Devote this talk to

Scarlett's 2^{nd} Week Birthday

Sparse SOS Relaxation and Applications

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Outline of Talk

- Motivation

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- Sparse SOS relaxations

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Motivation

- Let $f_1(x_{\Delta_1}), \cdots, f_m(x_{\Delta_m}) \in \mathbb{R}[x_1, \cdots, x_n]$

$$\Delta_1 \cup \cdots \cup \Delta_m = [n] := \{1, 2, \cdots, n\}$$

$$x_{\Delta_i} = \{x_k : k \in \Delta_i\}$$

- sparse polynomial optimization

$$\min_{x \in \mathbb{R}^n} \quad f(x) := \sum_{i=1}^{m} f_i(x_{\Delta_i})$$

- We want efficient SOS relaxations

SOS relaxation

- Dense SOS relaxation (Lasserre, Parrilo, ...)

$$f^*_{sos} := \max \quad \gamma \quad \text{s.t.} \quad f(x) - \gamma \ \text{ is SOS}$$

- Complexity bounds ($n = \#vars, \ \ 2d = \deg(f(x))$)
  - # variables $= \ \mathcal{O}(n^{2d})$
  - size of LMI $= \ \mathcal{O}(n^{d})$
  - comp. cost $= \ \mathcal{O}(n^{6d})$ (one IP iter.)

- Very expensive for medium or large size problems (e.g., $n = 30, 2d = 6$)

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Some Previous Work

- Finding the sparse support(Reznick, Parrilo, ...)

$$f(x) = \sum_i s_i^2(x) \Longrightarrow \quad \text{supp}(s_i(x)) \subset \ conv\{\frac{1}{2}\text{supp}(f(x))\}$$

Sometimes, this sparsity is too loose, e.g., $x_1^{2d} + \cdots + x_n^{2d}$

- Find the further sparse support (Muramatsu, Kojima, Kim, Waki, ...)
  - Use Cholesky factor of CSP matrix R
  - Can solve some large problems, but it might fail
  - Still expensive when R has dense Cholesky factor
- Convergent sparse SOS relaxations (Lasserre)
- Symmetry (Gaterman, Parrilo, Theobald, Scheiderer,...)

Sparse SOS relaxation

$$\min_{x \in \mathbb{R}^n} \quad f(x) := \sum_{i=1}^{m} f_i(x_{\Delta_i}) \qquad \|\Delta\| = \max_i |\Delta_i| \ll n$$

Sparse SOS relaxation

$$f_\Sigma^* := \max \quad \gamma \quad \textbf{s.t.} \quad f(x) - \gamma \in \sum_{i=1}^{m} s_i(x_{\Delta_i}) \in \sum \mathbb{R}_d[x_{\Delta_i}]^2.$$

The dual SDP is

$$\min_y \quad \sum_\alpha f_\alpha y_\alpha$$
$$s.t. \quad \mathcal{M}_d^{\Delta_i}(y) \succeq 0, \; i = 1, \cdots, m$$

Extracting Solutions

- Flat extension condition (Curto, Fialkow, Lasserre, Laurent, ...)
  rank$\mathcal{M}_d^{\Delta_i}(y) =$ rank$\mathcal{M}_{d-1}^{\Delta_i}(y)$
- For each Δ_i, find partial solution set $\mathcal{V}_i \subset \mathbb{R}^{\Delta_i}$, the projection $\mathbb{R}^n \to \mathbb{R}^{\Delta_i}$ of solutions
- If rank condition fails, apply random perturbation:

$$\min_{x \in \mathbb{R}^n} \quad f(x) := \sum_{i=1}^{m} f_i(x_{\Delta_i}) \quad + \quad \sum_{i=1}^{m} \xi_i x_i$$

where $\xi = (\xi_1, \cdots, \xi_n)$ is tiny and random

Complexity Comparison

- # variables $= \mathcal{O}(m\|\Delta\|^{2d})$
(for dense SOS $\mathcal{O}(n^{2d})$)
- size of LMI $= \mathcal{O}(\|\Delta\|^{d})$, m copies
(for dense SOS, one single $\mathcal{O}(n^{d})$)
- comp. cost $= \mathcal{O}(m^3\|\Delta\|^{6d})$ (one IP iter.)
(for dense SOS $\mathcal{O}(n^{6d})$)

Nonlinear least square problem

$$g_1(x_{\Delta_1}) = g_2(x_{\Delta_2}) = \cdots = g_m(x_{\Delta_m}) = 0$$

Set $f_i(x_{\Delta_i}) = g_i^2(x_{\Delta_i})$, and then solve

$$\min_{x \in \mathbb{R}^n} f(x) := \sum_{i=1}^{m} f_i(x_{\Delta_i})$$

- If the polynomial system is consistent,

$$f_{\Delta}^* = f_{sos}^* = f^* = 0$$

- This lower bound is not interesting
- Dual optimal solution y^* recovers the real zeros

One example

Consider the sparse polynomial system

$$2x_1^2 - 3x_1 + 2x_2 - 1 = 0$$
$$2x_i^2 + x_{i-1} - 3x_i + 2x_{i+1} - 1 = 0 \, (i = 2, \cdots, n-1)$$
$$2x_n^2 + x_{n-1} - 3x_n - 1 = 0.$$

Set it as an LS problem, and then solve by sparse SOS

$$\hat{x} = (1.8327, \, -0.1097, \, -0.5929, \cdots, -0.6658, \, -0.5960, \, -0.4164)$$
$$\tilde{x} = (-0.5708, \, -0.6819, \, -0.7025, \cdots, -0.6658, \, -0.5960, \, -0.4164).$$

A few seconds for $n = 100$

The sparse SOS might fail

Consider problem

$$\min_{x\in\mathbb{R}^3} \quad \underbrace{x_1^4 + (x_1x_2 - 1)^2}_{f_1(x_{\Delta_1})} + \underbrace{x_2^2x_3^2 + (x_3^2 - 1)^2}_{f_2(x_{\Delta_2})}$$

$$\Delta_1 = \{1,2\}, \qquad \Delta_2 = \{2,3\}$$

Numerical computation shows

$$f_\Delta^* \approx 5\cdot 10^{-5}$$

$$f_{sos}^* \approx 0.8499$$

$$f^* \approx 0.8650$$

$$f_\Delta^* < f_{sos}^* < f^*$$

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Randomly generated problems

Choose random $f(x) = \sum_i f_i(x_{\Delta_i})$ where

$$f_i(x_{\Delta_i}) = \mathbf{m}_d(x_{\Delta_i})^T \cdot A_i \cdot \mathbf{m}_d(x_{\Delta_i}) + b_i^T \mathbf{m}_{2d-1}(x_{\Delta_i}).$$

where Δ_i are chosen to be random subsets of $[n]$ such that $\|\Delta_i\| \leq \|\Delta\| \ll n$. Here A_i is

$$A_i = \sqrt{n} I_{N_i} + u_i u_i^T$$

This makes the global minimizers of $f(x)$ are contained in some compact set.

$$accuracy = \frac{|f(\hat{x}) - f_\Delta^*|}{\max\{1, |f(\hat{x})|\}}.$$

Computational Result (1)

	$\|\Delta\| = 3$				$\|\Delta\| = 4$			
	CPU seconds			accu	CPU seconds			accu
n	max	avr.	min	max	max	avr.	min	max
20	0.85	0.62	0.54	4.1e-9	1.46	1.15	0.91	2.4e-9
40	1.22	1.07	0.91	1.9e-9	2.86	2.49	2.25	2.9e-9
60	1.80	1.55	1.45	2.9e-9	4.43	4.17	3.91	3.1e-9
80	2.30	2.18	2.02	2.3e-9	6.26	5.94	5.24	3.7e-9
100	3.02	2.70	2.33	2.8e-9	7.85	7.41	7.01	5.0e-9

Table 1: Computational results for quartic polynomials with different sizes (deg = 4)

Computational Result (2)

	$\|\Delta\| = 3$				$\|\Delta\| = 4$			
	CPU seconds			accu	CPU seconds			accu
$2d$	max	avr	min	max	max	avr	min	max
4	1.01	0.87	0.77	2.5e-9	2.33	1.93	1.65	2.4e-9
6	3.22	2.96	2.67	1.8e-9	17.16	14.92	11.71	2.2e-9
8	13.07	11.44	10.13	1.7e-8	136.67	119.90	107.28	9.4e-8

Table 2: Computational results for polynomials of size $n = 30$ with different degrees

Solving Nonlinear Differential Equations

Consider the two-point boundary value problem (BVP)

$$F(t, x, x', x'') = 0, \;\; x(a) = \alpha, \; x(b) = \beta$$

where $F(t, x, x', x'') \in \mathbb{R}[t, x, x', x'']$. Use step size $h = \frac{b-a}{N+1}$ to discretize, then we get

$$F\left(t_k, x_k, \frac{x_{k+1} - x_{k-1}}{2h}, \frac{x_{k-1} - 2x_k + x_{k+1}}{h^2}\right) = 0, \; k = 1, \cdots, N$$

Solve this polynomial system by applying the sparse SOS relaxation method.

Boundary Value Problem (1)

$$x'' - 2x^3 = 0, \quad x(0) = \frac{1}{2},\ x(1) = \frac{1}{3}$$

N	eqn. error	$\|x_k - x(t_k)\|_\infty$	$\|x_k - x(t_k)\|_\infty / h^2$	time(sec)
20	5.2879e-07	1.5041e-05	6.6331e-003	1.18
30	2.6194e-07	1.9413e-05	1.8656e-002	2.09
40	3.0304e-07	4.3344e-05	7.2861e-002	3.99
50	6.5375e-07	1.5124e-04	3.9338e-001	6.82
60	1.5271e-06	4.8695e-04	1.8119e+00	7.77
70	1.2555e-06	5.2428e-04	2.6429e+00	9.16
80	9.7315e-07	6.1330e-04	4.0239e+00	9.78
90	2.7519e-06	1.9311e-03	1.5991e+01	10.81
100	1.8628e-06	8.1425e-04	8.3062e+00	8.79

Boundary Value Problem (2)

$$x'' + \frac{1}{2}(x+t)^3 = 0, \;\; x(0) = 0, \, x(1) = 0.$$

Use step size $h = \frac{1}{N+1}$ to discretize, then we get

$$2x_1 - x_2 + \frac{1}{2}h^2(x_1 + t_1)^3 = 0$$

$$2x_i - x_{i-1} - x_{i+1} + \frac{1}{2}h^2(x_i + t_i)^3 = 0, \, i = 2, \cdots, N-1$$

$$2x_N - x_{N-1} + \frac{1}{2}h^2(x_N + t_N)^3 = 0$$

For $N = 30$, we get the real solution in 2.5 CPU seconds

$$(-0.0159, -0.0312, -0.0459, -0.0600, -0.0735, -0.0864, -0.0985, -0.1099, -0.1205, -0.1302,$$
$$-0.1391, -0.1470, -0.1540, -0.1599, -0.1646, -0.1682, -0.1705, -0.1715, -0.1710, -0.1689,$$
$$-0.1651, -0.1596, -0.1521, -0.1425, -0.1307, -0.1164, -0.0995, -0.0796, -0.0567, -0.0302).$$

Sensor Network Localization

- Given Data:
  - anchors: $a_1, \cdots, a_m \in \mathbb{R}^d$
  - # sensors: n ($u_1, u_2, \cdots, u_n$ not given)
  - edge set: $\mathcal{A} = \{(i,j) : \ u_i \text{ is connected to } u_j\}$
    $\mathcal{B} = \{(i,k) : \ u_i \text{ is connected to } a_k\}$
  - distance: $d_{ij} \, ((i,j) \in \mathcal{A}), \ \ e_{ik} \, ((i,k) \in \mathcal{B})$
- Problem: Find sensors $u_1, u_2, \cdots, u_n \in \mathbb{R}^d$ such that

$$\|u_i - u_j\|_2 = d_{ij} \ \ \forall (i,j) \in \mathcal{A}$$

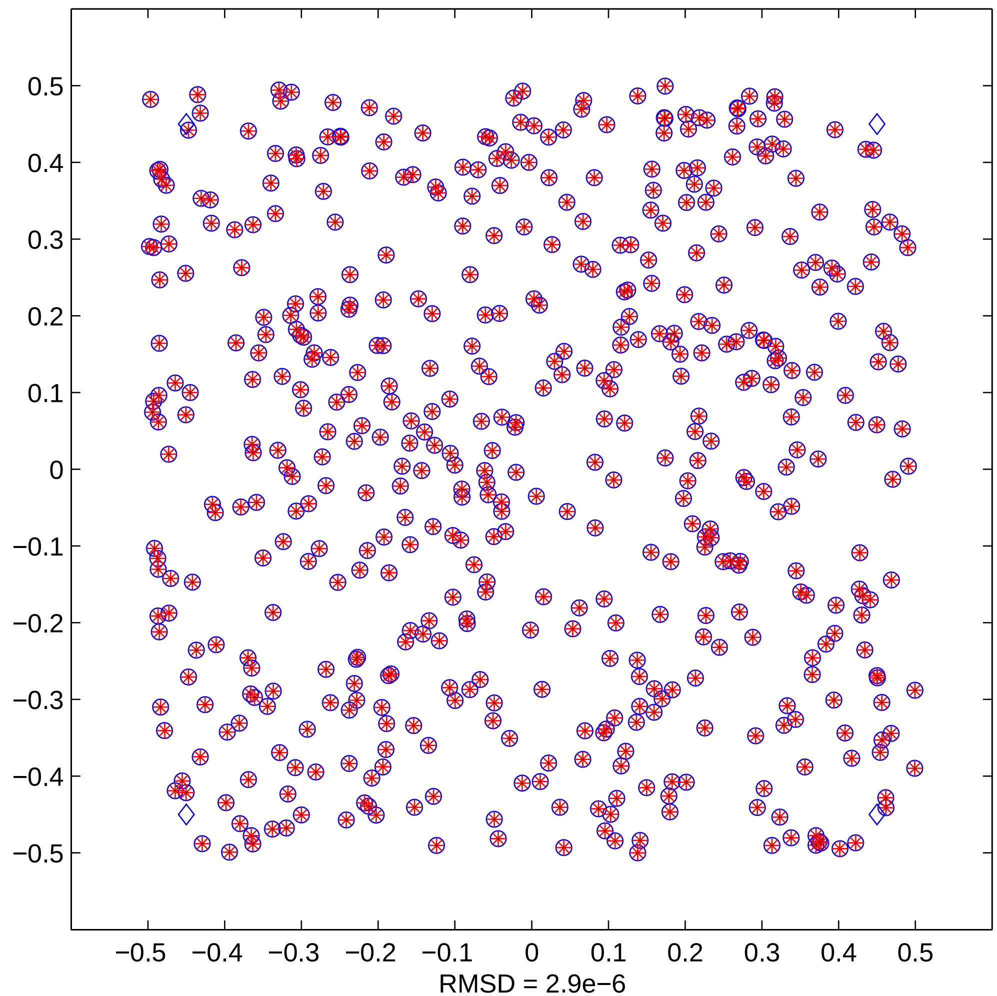
$$\|u_i - a_k\|_2 = e_{ik} \ \ \forall (i,k) \in \mathcal{B}$$

- NP-hard problem

Example

500 sensors, 4 anchors, range dist 0.3

This poly. system
has 1000 unknowns
and 11000 equations,
and was solved
in about 18 CPU mins



Perturbation Error Bound

$$f(X) = \sum_{(i,j)\in\mathcal{A}} \left(\|u_i - u_j\|_2^2 - d_{ij}^2\right)^2 + \sum_{(i,k)\in\mathcal{B}} \left(\|u_i - a_k\|_2^2 - e_{ik}^2\right)^2$$

Sparse SOS relax. returns true sensor location X^*

$$(d_{ij}, e_{ik}) \mapsto U^* = [u_1^*, \cdots, u_n^*]$$

$$U^* = \Psi((d_{ij}, e_{ik}))$$

Under some technical assump., it holds that

$$\underbrace{|\Psi((d_{ij}, e_{ik})) - \Psi((\hat{d}_{ij}, \hat{e}_{ik}))|}_{\text{sensor location error}} \le C \cdot \underbrace{\max\{|d_{ij} - \hat{d}_{ij}|, |e_{ik} - \hat{e}_{ik}|\}}_{\text{distance error}}$$

$$\varepsilon_U \le C \cdot \varepsilon$$

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Constrained Problem

$$f^* := \min_{x \in \mathbb{R}^n} \quad \sum_{i=1}^{m} f_i(x_{\Delta_i})$$

$$s.t. \quad g_{i,j}(x_{\Delta_i}) \geq 0,\ j \in \Omega_i,\ 1 \leq i \leq m$$

A nature sparse SOS relaxation is

$$f_N^* := \max \quad \gamma$$

$$s.t. \quad f(x) - \gamma - \sum_{i=1}^{m} \sum_{j \in \Omega_i} \sigma_{i,j}(x_{\Delta_i}) g_{i,j}(x_{\Delta_i}) \in \sum_i \sum \mathbb{R}_N[x_{\Delta_i}]^2$$

$$\deg(\sigma_{i,j} g_{i,j}) \leq 2N, \ \ \sigma_{i,j} \text{ is SOS},\ j \in \Omega_i,\ 1 \leq i \leq m.$$

Sparse Putinar's Theorem

Running Intersection Property (RIP):

$$\forall\, 1 \le i \le m,\ \exists\, k \le i-1 \ \text{ s.t. } \quad \Delta_i \cap (\cup_{j=1}^{i-1} \Delta_j) \subset \Delta_k.$$

Theorem (Lasserre): Under RIP and archimedean assumption, the lower bounds of sparse SOS relaxation converge to minimum

$$f_N^* \to f^*.$$

RIP is necessary

Running Intersection Property (RIP):

$$\forall\, 1 \le i \le m-1,\ \exists\, k \le i \ \text{ s.t. } \quad \Delta_{i+1} \cap \left(\cup_{j=1}^{i} \Delta_j\right) \subsetneq \Delta_k.$$

Without RIP, the convergence is not guaranteed, e.g.,

$$\min_{x \in \mathbb{R}^3} \quad \underbrace{\frac{1}{2}(x_1 + x_2)}_{f_1(x_{\Delta_1})} + \underbrace{\frac{1}{2}(x_1 + x_3)}_{f_2(x_{\Delta_2})} + \underbrace{\frac{1}{2}(x_2 + x_3)}_{f_3(x_{\Delta_2})}$$

$$s.t. \quad x_1^2 = x_1,\ x_2^2 = x_2,\ x_3^2 = x_3,$$

$$x_1 + x_2 \ge 1,\ x_1 + x_3 \ge 1, x_2 + x_3 \ge 1$$

$$\Delta_3 \cap \cup_{j=1}^{2} \Delta_j \not\subseteq \Delta_1, \quad \Delta_3 \cap \cup_{j=1}^{2} \Delta_j \not\subseteq \Delta_2.$$

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  Algebraic degree of Semidefinite Programming

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Thank you very much !

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