s - x)^+ = s - \min(x, s) \Rightarrow$ extension to third moment info!
Example

Let $x$: inventory of a single product, $c$: product’s unit cost, and $r$: product’s unit price. What is the minimum expected profit over demand distributions with given mean and variance?

Theorem (Scarf)

$$\inf_{\pi} \mathbb{E}_{\pi}(r \min\{x, s\} - cx)$$

s.t.

$$\mathbb{E}_{\pi}(1) = 1, \quad \mathbb{E}_{\pi}(s) = \mu, \quad \mathbb{E}_{\pi}(s^2) = \gamma \mu^2$$

$\pi$ a p.d. in $\mathbb{R}_+$,

$$\begin{cases} 
  x \left( \frac{r}{\gamma} - c \right) & x \leq \frac{\gamma \mu}{2}, \\
  \frac{r}{2} \left( \frac{\mu + x - \sqrt{\gamma \mu^2 - 2\mu x + x^2}}{\gamma \mu^2} \right) - cx & x \geq \frac{\gamma \mu}{2}.
\end{cases}$$

- Scarf finds $x^*$ that maximizes the min. expected profit.
- $(s - x)^+ = s - \min(x, s) \Rightarrow$ extension to third moment info!
Bounds on “risk” associated to European call payoff

Consider bound on “risk” of payoff given its expectation:

\[
\sup \frac{1}{\inf} \text{Var}_\pi((s - 1)^+) \equiv \mathbb{E}_\pi(((s - 1)^+)^2) - \tilde{\mu}^2
\]

s.t.

\[
\begin{align*}
\mathbb{E}_\pi(1) &= 1 \\
\mathbb{E}_\pi(s) &= \mu \\
\mathbb{E}_\pi(s^2) &= \gamma \mu^2 \\
\mathbb{E}_\pi((s - 1)^+) &= \tilde{\mu} \\
\pi &\text{ a p.d. in } \mathbb{R}_+.
\end{align*}
\]

From “Lo’s” 2nd moment upper–lower bds., this problem is feasible iff:

\[
\begin{align*}
\gamma &\geq 1, \mu \geq 0 \text{ and} \\
(\mu - 1)^+ &\leq \tilde{\mu} \leq \\
\text{if } \mu &\geq \frac{2}{\gamma}:
\begin{cases}
\mu - \frac{1}{\gamma} \\
\frac{1}{2}(\mu - 1) + \frac{1}{\gamma}(\gamma - 1)\mu^2 + (\mu - 1)^2
\end{cases}
\text{if } \mu \leq \frac{2}{\gamma}.
\end{align*}
\]
Bounds on “risk” associated to European call payoff

Consider bound on “risk” of payoff given its expectation:

\[
\sup \ / \ \inf \ \ \text{Var}_\pi((s - 1)^+) \equiv \mathbb{E}_\pi(((s - 1)^+)^2) - \tilde{\mu}^2
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\[
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\mathbb{E}_\pi(1) &= 1 \\
\mathbb{E}_\pi(s) &= \mu \\
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\mathbb{E}_\pi((s - 1)^+) &= \tilde{\mu} \\
\pi &\text{ a p.d. in } \mathbb{R}_+.
\end{align*}
\]

From “Lo’s” 2\textsuperscript{nd} moment upper–lower bds., this problem is feasible iff:

- \( \gamma \geq 1, \mu \geq 0 \) and

\[
\mu - \frac{1}{\gamma} \leq \tilde{\mu} \leq \begin{cases} \\
w(\mu - 1) \\
\frac{1}{2}(\mu - 1) + \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2}
\end{cases}
\]

if \( \mu \geq \frac{2}{\gamma} \),

if \( \mu \leq \frac{2}{\gamma} \).
Bounds on “risk” associated to European call payoff

Consider bound on “risk” of payoff given its expectation:

$$\sup / \inf \ Var_{\pi}((s - 1)^+) \equiv \mathbb{E}_{\pi}(((s - 1)^+)^2) - \tilde{\mu}^2$$

s.t. $\mathbb{E}_{\pi}(1) = 1$

$\mathbb{E}_{\pi}(s) = \mu$

$\mathbb{E}_{\pi}(s^2) = \gamma \mu^2$

$\mathbb{E}_{\pi}((s - 1)^+) = \tilde{\mu}$

$\pi$ a p.d. in $\mathbb{R}_+$.

From “Lo’s” 2nd moment upper–lower bds., this problem is feasible iff:

- $\gamma \geq 1$, $\mu \geq 0$ and

$$\frac{\mu - 1}{\gamma} \quad \text{if } \mu \geq \frac{2}{\gamma},$$

$$\left(\frac{1}{2}(\mu - 1) + \sqrt{(\gamma - 1)\mu^2 + (\mu - 1)^2}\right) \quad \text{if } \mu \leq \frac{2}{\gamma}.$$
Theorem (Du, Peña, Z.)

For \( \gamma \geq 1, \mu \geq 0, \) and \( (\mu - 1)^+ \leq \widetilde{\mu} \leq \) Lo’s bound, let

\[
\xi^{\text{inf}} = \inf \mathbb{V} \mathbb{a} \mathbb{r}_{\pi} ((s - 1)^+)
\]

s.t. \( \mathbb{E}_{\pi}(1) = 1 \)

\( \mathbb{E}_{\pi}(s) = \mu \)

\( \mathbb{E}_{\pi}(s^2) = \gamma \mu^2 \)

\( \mathbb{E}_{\pi}((s - 1)^+) = \widetilde{\mu} \)

\( \pi \) a p.d. in \( \mathbb{R}_+ \).

\[
\xi^{\text{inf}} = \begin{cases} 
\mu(\gamma \mu - 1) - \widetilde{\mu} - \widetilde{\mu}^2 & \text{if } \widetilde{\mu} \leq (\mu - \frac{1}{\gamma}), \\
\frac{1}{2} \left( 2\widetilde{\mu}(\mu - 1) + \mu \left( (\gamma - 1)\mu - \sqrt{(\gamma - 1)((\gamma - 1)\mu^2 + 4\widetilde{\mu}(\mu - 1) - 4\widetilde{\mu}^2))} \right) - \widetilde{\mu}^2 & \text{if } \widetilde{\mu} \geq (\mu - \frac{1}{\gamma}),
\end{cases}
\]
Maximum “risk”

Theorem (Du, Peña, Z.)

For $\gamma \geq 1$, $\mu \geq 0$, and $(\mu - 1)^+ \leq \tilde{\mu} \leq \text{Lo’s bound}$, let

$$\xi^{\sup} = \sup \begin{array}{l}
\text{Var}_{\pi}((s - 1)^+)
\end{array}$$

s.t. $\begin{align*}
E_{\pi}(1) &= 1 \\
E_{\pi}(s) &= \mu \\
E_{\pi}(s^2) &= \gamma \mu^2 \\
E_{\pi}((s - 1)^+) &= \tilde{\mu} \\
\pi &\text{ a p.d. in } \mathbb{R}_+.
\end{align*}$

$$\xi^{\sup} = \frac{1}{2}(2\tilde{\mu}(\mu - 1) + \mu((\gamma - 1)\mu$$

$$+ \sqrt{(\gamma - 1)((\gamma - 1)\mu^2 + 4\tilde{\mu}(\mu - 1) - 4\tilde{\mu}^2))}) - \tilde{\mu}^2$$
Maximum and Minimum “risk” ($\mu = 1.2, \gamma = 1.1$)

$$\sqrt{\text{Var}[\max (s - 1, 0)]}$$
“Skewness” of optimal distributions \((\mu = 1.2, \gamma = 1.1)\)

\[
\sqrt[3]{\mathbb{E}[(s - \mu)^3]}
\]
Consider bound on “risk” of payoff without given its expectation:

\[
\begin{align*}
\sup / \inf \ Var_\pi ((s - 1)^+) \\
\text{s.t.} \\
& \mathbb{E}_\pi (1) = 1 \\
& \mathbb{E}_\pi (s) = \mu \\
& \mathbb{E}_\pi (s^2) = \gamma \mu^2 \\
& \mathbb{E}_\pi ((s-1)^+) = \tilde{\mu} \\
& \pi \text{ a p.d. in } \mathbb{R}_+. \\
\end{align*}
\]

By optimizing over the parameter \( \tilde{\mu} \) previous risk bounds:

- sup bound = \( (\gamma - 1)\mu^2 \)
- inf bound = \( \frac{(\gamma - 1)}{\gamma^2} ((\gamma \mu - 1)^+)^2 \)

Currently working on applications to Two newsvendor problem under incomplete demand information.
Consider bound on “risk” of payoff without given its expectation:

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\sup / \inf \quad \text{Var}_\pi((s - 1)^+) \\
\text{s.t.} \\
\mathbb{E}_\pi(1) = 1 \\
\mathbb{E}_\pi(s) = \mu \\
\mathbb{E}_\pi(s^2) = \gamma \mu^2 \\
\pi \text{ a p.d. in } \mathbb{R}_+.
\]

By optimizing over the parameter \(\tilde{\mu}\) previous risk bounds:

- \(\sup\) bound = \((\gamma - 1)\mu^2\)
- \(\inf\) bound = \(\frac{(\gamma - 1)}{\gamma^2}((\gamma \mu - 1)^+)\)

Currently working on applications to Two newsvendor problem under incomplete demand information.
Bounds on “risk” independent of Expected return

Consider bound on “risk” of payoff without given its expectation:

\[
\sup / \inf \ \text{Var}_\pi ((s - 1)^+)
\]
\[
\text{s.t.} \quad \mathbb{E}_\pi (1) = 1 \\
\mathbb{E}_\pi (s) = \mu \\
\mathbb{E}_\pi (s^2) = \gamma \mu^2 \\
\pi \quad \text{a p.d. in } \mathbb{R}_+.
\]

By optimizing over the parameter \(\tilde{\mu}\) previous risk bounds:

- sup bound = \((\gamma - 1)\mu^2\)
- inf bound = \(\frac{\gamma^2 - 1}{\gamma^2} (\gamma \mu - 1)^+)^2\)

Currently working on applications to Two newsvendor problem under incomplete demand information
How are the solutions obtained?

\begin{align*}
\sup / \inf \ & \mathbb{E}_\pi ((s - 1)^+) \\
\text{s.t.} \ & \mathbb{E}_\pi (1) = 1 \quad \mathbb{E}_\pi (s^2) = \gamma \mu^2 \\
\ & \mathbb{E}_\pi (s) = \mu \quad \mathbb{E}_\pi (s^3) = \beta \mu^3 \\
\pi \ & \text{a p.d. in } \mathbb{R}_+.
\end{align*}

- From moment theory, can restrict to atomic p.d. with 5 (or less) atoms
- Suppose we try two atoms:
  \begin{align*}
  \pi = \begin{cases} 
  0 & p_0 \\
  s_1 (< 1) & p_1 \\
  s_2 (\geq 1) & p_2
  \end{cases}
  \end{align*}
- Get non-linear problem

\begin{align*}
\sup / \inf \ & p_2 (s_2 - 1) \\
\text{s.t.} \ & p_0 + p_1 + p_2 = 1, \quad p_1 s_1^2 + p_2 s_2^2 = \gamma \mu^2 \\
\ & p_1 s_1 + p_2 s_2 = \mu, \quad p_1 s_1^3 + p_2 s_2^3 = \beta \mu^3
\end{align*}
How are the solutions obtained?

\[
\begin{align*}
\text{sup} / \text{inf} & \quad \mathbb{E}_\pi ((s - 1)^+) \\
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& \quad \mathbb{E}_\pi (s) = \mu, \quad \mathbb{E}_\pi (s^3) = \beta \mu^3 \\
\pi & \quad \text{a p.d. in } \mathbb{R}_.
\end{align*}
\]

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\pi = \begin{cases} 
0 & p_0 \\
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\end{cases}
\]

- Get non-linear problem

\[
\begin{align*}
\text{sup} / \text{inf} & \quad p_2 (s_2 - 1) \\
\text{s.t.} & \quad p_0 + p_1 + p_2 = 1, \quad p_1 s_1^2 + p_2 s_2^2 = \gamma \mu^2 \\
& \quad p_1 s_1 + p_2 s_2 = \mu, \quad p_1 s_1^3 + p_2 s_2^3 = \beta \mu^3
\end{align*}
\]
How are the solutions obtained?

\[ \begin{align*}
\sup / \inf & \quad \mathbb{E}_\pi ((s - 1)^+) \\
\text{s.t.} & \quad \mathbb{E}_\pi (1) = 1, \quad \mathbb{E}_\pi (s^2) = \gamma \mu^2 \\
& \quad \mathbb{E}_\pi (s) = \mu, \quad \mathbb{E}_\pi (s^3) = \beta \mu^3 \\
\pi & \quad \text{a p.d. in } \mathbb{R}_+.
\end{align*} \]

- From moment theory, can restrict to atomic p.d. with 5 (or less) atoms.
- Suppose we try two atoms:

\[ \pi = \begin{cases} 
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 s_1 (< 1) & p_1 \\
 s_2 (\geq 1) & p_2
\end{cases} \]

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\sup / \inf & \quad p_2 (s_2 - 1) \\
\text{s.t.} & \quad p_0 + p_1 + p_2 = 1, \quad p_1 s_1^2 + p_2 s_2^2 = \gamma \mu^2 \\
& \quad p_1 s_1 + p_2 s_2 = \mu, \quad p_1 s_1^3 + p_2 s_2^3 = \beta \mu^3
\end{align*} \]
How are the solutions obtained? (cont.)

\[
\sup / \inf \quad p_2(s_2 - 1) \\
\text{s.t.} \quad p_0 + p_1 + p_2 = 1, \quad p_1s_1^2 + p_2s_2^2 = \gamma \mu^2 \\
\quad p_1s_1 + p_2s_2 = \mu, \quad p_1s_1^3 + p_2s_2^3 = \beta \mu^3
\]

- The lagrangian is \( \mathcal{L}(p_0, p_1, p_2, s_1, s_2, y_1, y_2, y_3) \)

\[
= p_2(s_2 - 1) + y_0(1 - (p_0 + p_1 + p_2)) + y_1(\mu - (p_1s_1 + p_2s_2)) + y_2(\gamma \mu^2 - (p_1s_1^2 + p_2s_2^2)) + y_3(\beta \mu^3 - (p_1s_1^3 + p_2s_2^3))
\]

- First order optimality conditions:

\[
\begin{align*}
p_0 + p_1 + p_2 &= 1 & y_0 &= 0 \\
p_1s_1 + p_2s_2 &= \mu & y_0 + y_1s_1 + y_2s_1^2 + y_3s_1^3 &= 0 \\
p_1s_1^2 + p_2s_2^2 &= \gamma \mu^2 & y_0 + y_1s_2 + y_2s_2^2 + y_3s_2^3 &= (s_2 - 1) \\
p_1s_1^3 + p_2s_2^3 &= \beta \mu^3 & y_1 + 2y_2s_1 + 3y_3s_1^2 &= 0 \\
\quad & y_1 + 2y_2s_2 + 3y_3s_2^2 &= 1
\end{align*}
\]
How are the solutions obtained? (cont.)

\[
\begin{align*}
\sup / \inf & \quad \sup / \inf p_2(s_2 - 1) \\
\text{s.t.} & \quad p_0 + p_1 + p_2 = 1, \quad p_1 s_1^2 + p_2 s_2^2 = \gamma \mu^2 \\
& \quad p_1 s_1 + p_2 s_2 = \mu, \quad p_1 s_1^3 + p_2 s_2^3 = \beta \mu^3
\end{align*}
\]

- The lagrangian is \( L(p_0, p_1, p_2, s_1, s_2, y_1, y_2, y_3) \)

\[
= \quad p_2(s_2 - 1) + y_0(1 - (p_0 + p_1 + p_2)) + y_1(\mu - (p_1 s_1 + p_2 s_2)) + y_2(\gamma \mu^2 - (p_1 s_1^2 + p_2 s_2^2)) + y_3(\beta \mu^3 - (p_1 s_1^3 + p_2 s_2^3))
\]

- First order optimality conditions:

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p_0 + p_1 + p_2 &= 1 & y_0 &= 0 \\
p_1 s_1 + p_2 s_2 &= \mu & y_0 + y_1 s_1 + y_2 s_1^2 + y_3 s_1^3 &= 0 \\
p_1 s_1^2 + p_2 s_2^2 &= \gamma \mu^2 & y_0 + y_1 s_2 + y_2 s_2^2 + y_3 s_2^3 &= (s_2 - 1) \\
p_1 s_1^3 + p_2 s_2^3 &= \beta \mu^3 & y_1 + 2y_2 s_1 + 3y_3 s_1^2 &= 0 \\
& & y_1 + 2y_2 s_2 + 3y_3 s_2^2 &= 1
\end{align*}
\]
How are the solutions obtained? (cont.)

\[ p_0 + p_1 + p_2 = 1 \quad y_0 = 0 \]
\[ p_1 s_1 + p_2 s_2 = \mu \quad y_0 + y_1 s_1 + y_2 s_1^2 + y_3 s_1^3 = 0 \]
\[ p_1 s_1^2 + p_2 s_2^2 = \gamma \mu^2 \quad y_0 + y_1 s_2 + y_2 s_2^2 + y_3 s_2^3 = (s_2 - 1) \]
\[ p_1 s_1^3 + p_2 s_2^3 = \beta \mu^3 \quad y_1 + 2y_2 s_1 + 3y_3 s_1^2 = 0 \]
\[ y_1 + 2y_2 s_2 + 3y_3 s_2^2 = 1 \]

- But dual of original problem is:

\[
\inf \quad y_0 + \mu y_1 + \gamma \mu^2 y_2 + \beta \mu^3 y_3 \\
\text{s.t.} \quad y(s) := y_0 + y_1 s + y_2 s^2 + y_3 s^3 \geq (s - 1)^+ \quad \text{for all } s \in \mathbb{R}_+ 
\]
How are the solutions obtained? (cont.)

\[ p_0 + p_1 + p_2 = 1 \quad y_0 = 0 \]
\[ p_1s_1 + p_2s_2 = \mu \quad y_0 + y_1s_1 + y_2s_1^2 + y_3s_1^3 = 0 \quad y(s_1) = 0 \]
\[ p_1s_1^2 + p_2s_2^2 = \gamma \mu^2 \quad y_0 + y_1s_2 + y_2s_2^2 + y_3s_2^3 = (s_2 - 1) \quad y(s_2) = (s_2 - 1) \]
\[ p_1s_1^3 + p_2s_2^3 = \beta \mu^3 \quad y_1 + 2y_2s_1 + 3y_3s_1^2 = 0 \quad y'(s)|_{s_1} = 0 \]
\[ y_1 + 2y_2s_2 + 3y_3s_2^2 = 1 \quad y'(s)|_{s_2} = 1 \]

- But dual of original problem is:

\[
\begin{align*}
\inf & \quad y_0 + \mu y_1 + \gamma \mu^2 y_2 + \beta \mu^3 y_3 \\
\text{s.t.} & \quad y(s) := y_0 + y_1s + y_2s^2 + y_3s^3 \geq (s - 1)^+ \quad \text{for all } s \in \mathbb{R}_+ 
\end{align*}
\]
How are the solutions obtained? (cont.)

\[ p_0 + p_1 + p_2 = 1 \quad y_0 = 0 \]
\[ p_1 s_1 + p_2 s_2 = \mu \quad y_0 + y_1 s_1 + y_2 s_1^2 + y_3 s_1^3 = 0 \]
\[ p_1 s_1^2 + p_2 s_2^2 = \gamma \mu^2 \quad y_0 + y_1 s_2 + y_2 s_2^2 + y_3 s_2^3 = (s_2 - 1) \]
\[ p_1 s_1^3 + p_2 s_2^3 = \beta \mu^3 \quad y_1 + 2y_2 s_1 + 3y_3 s_1^2 = 0 \]
\[ y_1 + 2y_2 s_2 + 3y_3 s_2^2 = 0 \]
\[ y(s_1) = 0 \quad y(s_2) = (s_2 - 1) \]
\[ y'(s)|_{s_1} = 0 \quad y'(s)|_{s_2} = 1 \]

So “location” of atoms can be determined.

Now use Mathematica to get closed-form solutions to optimality conditions, and compare them against numerical solution of the problem (which can be solved using SDP).
How are the solutions obtained? (cont.)

\[ p_0 + p_1 + p_2 = 1 \]
\[ y_0 = 0 \]
\[ p_1 s_1 + p_2 s_2 = \mu \]
\[ y_0 + y_1 s_1 + y_2 s_1^2 + y_3 s_1^3 = 0 \]
\[ y(s_1) = 0 \]
\[ p_1 s_2^2 + p_2 s_2^2 = \gamma \mu^2 \]
\[ y_0 + y_1 s_2 + y_2 s_2^2 + y_3 s_2^3 = (s_2 - 1) \]
\[ y(s_2) = (s_2 - 1) \]
\[ p_1 s_1^3 + p_2 s_2^2 = \beta \mu^3 \]
\[ y_1 + 2y_2 s_1 + 3y_3 s_1^2 = 0 \]
\[ y(s_1) = 0 \]
\[ y(s_2) = (s_2 - 1) \]

So “location” of atoms can be determined.

Now use Mathematica to get closed-form solutions to optimality conditions, and compare them against numerical solution of the problem (which can be solved using SDP).
Now consider more general options:

**Basket option Payoff**

\[(w \cdot S - K)^+ := \max\{0, w \cdot S - K\}\]

- \(S \in \mathbb{R}^n\): assets’ price (at maturity).
- \(w \in \mathbb{R}^n\): weights of the assets.
- \(K \in \mathbb{R}\): strike price.

**Arbitrage Bounds**: compute sharp bounds on the price of an European basket option based only on information about the prices of other basket options.
Arbitrage bound problem

Primal formulation

\[ \sup_{\pi} \mathbb{E}_\pi[(w_0 \cdot S - K_0)^+] \]
\[ \text{s.t. } \mathbb{E}_\pi[(w_j \cdot S - K_j)^+] = p_j, \quad j = 1, \ldots, r \]
\[ \pi \quad \text{probability distribution in } \mathbb{R}^n_+ . \]

- \( S \): prices of the \( n \) underlying assets (at maturity).
- \( w_j \in \mathbb{R}^n \): weights of the underlying assets in each basket.
- \( K_j \in \mathbb{R} \): strike price of each basket.
- \( p_j \in \mathbb{R} \): price of each basket.

These bounds can be seen as \textit{robust} bounds that any reasonable pricing model must satisfy.
Fair amount of attention in recent years:

- D’Aspremont and El Ghaoui [05]:
  - Same general problem.
  - Use integral transformation of the options’ price functions.
  - Derive LP relaxations (polynomial size).
  - Provide LP (polynomial size), or closed-form expressions in special cases.

- Hobson, Laurence and Wang [05,05]
  - More general results for the upper bound when call options prices on every asset are given for a continuum of strikes.
  - Use a Lagrangian programming formulation and the fact that the continuum of options determines the full marginal distributions of each of the assets.

Goal
Obtain more general results via simple LP-machinery.
Previous work

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- D’Aspremont and El Ghaoui [05]:
  - Same general problem.
  - Use integral transformation of the options’ price functions.
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  - More general results for the upper bound when call options prices on every asset are given for a continuum of strikes.
  - Use a Lagrangian programming formulation and the fact that the continuum of options determines the full marginal distributions of each of the assets.

Goal

Obtain more general results via simple LP-machinery.
Writing the problem as an LP

\[
\sup_{\pi} \mathbb{E}_{\pi} [(w_0 \cdot S - K_0)^+] \\
\text{s.t.} \quad \mathbb{E}_{\pi} [(w_j \cdot S - K_j)^+] = p_j, \quad j = 1, \ldots, r \\
\pi \quad \text{probability distribution in} \ R^n_+.
\] (P)

Dual formulation

\[
\inf_{z \in \mathbb{R}, y \in \mathbb{R}^r} \quad z + p \cdot y \\
\text{s.t.} \quad z + \sum_{j=1}^{r} y_j (w_j \cdot s - K_j)^+ \geq (w_0 \cdot s - K_0)^+ \quad \text{for all } s \in \mathbb{R}^n_+ \\
\]

\underline{super-replication constraint}

(D)

- Finds cheapest portfolio of basket options and cash that \textit{super-replicates} the target basket payoff.
Strong Duality

\[
\begin{align*}
\sup_{\pi} \quad & \mathbb{E}_{\pi} \left[ (w_0 \cdot S - K_0)^+ \right] \\
\text{s.t.} \quad & \mathbb{E}_{\pi} \left[ (w_j \cdot S - K_j)^+ \right] = p_j, \quad j = 1, \ldots, r \\
& \pi \text{ probability distribution in } \mathbb{R}^n_+.
\end{align*}
\] (P)

\[
\begin{align*}
\inf_{z \in \mathbb{R}, y \in \mathbb{R}^r} \quad & z + p \cdot y \\
\text{s.t.} \quad & z + \sum_{j=1}^{r} y_j (w_j \cdot s - K_j)^+ \geq (w_0 \cdot s - K_0)^+ \quad \text{for all } s \in \mathbb{R}^n_+ \\
& \text{(D)}
\end{align*}
\]

- **Slater condition** in either (P) or (D). In practice holds if:
  - If given option prices are *arbitrage-free* (prices are convex on strike), and remain *arbitrage-free* after slight perturbations
  - If prices of forward (strike := 0) options are given for all assets
Given baskets $b_0, b_1, b_2, b_3$

We want linear conditions on $z$ and $y_1, y_2, y_3$ equivalent to

$$z + y_1 b_1(s) + y_2 b_2(s) + y_3 b_3(s) \geq b_0(s)$$

for all $s \in \mathbb{R}^n_+$. 

**Focus**

Write the super-replication constrain as a set of linear constraints.
Focusing on the **super-replication** constraint, instead of the "complete" problem, allow us to include important features *not considered before* into the problem:

- Negative weights
- Bid/ask spreads
- Limits on the sizes of long/short positions
- Transaction costs.

This greatly broadens the practical applications of this type of *arbitrage pricing* techniques.
LP representation of super-replication

- Each basket divides the domain $\mathbb{R}^n_+$ in two regions:

- All the baskets together divide $\mathbb{R}^n_+$ in polyhedral cells:
Since

- Cells are *polyhedrons*
- Basket option payoffs are linear functions when restricted to a cell

We can rewrite the condition

$$z + y_1 b_1(s) + y_2 b_2(s) + y_3 b_3(s) \geq b_0(s) \text{ for all } s \in \text{Cell}$$

$$\iff$$

$$\alpha(y) \cdot s + \beta(y, z) \geq 0 \text{ for all } Cs \geq d$$

as a set of linear equations on $z$ and $y_1, y_2, y_3$ using *Farkas’s Lemma*

$$\alpha(y) = C\gamma, \quad \beta(y, z) \geq -d\gamma, \quad \gamma \geq 0$$

- $\alpha$ and $\beta$ are linear
Theorem (Peña, Vera, Z.)

1. Dual (D) can be rewritten as an LP:

\[
\begin{align*}
\min & \quad z + p \cdot y \\
\text{s.t.} & \quad \begin{bmatrix} -1 \\ y \end{bmatrix}_J \cdot W_J \geq \gamma^J \cdot W_J - \beta^J \cdot W_{J^c} \quad J \in \mathcal{J} \\
& \quad -z + \begin{bmatrix} -1 \\ y \end{bmatrix}_J \cdot K_J \leq \gamma^J \cdot K_J - \beta^J \cdot K_{J^c} \quad J \in \mathcal{J}
\end{align*}
\]

(\text{LD})

where \( y \in \mathbb{R}^r, \ z \in \mathbb{R}, \ \gamma^J \in \mathbb{R}_+, \ \beta^J \in \mathbb{R}_+^c, \ J \in \mathcal{J}. \)

2. LP dual of (LD) gives atomic underlying asset price distributions that is or converges to optimal solution of arbitrage bound problem.
Problem size:
- LP formulation has \((r + 1)(k + 1)\) variables and \(2nk\) constraints where \(k\) is the number of cells.
- But \(k \leq n\left(\frac{r+2}{n}\right)\) if \(n \leq r/2\) and \(k \leq 2^r\) for all \(r\).
- If \(r\) or \(n\) is small, then the problem (LD) is of manageable size.

Problem variations: by adding linear constraints or changing linear objective
- Bid/ask spreads
- Limits on the sizes of long/short positions
- Transaction costs.

Similar results follow for the lower bound case.
Problem size:

- LP formulation has \((r + 1)(k + 1)\) variables and \(2nk\) constraints where \(k\) is the number of cells.
- But \(k \leq n\binom{r+2}{n}\) if \(n \leq r/2\) and \(k \leq 2^r\) for all \(r\).
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Problem variations: by adding linear constraints or changing linear objective

- Bid/ask spreads
- Limits on the sizes of long/short positions
- Transaction costs.

Similar results follow for the lower bound case.
Upper bound given calls on single assets

Special version of problem

\[
\begin{align*}
\sup_{\pi} & \quad \mathbb{E}_\pi[(w_0 \cdot S - k_0)^+] \\
\text{s.t.} & \quad \mathbb{E}_\pi[S] = p^0 \\
& \quad \mathbb{E}_\pi[(S - K^j)^+] = p^j, \\
& \quad \pi \text{ a probability distribution in } \mathbb{R}_+^n.
\end{align*}
\]

- Prices of a forward \( p_0 = \mathbb{E}_\pi[S] \) and \( m \) calls \( p^j = \mathbb{E}_\pi[(S - K^j)^+] \), \( j = 1, \ldots, r \) for each asset are known.
- In this case # of cells is \( k = rn \). Our original LP-formulation has exponential size on \( n \) (and \( r \)).
- However, “facets” of LP can be found to obtain polynomial formulation.
  - Such formulation can be solved in closed-form.
Upper bound given calls on single assets (cont.)

LP (Primal) reformulation of special problem

\[
\max_{v,T,\tau,\tilde{v},\tilde{T},\tilde{\tau}} w_0 \cdot \sum_{i=0}^{m} v^i - K_0 \tau
\]

s.t.
\[
\tau + \tilde{\tau} = 1
\]
\[
\sum_{i=j}^{m} (v^i + \tilde{v}^i - (T^i + \tilde{T}^i) \circ K^j) = p^j, \quad j = 0, \ldots, r
\]

\[(v, T, \tau), (\tilde{v}, \tilde{T}, \tilde{\tau}) \in \text{cone}(K).\]

\[\text{cone}(K) = \left\{ (v, T, \tau) : \begin{array}{l}
T^i \circ K^i \leq v^i, \quad i = 0, \ldots, r, \\
v^i \leq T^i \circ K^{i+1}, \quad i = 0, \ldots, r - 1, \\
\sum_{i=0}^{m} T^i - \tau e = 0.
\end{array} \right\}\]
Upper bound given calls on single assets (cont.)

LP (Primal) reformulation of special problem

\[
\begin{align*}
\max_{v, T, \tau, \tilde{v}, \tilde{T}, \tilde{\tau}} & \quad w_0 \cdot \sum_{i=0}^{m} v^i - K_0 \tau \\
\text{s.t.} & \quad \tau + \tilde{\tau} = 1 \\
& \quad \sum_{i=j}^{m} (v^j + \tilde{v}^j - (T^i + \tilde{T}^i) \circ K^j) = p^j, \quad j = 0, \ldots, r \\
& \quad (v, T, \tau), (\tilde{v}, \tilde{T}, \tilde{\tau}) \in \text{cone}(K).
\end{align*}
\]

Once \( \tau \) is fixed, the LP decomposes into \( n \) one-dimensional problems of similar structure. Each of these problems has a closed-form solution.
Once $\tau$ is fixed, the LP decomposes into $n$ one-dimensional problems of similar structure. Each of these problems has a closed-form solution.
Upper bound given calls on single assets

- If *arbitrage free* condition holds:

**Theorem**

Assume $w_0 \geq 0$ and arbitrage-free on the prices of the given vanilla options condition. Then the optimal value of the especial super-replicating problem is

$$\max_{\tau \in [0,1]} \left( w_0 \cdot \nu(\tau) - \tau K_0 \right)$$

where

$$\nu(\tau)_i := \min_{j=0,\ldots,m} \left\{ p^j_i + \tau K^j_i \right\}, \quad i = 1, \ldots, n.$$ 

- As the components of $\nu(\tau)$ are concave piece-wise linear, the value (??) can be found by evaluating $mn + 2$ points.
Other features of this approach

- Optimal super-replicating strategy (i.e., optimal portfolio of cash and call options) can be computed in closed-form.
- A similar result holds when \( w_0 \) (weights of assets in target basket option) is not necessarily \( \geq 0 \); e.g. can compute bounds on exchange options in closed form:

  \[
  \text{Exchange Option Payoff} \; := \; (s_1 - s_2)^+
  \]

- By a simple change in the objective of the LP’s and the addition of extra linear constraints, we can deal:
  - Bid-ask spread
  - Transaction costs
The use of bid-ask prices solves one of the main limitations of arbitrage pricing techniques, which is to find *arbitrage-free* prices.

LP approach makes understanding and implementation of arbitrage pricing techniques very straightforward.

Previous results (e.g., d’Aspremont & El Ghaoui, Hobson et. al.) are greatly generalized.
Difficulties in obtaining good estimates for the problem parameters.

High sensitivity of classical portfolio allocation models to estimates in mean asset returns and correlations.

Use of measures of risk that take into account the desire of investors to minimize downside risk.

Classical portfolio allocation models many times produce portfolios that are not intuitively correct in the eyes of practitioners.
Nominal Portfolio Problem

- $r = (r_1, \ldots, r_n)^T \in \mathbb{R}^n$: uncertain returns of $n$ risky assets from the current time $t = 0$ to a fixed future time $t = T$.
- $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$: percentage of money allocated in each of the $n$ risky assets.

$$\max_{x \in \mathbb{R}_+^n} \mathbb{E}(r^T x) - \lambda \text{Risk}(r^T x)$$

s.t. $e^T x = 1$

For a *simpler exposition* we will consider only portfolios with *long positions* ($x \geq 0$)

- **Buy and hold**: portfolio bought at time $0$, hold until time $T$. 
Conditional Value-at-Risk (CVaR)

- We will use $CVaR_\beta$ as the measure of risk
  - Downside risk
  - Coherent measure (i.e., “nice” math properties)
  - Widely accepted in academia
- $CVaR_\beta$ is the expected value of the worst $(1 - \beta) \times 100\%$ losses; e.g., at $\beta = 95\%$ CVaR is the expected value of the 5% worst losses.
Rockafellar & Uryasev show that:

\[ \text{CVaR}_\beta(x, \pi(r)) = \min_{\alpha \in \mathbb{R}} F_\beta(x, \alpha, \pi(r)) \]

where

\[ F_\beta(x, \alpha, \pi(r)) := \alpha + (1 - \beta)^{-1} \mathbb{E}_{\pi(r)}((-r^T x - \alpha)^+) \]

is convex on \( \alpha \) (\( (x)^+ := \max\{x, 0\} \)).

With this, the nominal CVaR portfolio can be computed efficiently with LP when return distribution is estimated by the discrete distribution of past returns.
Robust CVaR portfolio allocation model

Following Zhu & Fukushima (among others), the *robust* counterpart of the nominal problem can be obtained by max *worst-case* mean-to-risk return over all asset return distributions in a uncertainty set.

\[
\max_{x \in \mathbb{R}^n_+} \inf_{\pi(r) \in \mathcal{P}} \mathbb{E}_{\pi(r)} (r^T x) - \lambda \text{CVaR}_\beta(x, \pi(r)) \\
\text{s.t.} \quad e^T x = 1
\]

*Uncertainty set* \(\mathcal{P}\) will be different from previous related work (e.g., Zhu & Fukushima, El Ghaoui et al., Goldfarb & Iyengar)

*Aim*: Choose \(\mathcal{P}\) s.t. simple to solve, no “estimation".
Uncertainty set for asset return distributions

- Idea is to use prices of options (call and forward) at $t = 0$, that mature at time $T$ to define the uncertainty set.
- This type of uncertainty set is typically used in "robust option pricing", commonly known as arbitrage-free pricing.
- First notice the relationship between returns and asset prices:

$$r_i = \frac{S_{T,i} - S_{0,i}}{S_{0,i}} = \frac{S_{T,i}}{S_{0,i}} - 1,$$

where:
- $S_{T,i}$: Price of asset $i$ at maturity $t = T$ (random)
- $S_{0,i}$: Price of asset $i$ currently $t = 0$ (known)

- So we can define the uncertainty set in terms of the asset price distributions at maturity $\pi(S_T)$. 

Uncertainty set for asset return distributions (cont.)

- **Uncertainty set**: Asset price distributions at maturity $\pi(S_T)$ that replicate prices of forward and call options with maturity $T$:

  \[
  \begin{align*}
  p^0 & : \text{ vector with forward option prices} \\
  p^i & : \text{ vector with European call option prices with strikes given by } K^i, i = 1 : m
  \end{align*}
  \]

  \[
  \mathcal{P} = \left\{ \pi(S_T) : \begin{array}{l}
  \mathbb{E}_{\pi(S_T)}(1) = 1 \\
  \mathbb{E}_{\pi(S_T)}(S_T) = p^0 \\
  \mathbb{E}_{\pi(S_T)}((S_T - K^i)^+) = p^i, i = 1, \ldots, m \\
  \pi(S_T) \text{ is a distribution in } \mathbb{R}^n_+
  \end{array} \right\}
  \]
**LP formulation of Robust CVaR problem**

- Recall the Robust CVaR problem:

  \[
  \max_{x \in \mathbb{R}_+^n} \inf_{\pi(S_T) \in \mathcal{P}} \mathbb{E}_{\pi(S_T)}(r^T x) - \lambda \ \text{CVaR}_\beta(x, \pi(S_T))
  \]

  \[\text{s.t.} \quad e^T x = 1\]

- Notice that for \(\pi(S_T) \in \mathcal{P}\)

  \[
  \mathbb{E}_{\pi(S_T)}(r^T x) = \mathbb{E}_{\pi(S_T)} \left( \left( \frac{S_T}{S_0} - e \right)^T x \right) = \left( \frac{p^0}{S_0} - e \right)^T x
  \]

- So the robust problem becomes:

  \[
  \max_{x \in \mathbb{R}_+^n} \left( \frac{p^0}{S_0} - e \right)^T x - \lambda \ \sup_{\pi(S_T) \in \mathcal{P}} \ \text{CVaR}_\beta(x, \pi(S_T))
  \]

  \[\text{s.t.} \quad e^T x = 1\]
Following Zhu & Fukushima, and the similar definition for Worst case VaR of El Ghaoui et al., define \textit{worst case CVaR}:

\[
\text{WCVaR}_\beta(x, \pi(S_T)) := \sup_{\pi(S_T) \in \mathcal{P}} \text{CVaR}_\beta(x, \pi(S_T))
\]

With an argument similar to Zhu & Fukushima, it follows that if \(\mathcal{P} \neq \emptyset\) (the uncertainty set is non-empty):

\[
\text{WCVaR}_\beta(x, \pi(S_T)) := \min_{\alpha \in \mathbb{R}} \sup_{\pi(S_T) \in \mathcal{P}} F_\beta(x, \alpha, \pi(S_T)) := \min_{\alpha \in \mathbb{R}} \alpha + (1 - \beta)^{-1} \sup_{\pi(S_T) \in \mathcal{P}} \mathbb{E}_{\pi(S_T) \in \mathcal{P}} \left( \left( - \left( \frac{S_T}{S_0} - e \right)^{\mathbf{T}} x - \alpha \right)^+ \right)
\]
Proposition (Peña, Vera, Z.)

If \( x \geq 0 \) and option prices are arbitrage-free, then \( \text{WCVaR}_\beta(x) = \)

\[
\min_{y, \alpha \in \mathbb{R}} (1 - \beta)^{-1} ((1 - \beta \alpha) - (x/S_0)^T p^0 + y)
\]

s.t.

\[
\begin{align*}
    y &\geq (x/S_0)^T \nu(\tau_{ij}) - \tau_{ij}(1 - \alpha), \ i = 1 : n, j = 1 : m, \\
    y &\geq (x/S_0)^T \nu(0) \\
    y &\geq (x/S_0)^T \nu(1) - (1 - \alpha)
\end{align*}
\]

where \( \nu(\tau)_i = \min_{j=0:m} \left\{ p^j_i + \tau K^j_i \right\}, \ \nu(\tau) = \frac{p^j_i - p^j_{i-1}}{K^j_i - K^j_{i-1}} \)

- Follows from previous closed-form result on the semiparametric bound \( \sup_{\pi(S_T) \in \mathcal{P}} \mathbb{E}_{\pi}(S_T) \left( (x^T S_T - K_0)^+ \right) \)
The final LP formulation of the Robust CVaR problem is:

\[
\begin{align*}
\max_{x \in \mathbb{R}^n_+, y \in \mathbb{R}, \alpha \in \mathbb{R}} & \quad (1 - \beta)^{-1} \left( (1 - \beta + \lambda) ((x/S_0)^T p^0 - 1) - \lambda y + \lambda \beta \alpha \right) \\
\text{s.t.} & \quad y \geq (x/S_0)^T \nu(\tau_{ij}) - \tau_{ij} (1 - \alpha), \quad i = 1 : n, j = 1 : m \\
& \quad y \geq (x/S_0)^T \nu(0) \\
& \quad y \geq (x/S_0)^T \nu(1) - (1 - \alpha) \\
& \quad e^T x = 1
\end{align*}
\]

Parameters

- \( \beta \): CVaR quantile (typically 95\%)
- \( \lambda \): Measures risk-return reward
- \( S_0 \): Initial asset prices
- \( p^0 \): Forward asset prices
- \( \tau_{ij}, \nu(\tau_{ij}) \): Depend on option prices

Characteristics

- No “estimation”
- CVaR as measure of risk
- LP: Simple to solve
- Robustness (theoretical)
Disclaimer: We have just started to experiment with this Robust CVaR portfolio allocation model!

To illustrate the model let’s consider a simple instance of the portfolio allocation model

Consider the problem of how strategically divide funds into five possible market portfolios:

<table>
<thead>
<tr>
<th>Ticker</th>
<th>Portfolio</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>OEX</td>
<td>S&amp;P 100</td>
<td>US Biggest Cap</td>
</tr>
<tr>
<td>SPX</td>
<td>S&amp;P 500</td>
<td>US Big Cap</td>
</tr>
<tr>
<td>MID</td>
<td>S&amp;P 400</td>
<td>US Mid Cap</td>
</tr>
<tr>
<td>RUT</td>
<td>RUSSELL 2000</td>
<td>US Small Cap</td>
</tr>
<tr>
<td>TYX</td>
<td>CBOE Treasury</td>
<td>US Bonds</td>
</tr>
</tbody>
</table>
We will compare the performance of our Robust CVaR portfolio model with the “1/N" asset allocation rule.

“1/N" asset allocation rule: If investing in $N$ risky assets, invest equally $100\% (1/N)$ on each of the assets

- No optimization
- No estimation

In a very comprehensive study, DeMiguel & Garlappi & Uppal\(^1\) show that:

- “1/N" rule is more robust than many other optimal portfolio allocation models.
- Intuition: Gain from optimal diversification is smaller than loss due to estimation error.
- One of their robust measures is out-of-sample Sharpe ratio

\(^1\)How inefficient are simple asset-allocation strategies?
### Data

- **Today** \((t = 0)\): \(\rightarrow 01/12/04\)
- **Maturity** \((T)\): \(\rightarrow 19/03/05\)
- **Out of sample data**: portfolios daily returns from 01/12/04 to 18/03/05
- **Portfolio Prices**: on 01/12/04
- **Options Prices**: prices of European call options on portfolios on 01/12/04 with maturity 19/03/05

<table>
<thead>
<tr>
<th>portfolio</th>
<th>date</th>
<th>maturity</th>
<th>Option</th>
<th>strike</th>
<th>mid-price</th>
<th>volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 400</td>
<td>1-Dec-04</td>
<td>19-Mar-05</td>
<td>Call</td>
<td>630</td>
<td>31.05</td>
<td>2</td>
</tr>
<tr>
<td>S&amp;P 400</td>
<td>1-Dec-04</td>
<td>19-Mar-05</td>
<td>Call</td>
<td>650</td>
<td>19.2</td>
<td>2</td>
</tr>
<tr>
<td>S&amp;P 100</td>
<td>1-Dec-04</td>
<td>19-Mar-05</td>
<td>Call</td>
<td>580</td>
<td>9.4</td>
<td>605</td>
</tr>
<tr>
<td>S&amp;P 100</td>
<td>1-Dec-04</td>
<td>19-Mar-05</td>
<td>Call</td>
<td>560</td>
<td>19.85</td>
<td>400</td>
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<tr>
<td>portfolio</td>
<td>out-sample mean return</td>
<td>std. dev.</td>
<td>sharpe ratio</td>
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<tr>
<td>1/N</td>
<td>0.0065%</td>
<td>0.0055</td>
<td>0.0125</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>S&amp;P 100 20%</td>
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<tr>
<td></td>
<td>S&amp;P 500 20%</td>
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<tr>
<td></td>
<td>S&amp;P 400 20%</td>
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</tr>
<tr>
<td></td>
<td>RUS 2000 20%</td>
<td></td>
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</tr>
<tr>
<td></td>
<td>BONDS 20%</td>
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</tr>
<tr>
<td>WCVaR</td>
<td>0.0065%</td>
<td>0.0046</td>
<td>0.0143</td>
<td></td>
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<td></td>
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<tr>
<td></td>
<td>S&amp;P 100 0%</td>
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<tr>
<td></td>
<td>S&amp;P 500 74%</td>
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<tr>
<td></td>
<td>S&amp;P 400 0%</td>
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</tr>
<tr>
<td></td>
<td>RUS 2000 26%</td>
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<tr>
<td></td>
<td>BONDS 0%</td>
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</tbody>
</table>

We set parameter $\lambda$ (mean-risk reward) such that out-of-sample mean return matches with $1/N$ rule.
Return Distribution

- 1/N
- WCVaR
Final Remarks

- Closed-form solutions can be found for some moment problems with important applications in Math Finance, and Inventory Theory.

- Using moments is another valid alternative to obtain robust formulations of problems:
  - Option pricing problems
  - Portfolio allocation problems

- LP techniques can also be used to obtain important “moment” results, with the benefit of easy implementation.

- If interested, check [www.optimization-online.org](http://www.optimization-online.org)
  - “Extensions of Lo’s semiparametric bound for European call options”
  - “Static-arbitrage bounds on the prices of basket options via linear programming”