SALMA, HERE IS THE BANFF GONDOLA UPDATE!

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This is a continuation and extension of my previous mail, and will also appear in [1] This family of examples was inspired by looking at the trees on Sulphur Mountain from the descending gondola and imagining them as lattice points. Viva Banff!

Let $A_d = \{(i,j) : 0 \le i, j \le d-1, i+j \le d\}$. There are two ways to think of A_d. One is that it is the set of lattice points in the triangle with vertices $(0,0), (d,0), (0,d)$, with the two vertices $(d,0)$ and $(0,d)$ removed. It's clear in this way that $|A_d| = \binom{d+2}{2} - 2 = \frac{d^2+3d-2}{2}$. The other way of thinking of A_d is that it consists of the southwest part of a $d \times d$ square, including the diagonal and the first superdiagonal. For example,

$$A_5 = \begin{matrix} \circ & \circ & x & x & x \\ \circ & \circ & \circ & x & x \\ \circ & \circ & \circ & \circ & x \\ \circ & \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ & \circ \end{matrix} \tag{1}$$

It can be shown that if $deg(p) = d$ and $p|_{A_d} = 0$, then $p = \alpha f + \beta g$, where

$$f_d(x,y) = x(x-1)\dots(x-(d-1)), \qquad g_d(x,y) = y(y-1)\dots(y-(d-1)) \tag{2}$$

so that p also vanishes on the other $\frac{d^2-3d+2}{2}$ points of the square. (This is the maximum by Bezout's theorem.) On the other hand, the polynomial

$$L_d(x,y) = xy(x-1)(y-1)(x+y-2)(x+y-3)(x+y-4)^2(x+y-5)^2\dots(x+y-d)^2 \tag{3}$$

has the property that it is singular on A_d and positive on the points of the NE corner.

It follows from the Hilbert construction that for some $c(d) > 0$,

$$f_(x,y)^2 + g(x,y)^2 + cL(x,y) \tag{4}$$

is positive if $0 \le c \le c(d)$ and not sos if $0 < c$. In particular, I have computed that $c(3) = 4/3$ (exactly) and that $c(d) \le 12d^{-2}$, so $c(d) \to 0$.

References

[1] Reznick, B. The Hilbert construction of positive polynomials that are not sos, in preparation.

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