Semidefinite characterization and computation of real radical ideals

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The problem

Given an ideal I generated by $h_1, \ldots, h_m \in \mathbb{R}[x_1, \ldots, x_n]$
with $|V_{\mathbb{R}}(I)| < \infty$
Find $V_{\mathbb{R}}(I)$ and the **real radical ideal** $I(V_{\mathbb{R}}(I))$

$$V_{\mathbb{R}}(I) := \{v \in \mathbb{R}^n \mid f(v) = 0 \; \forall f \in I\}$$

$$I(V_{\mathbb{R}}(I)) := \{f \in \mathbb{R}[x] \mid f(v) = 0 \; \forall v \in V_{\mathbb{R}}(I)\}$$

$$= \sqrt[\mathbb{R}]{I} := \{f \in \mathbb{R}[x] \mid f^{2m} + s \in I \text{ for some } m \in \mathbb{N}, s \text{ s.o.s.}\}$$

Our contribution

- A *semidefinite characterization of* $I(V_{\mathbb{R}}(I))$ [as the kernel of a suitable positive semidefinite moment matrix]

- An algorithm for finding a (border / Gröbner for any total degree term ordering) basis of $I(V_{\mathbb{R}}(I))$, a linear basis of $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$, and $V_{\mathbb{R}}(I)$

Remarks about the method:

- *real algebraic* in nature: no complex roots computed
- no Gröbner basis of I is needed
- *numerical*, based on SDP
- works if $V_{\mathbb{R}}(I)$ is finite (even if $V(I)$ is not)
- works for $I(V_{\mathbb{R}}(I) \cap S)$, where $S = \{x \mid g_j(x) \geq 0 \; j = 1, \ldots, k\}$
- extends to the complex case: for $V(I)$ and $I(V(I))$

The complex case is well understood

Given an ideal $I \subseteq \mathbb{C}[x]$ with $|V(I)| < \infty$,
find $V(I)$ and the **radical ideal** $I(V(I))$

Let p_i be the monic generator of $I \cap \mathbb{C}[x_i]$
(easy to find if we have a basis of $\mathbb{C}[x]/I$)

Let q_i be the square-free part of p_i, $q_i = \frac{p_i}{\text{g.c.d.}(p_i, p_i')}$

Seidenberg [1974]: $I(V(I)) = \langle I \cup \{q_1, \ldots, q_n\} \rangle$.

Several performant methods for computing $V(I)$; e.g. RUR method [Rouillier], homotopy methods [Verschelde], eigenvalue method, ...

Some notation

- $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$
- Identify a polynomial $p \in \mathbb{R}[x]$, $x \mapsto p(x) = \sum_\alpha p_\alpha x^\alpha$, with its sequence of coefficients $\mathrm{vec}(p) := (p_\alpha)_\alpha$
- $v \in \mathbb{R}^n \rightsquigarrow \boxed{\zeta_v := (v^\alpha)_\alpha}$

Thus: $p(v) = \mathrm{vec}(p)^T \zeta_v$

- $\mathbb{N}_t^n := \{\alpha \in \mathbb{N}^n \mid |\alpha| = \sum_i \alpha_i \leq t\}$
- $\mathbb{T}_n := \{x^\alpha \mid \alpha \in \mathbb{N}^n\}$, $\mathbb{T}_{n,t} := \{x^\alpha \mid \alpha \in \mathbb{N}_t^n\}$

Moment matrices

$y \in \mathbb{R}^{\mathbb{N}^n} \rightsquigarrow M(y)$ matrix indexed by $\mathbb{N}^n$

$$\boxed{[M(y)]_{\alpha,\beta} := y_{\alpha+\beta} \quad (\alpha, \beta \in \mathbb{N}^n)}$$

Fact: If $M(y) \succeq 0$, then $\mathrm{Ker} M(y)$ is a real radical ideal in $\mathbb{R}[x]$

Notation: Say that "$p \in \mathrm{Ker} M(y)$" if $\mathrm{vec}(p) \in \mathrm{Ker} M(y)$

Sketch of proof: Use identity:

$$\boxed{\mathrm{vec}(f)^T M(y) \mathrm{vec}(gh) = \mathrm{vec}(fg)^T M(y) \mathrm{vec}(h) \ \ \forall f, g, h \in \mathbb{R}[x]}$$

$$p^2 \in \mathrm{Ker} M(y) \Longrightarrow 0 = \mathrm{vec}(1)^T M(y) \mathrm{vec}(p^2) = \mathrm{vec}(p)^T M(y) \mathrm{vec}(p)$$

$$\Longrightarrow p \in \mathrm{Ker} M(y)$$

$$p^{2m} + \sum_i g_i^2 \in \mathrm{Ker} M(y) \Longrightarrow p^m \in \mathrm{Ker} M(y) \Longrightarrow p \in \mathrm{Ker} M(y)$$

Positive semidefinite moment matrices

Fact: If y has a **representing measure** μ, i.e.,

$$\boxed{\; y_\alpha = \int x^\alpha d\mu(x) \;\; (\alpha \in \mathbb{N}^n) \;}$$

and μ is positive, then $M(y) \succeq 0$ and $\mathrm{supp}(\mu) \subseteq V(\mathrm{Ker} M(y))$

Proof: $\mathrm{vec}(p)^T M(y) \mathrm{vec}(p) = \sum_{\alpha,\beta} p_\alpha p_\beta y_{\alpha+\beta} = \sum_{\alpha,\beta} p_\alpha p_\beta \int x^{\alpha+\beta} d\mu(x) = \int p(x)^2 d\mu(x) \geq 0$

Say that μ is r-atomic if $|\mathrm{supp}(\mu)| = r$,
i.e., $\mu = \sum_{i=1}^r \lambda_i \delta_{v_i}$ for some $v_i \in \mathbb{R}^n$, $\lambda_i > 0$.

Then, $M(y) = \sum_{i=1}^r \lambda_i \zeta_v \zeta_v^T$ has rank r.

Finite rank moment matrices

Thm CF1 [Curto-Fialkow 1996]: $M(y) \succeq 0$, $\mathrm{rank} M(y) < \infty$

$\iff y$ has a finitely atomic positive representing measure μ

Sketch of proof [La 2005]:

- $J := \mathrm{Ker} M(y)$ is real radical ideal $\Longrightarrow V(J) \subseteq \mathbb{R}^n$
- $\mathcal{B} \subseteq \mathbb{T}_n$ basis of column space of $M(y) \iff \mathcal{B}$ basis of $\mathbb{R}[x]/J$

Hence: $|V(J)| = \dim \mathbb{R}[x]/J = \mathrm{rank} M(y) < \infty$

- p_v $(v \in V(J))$ interpolation polynomials $\rightsquigarrow$ basis of $\mathbb{R}[x]/J$
- $M(y) = \sum_{v \in V(J)} \underbrace{\mathrm{vec}(p_v)^T M(y) \mathrm{vec}(p_v)}_{\lambda_v > 0} \zeta_v \zeta_v^T$

Hence: $\mu = \sum_{v \in V(J)} \lambda_v \delta_v$ is a representing measure for y,

with $\mathrm{supp}(\mu) = V(J)$ and $\mathrm{rank} M(y) = |V(J)|$

Semidefinite characterization of $I(V_{\mathbb{R}}(I))$

$$I = \langle h_1, \ldots, h_m \rangle \text{ with } |V_{\mathbb{R}}(I)| < \infty$$

$$\boxed{K := \{y \in \mathbb{R}^{\mathbb{N}^n} \mid M(y) \succeq 0,\ M(h_j y) = 0\ (j = 1, \ldots, m)\}}$$

setting $hy := M(y)\mathrm{vec}(h)$, i.e., $(hy)_\alpha = \sum_\beta h_\beta y_{\alpha+\beta}$

Theorem 1:

- $K = \mathrm{cone}(\zeta_v \mid v \in V_{\mathbb{R}}(I))$
- $I(V_{\mathbb{R}}(I)) \subseteq \mathrm{Ker} M(y)$ for all $y \in K$, with ***equality*** iff $\mathrm{rank} M(y)$ is maximum ($= |V_{\mathbb{R}}(I)|$), i.e., $y \in \mathrm{int}(K)$

Proof: $y \in K \Longrightarrow I \subseteq \mathrm{Ker} M(y) \quad \Longrightarrow V(\mathrm{Ker} M(y)) \subseteq V_{\mathbb{R}}(I)$

$$\Longleftrightarrow I(V_{\mathbb{R}}(I)) \subseteq \mathrm{Ker} M(y)$$

Thus: $\mathrm{rank} M(y) = |V(\mathrm{Ker} M(y)| \leq |V_{\mathbb{R}}(I)|$

How to turn this into an algorithm?

Answer: Use *truncated* moment matrices $M_t(y)$ [indexed by $\mathbb{N}^n_t$]

$$\boxed{K_t := \{y \in \mathbb{R}^{\mathbb{N}^n_{2t}} \mid M_t(y) \succeq 0,\ M_{t-d_j}(h_j y) = 0\ (j = 1, \ldots, m)\}}$$

for $t \geq d$, setting $d_j := \lceil \deg(h_j)/2 \rceil$, $d := \max_j d_j$

Theorem 2: Let $y \in K_t$ with $\mathrm{rank} M_t(y)$ maximum.
Consider the conditions:
(F1) $\mathrm{rank} M_s(y) = \mathrm{rank} M_{s-d}(y)$ for some $d \leq s \leq t$,
(F2) $\mathrm{rank} M_s(y) = \mathrm{rank} M_{s-1}(y)$ for some $2d \leq s \leq t$.
If (F1) or (F2) holds, then $I(V_{\mathbb{R}}(I)) = \langle \mathrm{Ker} M_s(y) \rangle$.

We say that '$M_s(y)$ **is a flat extension of** $M_{s-1}(y)$' when $\mathrm{rank} M_s(y) = \mathrm{rank} M_{s-1}(y)$.

Proof

Claim 1: $\mathrm{Ker} M_s(y) \subseteq I(V_{\mathbb{R}}(I))$

Pf: [Geometric property of SDP]
$\mathrm{rank} M_t(y)$ maximum $\Longrightarrow \mathrm{Ker} M_t(y) \subseteq \mathrm{Ker} M_t(y') \quad \forall y' \in K_t$
$\Longrightarrow \mathrm{Ker} M_t(y) \subseteq \cap_{v \in V_{\mathbb{R}}(I)} \mathrm{Ker} M_t(\zeta_v) \subseteq I(V_{\mathbb{R}}(I))$ [as $\zeta_v \in K_t$]
As $M_t(\zeta_v) = \zeta_v \zeta_v^T, \quad p \in \mathrm{Ker} M_t(\zeta_v) \Longrightarrow p(v) = 0$

Claim 2: $I \subseteq \langle \mathrm{Ker} M_s(y) \rangle$

Pf: $h_1, \ldots, h_m \in \mathrm{Ker} M_s(y)$ [by (F1), or (F2)]

Claim 3: $\langle \mathrm{Ker} M_s(y) \rangle$ is real radical
[see *Flat extension theorem* below]

Thus:

$$I \subseteq \langle \mathrm{Ker} M_s(y) \rangle \subseteq I(V_{\mathbb{R}}(I)) \Longrightarrow V(\mathrm{Ker} M_s(y)) = V_{\mathbb{R}}(I)$$
$$\Longrightarrow \langle \mathrm{Ker} M_s(y) \rangle = I(V_{\mathbb{R}}(I)).$$

Curto-Fialkow flat extension theorem

Theorem CF2 [Curto-Fialkow 1996]:

$M_s(y) \succeq 0, \, \mathrm{rank} M_s(y) = \mathrm{rank} M_{s-1}(y)$

$\Longrightarrow y$ can be extended to $\tilde{y} \in \mathbb{R}^{\mathbb{N}^n}$ in such a way that

$\mathrm{rank} M(\tilde{y}) = \mathrm{rank} M_s(y), \;\; M(\tilde{y}) \succeq 0, \;\; \mathrm{Ker} M(\tilde{y}) = \langle \mathrm{Ker} M_s(y) \rangle$.

Property: For $\mathcal{B} \subseteq \mathbb{T}_{n,s}$

$\mathcal{B}$ indexes a maximum nonsingular principal submatrix of $M_s(y)$

[i.e., $\mathcal{B}$ is a basis of the column space of $M_s(y)$]

$\Longleftrightarrow \mathcal{B}$ is a basis of $\mathbb{R}[x]/\langle \mathrm{Ker} M_s(y) \rangle$

The method terminates

Theorem 3: (F1) holds for t large enough.

Claim 1: For $t \geq 2d$, $\mathrm{Ker} M_t(y) \supseteq h_1, \ldots, h_m$.

Claim 2: For $t \geq t_0$, $\mathrm{Ker} M_t(y) \supseteq \{g_1, \ldots, g_k\}$, a Gröbner basis of $I(V_{\mathbb{R}}(I))$ for a total-degree monomial ordering.

Pf: $g_l^{2m_l} + \sigma_l = \sum_{j=1}^m u_j^{(l)} h_j$, where σ_l s.o.s., $m_l \in \mathbb{N}$.

Claim 3: $\exists t_1$ s.t. $\forall t \geq t_1 + 1,\ \mathrm{rank} M_{t_1}(y) = \mathrm{rank} M_{t_1-d}(y)$.

Pf: $\mathcal{B} := \{b_1, \ldots, b_N\}$ Gröbner basis of $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$ for total deg. ordering, $d_{\mathcal{B}} := \max_i \deg(b_i)$, $t_1 := \max(d + d_{\mathcal{B}}, t_0)$.

For any $\alpha \in \mathbb{N}^n_{t_1}$, write $x^\alpha = \sum_{i=1}^N \lambda_i b_i + \sum_{l=1}^k u_l^{(\alpha)} g_l$.

As $\deg(u_l^{(\alpha)} g_l) \leq |\alpha| \leq t - 1$, $x^\alpha - \sum_{i=1}^N \lambda_i b_i \in \mathrm{Ker} M_t(y)$.

The optimum matrix $M_s(y)$ carries lots of information about $I(V_{\mathbb{R}}(I))$

(1) One can find a basis $\mathcal{B}$ of $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$: *Any basis of the column space of* $M_s(y)$

(2) One can compute the multiplication matrices M_{x_i} in $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$:

$$M_{x_i} = M_{\mathcal{B}}^{-1} P_{x_i}$$

- $M_{\mathcal{B}}$ is the principal submatrix of $M_s(y)$ indexed by $\mathcal{B}$
- P_{x_i} is the submatrix of $M_s(y)$ with rows in $\mathcal{B}$ and columns in $x_i\mathcal{B}$

(3) Hence one can compute $V_{\mathbb{R}}(I)$

[from the eigenvalues of M_{x_i}]

Finding a basis of $I(V_{\mathbb{R}}(I))$

(4) When $\mathcal{B} \subseteq \mathbb{T}_{n,s}$ is closed under division, one can compute a border basis of $I(V_{\mathbb{R}}(I))$:

Set $\mathcal{B} := \{b_1, \ldots, b_N\}$ and $M_{x_i} = (a_k^{(ij)})_{j,k=1}^N$.

That is, $g_{ij} := x_i b_j - \sum_{k=1}^N a_k^{(ij)} b_k \in I(V_{\mathbb{R}}(I))$. Then,

$$G_{\mathcal{B}} := \{g_{ij} \mid i = 1, \ldots, n, \ j = 1, \ldots, N\}$$

is a (border) basis of $I(V_{\mathbb{R}}(I))$.

(5) Let $\mathcal{B}_\prec$ be the set of standard monomials of $I(V_{\mathbb{R}}(I))$ for a total degree term ordering. Then, $\mathcal{B}_\prec \subseteq \mathbb{T}_{n,s}$,

- $\mathcal{B}_\prec$ can be found by a 'greedy sieve' algorithm using an indepedence oracle in $\mathbb{R}[x]/I(V_{\mathbb{R}}(I))$ [i.e., column space of $M_s(y)$]
- $G_{\mathcal{B}_\prec}$ is a Gröbner basis for $\prec$

A fundamental property of border bases

$\mathcal{B} := \{b_1, \ldots, b_N\} \subseteq \mathbb{T}_n$, **closed under division**.

$\partial\mathcal{B} := \cup_{i=1}^n x_i \mathcal{B}$, the **border** of $\mathcal{B}$

$G := \{g_{ij} := x_i b_j - \sum_{k=1}^N a_k^{(ij)} b_k \mid i = 1, \ldots, n, j = 1, \ldots, N\}$, a $\mathcal{B}$-border prebasis

$M_i := (a_k^{(ij)})_{j,k=1}^N$, the **formal multiplication matrices**

$J := \langle G \rangle$

Theorem [e.g. Kehrein, Kreuzer, Robbiano] [Mourrain]
$\mathcal{B}$ is a basis of $\mathbb{R}[x]/J \iff M_1, \ldots, M_n$ pairwise commute.
In that case, G is a (border) basis of J.

Application to finding a subideal of $I(V_{\mathbb{R}}(I))$ [without (F1), (F2)]

Let $y \in K_t$ with $\mathrm{rank} M_t(y)$ maximum, $1 \leq s \leq t$.

$\mathcal{B} := \{b_1, \ldots, b_N\} \subseteq \mathbb{T}_{n,s}$ index a maximum nonsingular principal submatrix of $M_{s-1}(y)$, and $\partial\mathcal{B}$ the border of $\mathcal{B}$.

If $\mathrm{rank} M_{\mathcal{B}\cup\partial\mathcal{B}}(y) = \mathrm{rank} M_{\mathcal{B}}(y)$, one can construct a $\mathcal{B}$-border prebasis G and the formal multiplication matrices M_i.

We have: $J := \langle G \rangle \subseteq \mathrm{Ker} M_s(y) \subseteq I(V_{\mathbb{R}}(I))$.

Theorem: Assume $\mathcal{B}$ is closed and the M_i's commute.
Then $\mathcal{B}$ is a basis of $\mathbb{R}[x]/J$, G is a border basis of J, $V(J) \supseteq V_{\mathbb{R}}(I)$, with equality: $J = I(V_{\mathbb{R}}(I))$ if $|V(J)| = |V_{\mathbb{R}}(I)| = \mathrm{rank} M_{s-1}(y)$.

Two small examples

Ex. 1: $I = \langle h := x_1^2 + x_2^2 \rangle \quad [d = 1]$

$V_{\mathbb{R}}(I) = \{0\}, V(I) = \{(x_1, \pm i x_1) \mid x_1 \in \mathbb{C}\}$.

$M_1(y) \succeq 0, M_0(hy) = 0 \Longrightarrow y_\alpha = 0 \;\; \forall \alpha \neq 0.$

If $\mathrm{rank} M_1(y)$ is maximum, then $y = (y_0, 0, \ldots, 0)$ with $y_0 > 0$.
Thus, $\langle \mathrm{Ker} M_1(y) \rangle = \langle x_1, x_2 \rangle = I(V_{\mathbb{R}}(I))$.

Ex. 2: $I = \langle h_i := x_i(x_i^2 + 1) \mid i = 1, \ldots, n \rangle \quad [d = 2]$

$V_{\mathbb{R}}(I) = \{0\}, |V(I)| = 3^n$.

$M_3(y) \succeq 0, M_1(h_i y) = 0 \Longrightarrow y_\alpha = 0 \;\; \forall \alpha \neq 0.$

If $\mathrm{rank} M_1(y)$ is maximum, then $y = (y_0, 0, \ldots, 0)$ with $y_0 > 0$.
Thus, $\langle \mathrm{Ker} M_1(y) \rangle = \langle x_1, \ldots, x_n \rangle = I(V_{\mathbb{R}}(I))$.

Some algorithmic issues

How to find $y \in K_t$ for which $\mathrm{rank} M_t(y)$ is maximum ?

We use a SDP solver with the 'extended self-dual embedding property'; e.g., SeDuMi-1.05

Then the central path converges to a solution in the relative interior of the optimum face, i.e., to a solution with maximum rank

How to compute ranks of matrices ?

We use SVD decomposition, but this is a sensitive numerical issue ...

Some numerical examples

$I = \langle x_2^4 x_1 + 3x_1^3 - x_2^4 - 3x_1^2,\ x_1^2 x_2 - 2x_1^2,\ 2x_2^4 x_1 - x_1^3 - 2x_2^4 + x_1^2 \rangle$, $d = 3$

order t	rank sequence	extract. order MON/SVD	accuracy MON/SVD	comm. error MON/SVD
3	1 3 5 9	—	—	—
4	1 2 2 2 7	4(2)/3(2)	1.9717e-9/0.00013144	9.676e-10/3.3908e-6
5	1 2 2 2 2 8	4(2)/4(2)	2.9557e-8/3.5325e-5	1.8781e-11/1.2291e-6

(F1) holds

Quotient linear basis: $\mathcal{B} = \{1, x_1\}$

Border basis: $G = \{-x_1 + \mathbf{x_1^2}, -2x_1 + \mathbf{x_2}, -2x_1 + \mathbf{x_1 x_2}\}$

Real solutions: $\begin{cases} x_1 = (2.12\text{e-}8, 1.91\text{e-}6), \\ x_2 = (1, 2). \end{cases}$

An example for $I(V_{\mathbb{R}}(I) \cap S)$

$I = \langle x_1 + x_2 - 2, x_1x_3 + x_2x_4, x_1x_3^2 + x_2x_4^2 - \frac{2}{3}, x_1x_3^3 + x_2x_4^3 \rangle$, $S = [-1, 1]^4$, $d = 2$

order t	rank sequence	extract. order MON/SVD	accuracy MON/SVD	comm. error MON/SVD
2	1 2 11	2(2)/—(—)	0.00010224/—	1.1124e-9/—
3	1 2 2 18	2(2)/2(2)	1.8985e-14/5.1015e-14	1.2212e-15/1.4155e-15
4	1 2 2 2 24	2(2)/2(2)	3.5527e-15/8.5487e-15	2.2204e-16/1.1102e-16

(F1) holds

Quotient basis: $\mathcal{B} = \{1, x_3\}$

Border basis:
$G = \{-1 + \mathbf{x_1}, -x_3 + \mathbf{x_1x_3}, -1 + \mathbf{x_2}, -x_3 + \mathbf{x_2x_3}, -0.33333 + \mathbf{x_3^2}, x_3 + \mathbf{x_4}, 0.33333 + \mathbf{x_3x_4}\}$

Real solutions: $\begin{cases} x_1 = (1, 1, -0.577, 0.577) \\ x_2 = (1, 1, 0.577, -0.577) \end{cases}$

An example from the Bini-Mourrain test suite: extraction possible without (F1) or (F2)

$I = \langle 5x_1^9 - 6x_1^5x_2 + x_1x_2^4 + 2x_1x_3, -2x_1^6x_2 + 2x_1^2x_2^3 + 2x_2x_3, x_1^2 + x_2^2 - 0.265625\rangle$
$d = 5$, $|V_{\mathbb{R}}(I)| = 8$, $|V(I)| = 20$

order t	rank sequence	extract. order MON/SVD	accuracy MON/SVD	comm. error MON/SVD
5	1 4 8 16 25 34	—	—	—
6	1 3 9 15 22 26 32	—	—	—
7	1 3 8 10 12 16 20 24	3(3)/—(—)	0.12786/—	0.00019754/—
8	1 4 8 8 8 12 16 20 24	4(3)/3(3)	4.6789e-5/0.00013406	4.7073e-5/0.00075005

Quotient basis: $\mathcal{B} = \{1, x_1, x_2, x_3, x_1^2, x_1x_2, x_1x_3, x_2x_3\} \rightsquigarrow$ border basis G of size 10

Real solutions: $$\left\{\begin{array}{ll} x_1 = (-0.515, -0.000153, -0.0124) & x_2 = (-0.502, 0.119, 0.0124) \\ x_3 = (0.502, 0.119, 0.0124) & x_4 = (0.515, -0.000185, -0.0125) \\ x_5 = (0.262, 0.444, -0.0132) & x_6 = (-2.07\text{e-}5, 0.515, -1.27\text{e-}6) \\ x_7 = (-0.262, 0.444, -0.0132) & x_8 = (-1.05\text{e-}5, -0.515, -7.56\text{e-}7) \end{array}\right.$$

Katsura 5 example: extraction possible only using a SVD quotient basis

$I = \langle 2x_6^2 + 2x_5^2 + 2x_4^2 + 2x_3^2 + 2x_2^2 + x_1^2 - x_1, x_6x_5 + x_5x_4 + 2x_4x_3 + 2x_3x_2 + 2x_2x_1 - x_2, 2x_6x_4 + 2x_5x_3 + 2x_4x_2 + x_2^2 + 2x_3x_1 - x_3, 2x_6x_3 + 2x_5x_2 + 2x_3x_2 + 2x_4x_1 - x_4, x_3^2 + 2x_6x_1 + 2x_5x_1 + 2x_4x_1 - x_5, 2x_6 + 2x_5 + 2x_4 + 2x_3 + 2x_2 + x_1 - 1\rangle$

$d = 1$, $|V_{\mathbb{R}}(I)| = 12$, $|V(I)| = 32$

order t	rank sequence	extract. order MON/SVD	accuracy MON/SVD	comm. error MON/SVD
2	1 6 16	—	—	—
3	1 6 12 12	—/3(3)	—/1.1928e-005	—/2.3073e-007

Real solutions:

$(0.277, 0.226, 0.162, 0.0858, 0.0115, -0.124)$

$(1, -2.8\text{e-}7, 4.7\text{e-}7, 8.81\text{e-}7, -2.79\text{e-}6, -3.69\text{e-}6)$

$(0.441, 0.151, 0.0225, 0.219, 0.0935, -0.207)$

$(0.462, 0.309, 0.0553, -0.102, -0.0844, 0.0917)$

$(0.753, 0.0532, 0.191, -0.114, -0.146, 0.139)$

$(0.68, 0.266, -0.154, 0.0323, 0.0897, -0.0735)$

$(0.59, 0.0422, 0.327, -0.0642, -0.0874, -0.0132)$

$(0.239, 0.0608, -0.0622, -0.0233, 0.186, 0.219)$

$(0.726, -0.0503, 0.122, 0.164, 0.11, -0.208)$

$(0.292, -0.101, 0.181, -0.0591, 0.193, 0.141)$

$(0.409, -0.0732, 0.0657, -0.127, 0.252, 0.178)$

$(0.136, 0.0428, 0.0417, 0.0404, 0.0964, 0.211)$

A complex example for $V(I)$ and $I(V(I))$

$I = \langle x_1^2 + x_2 + x_3 + 1, x_1 + x_2^2 + x_3 + 1, x_1 + x_2 + x_3^2 + 1\rangle$, 7 solutions, 1 real solution (mult. 2)

order t	rank sequence $M_s^{\mathbb{C}}(y)$, $M_s^{2\mathbb{C}}(y)$	extract. order MON/SVD	accuracy MON/SVD	comm. error MON/SVD
3	(1 4 7 7), (1 7 7 7)	3(3)/3(3)	0.0005719/0.00022538	0.00041241/0.00043871

Quotient basis: $\mathcal{B} = \{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3\} \rightsquigarrow$ border basis G of size 10

Complex solutions (with monomial basis $\mathcal{B}$)

$$\begin{cases} x_1 = (-1 - 8.15\text{e-}11i, -1 + 4.37\text{e-}11i, -1 - 4.24\text{e-}12i) & \approx (-1, -1, -1) \\ x_2 = (-1.16\text{e-}5 + 1.41i, 0.999 - 1.41i, 0.000654 + 1.41i) & \approx (\sqrt{2}i, 1 - \sqrt{2}i, \sqrt{2}i) \\ x_3 = (-4.8\text{e-}5 - 1.41i, 1 + 1.41i, 0.000147 - 1.41i) & \approx (-\sqrt{2}i, 1 + \sqrt{2}i, -\sqrt{2}i) \\ x_4 = (-9.92\text{e-}7 + 1.41i, 0.000713 + 1.41i, 0.999 - 1.41i) & \approx (\sqrt{2}i, \sqrt{2}i, 1 - \sqrt{2}i) \\ x_5 = (-3.98\text{e-}5 - 1.41i, 0.000146 - 1.41i, 1 + 1.41i) & \approx (-\sqrt{2}i, -\sqrt{2}i, 1 + \sqrt{2}i) \\ x_6 = (1 + 1.41i, -0.000149 - 1.41i, -0.000145 - 1.41i) & \approx (1 + \sqrt{2}i, -\sqrt{2}i, -\sqrt{2}i) \\ x_7 = (1 - 1.41i, 0.000103 + 1.41i, 0.000104 + 1.41i) & \approx (1 - \sqrt{2}i, \sqrt{2}i, \sqrt{2}i) \end{cases}$$

complex example continued

Complex solutions (with SVD quotient basis):

$$\left\{\begin{array}{ll} x_1 = (-1 - 2.51\text{e-}11i, -1 - 9.85\text{e-}11i, -1 - 5.99\text{e-}11i) & \approx (-1, -1, -1) \\ x_2 = (-1.73\text{e-}5 + 1.41i, 1 - 1.41i, -1.29\text{e-}5 + 1.41i) & \approx (\sqrt{2}i, 1 - \sqrt{2}i, \sqrt{2}i) \\ x_3 = (-3.78\text{e-}5 - 1.41i, 1 + 1.41i, -8.52\text{e-}5 - 1.41i) & \approx (-\sqrt{2}i, 1 + \sqrt{2}i, -\sqrt{2}i) \\ x_4 = (-2.83\text{e-}5 + 1.41i, 4.23\text{e-}5 + 1.41i, 1 - 1.41i) & \approx (\sqrt{2}i, \sqrt{2}i, 1 - \sqrt{2}i) \\ x_5 = (-4\text{e-}5 - 1.41i, 3.88\text{e-}5 - 1.41i, 1 + 1.41i) & \approx (-\sqrt{2}i, -\sqrt{2}i, 1 + \sqrt{2}i) \\ x_6 = (1 + 1.41i, -0.000191 - 1.41i, 6.2\text{e-}5 - 1.41i) & \approx (1 + \sqrt{2}i, -\sqrt{2}i, -\sqrt{2}i) \\ x_7 = (1 - 1.41i, -4.61\text{e-}5 + 1.41i, 0.000104 + 1.41i) & \approx (1 - \sqrt{2}i, \sqrt{2}i, \sqrt{2}i) \end{array}\right.$$