SALMA, HERE IS THE BANFF GONDOLA UPDATE!

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This is a continuation and extension of my previous mail, and will also appear in [1] This family of examples was inspired by looking at the trees on Sulphur Mountain from the descending gondola and imagining them as lattice points. Viva Banff!

Let \( A_d = \{(i, j) : 0 \leq i, j \leq d-1, i+j \leq d\} \). There are two ways to think of \( A_d \). One is that it is the set of lattice points in the triangle with vertices \((0,0), (d,0), (0,d)\), with the two vertices \((d,0)\) and \((0,d)\) removed. It’s clear in this way that \(|A_d| = \binom{d+2}{2} - 1 = \frac{d^2+3d-2}{2}\). The other way of thinking of \( A_d \) is that it consists of the southwest part of a \( d \times d \) square, including the diagonal and the first superdiagonal. For example,

\[
\begin{array}{ccc}
\circ & \circ & x & x \\
\circ & \circ & \circ & x \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\hline
1 \end{array}
\]

\( A_5 \quad = \quad \begin{array}{cccc}
\circ & \circ & \circ & \circ & x \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\end{array} \quad \begin{array}{cc}
\circ & \circ \\
\circ & \circ \\
\end{array} \]

It can be shown that if \( \text{deg}(p) = d \) and \( p|_{A_d} = 0 \), then \( p = \alpha f + \beta g \), where

\( f_d(x, y) = x(x-1) \ldots (x-(d-1)), \quad g_d(x, y) = y(y-1) \ldots (y-(d-1)) \)

so that \( p \) also vanishes on the other \( \frac{d^2-3d+2}{2} \) points of the square. (This is the maximum by Bezout’s theorem.) On the other hand, the polynomial

\( L_d(x, y) = xy(x-1)(y-1)(x+y-2)(x+y-3)(x+y-4)2(x+y-5)^2 \ldots (x+y-d)^2 \)

has the property that it is singular on \( A_d \) and positive on the points of the NE corner.

It follows from the Hilbert construction that for some \( c(d) > 0 \),

\( f(x, y)^2 + g(x, y)^2 + cL(x, y) \)

is positive if \( 0 \leq c \leq c(d) \) and not sos if \( 0 < c \). In particular, I have computed that \( c(3) = 4/3 \) (exactly) and that \( c(d) \leq 12d^{-2} \), so \( c(d) \to 0 \).

REFERENCES


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