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The transversal relative equilibria of a Hamiltonian system with symmetry

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Abstract. Let P be a symplectic manifold with a free symplectic action of a connected compact Lie group G. We show that, given a certain transversality condition, the set of relative equilibria $\mathcal{E}$ near $p_e \in \mathcal{E}$ of a G-invariant Hamiltonian system on P is locally Whitney-stratified by the conjugacy classes of the isotropy subgroups (under the product of the coadjoint and adjoint actions) of the momentum-generator pairs (μ, ξ) of the relative equilibria. The dimension of the stratum of the conjugacy class (K) is $\dim G + 2\dim Z(K) - \dim K$, where $Z(K)$ is the centre of K. Transverse to this stratum $\mathcal{E}$ is locally diffeomorphic to the set of commuting pairs of the Lie algebra of $K/Z(K)$. The stratum $\mathcal{E}_{(K)}$ is a symplectic submanifold of P near $p_e \in \mathcal{E}$ if and only if p_e is non-degenerate and K is a maximal torus of G. We also show that the set of G-invariant Hamiltonians on P for which all the relative equilibria are transversal is open and dense. Thus, generically, the types of singularities of the set of relative equilibria of a Hamiltonian system with symmetry are those types found amongst the singularities at zero of the sets of commuting pairs of certain Lie subalgebras of the symmetry group.

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Introduction

Let G be a connected, compact Lie group with Lie algebra $\mathfrak{g}$. The group G acts by its adjoint representation on $\mathfrak{g}$ and by the dual coadjoint representation on $\mathfrak{g}^*$. Suppose also that G acts freely and symplectically on a symplectic phase space (P, ω) endowed with an equivariant momentum mapping $J : P \to \mathfrak{g}^*$. Since G is acting freely, every point in P is a regular point of J.

Let H be a G-invariant Hamiltonian on P. A relative equilibrium of H with generator $\xi_e \in \mathfrak{g}$ is a point $p_e \in P$ for which the evolution under the Hamiltonian flow generated by H is $t \mapsto \exp(t\xi_e) \cdot p_e$. The main results of this paper describe the structure of the set of relative equilibria of H near a relative equilibrium which is *transversal*, a non-degeneracy condition that is defined in section 2.

Let G_{μ_e} and G_{ξ_e}, respectively, denote the isotropy subgroups of $\mu_e = J(p_e) \in \mathfrak{g}^*$ and $\xi_e \in \mathfrak{g}$ under the coadjoint and adjoint actions of G. It follows from an observation of Arnold (1978) that if G_{μ_e} is a principal isotropy subgroup of the coadjoint action, and hence a maximal torus T in G, and if p_e is a *non-degenerate* relative equilibrium, then for each μ near μ_e the momentum level set $J^{-1}(\mu)$ contains a unique relative equilibrium near p_e and the set of relative equilibria near p_e is a smooth submanifold of P of dimension $\dim G + \dim T$. The non-degeneracy condition used here is defined in section 3 and is stronger than transversality when $G_{\mu_e} = T$ (see the remark following corollary 3).

Patrick (1995) generalized this result to relative equilibria p_e for which the isotropy subgroup $G_{(\mu_e,\xi_e)} = G_{\mu_e} \cap G_{\xi_e}$ of the momentum-generator pair (μ_e, ξ_e) under the product of the coadjoint and adjoint actions is a maximal torus. Again, if p_e is non-degenerate, then the set of relative equilibria near p_e is a submanifold of dimension $\dim G + \dim T$. Moreover, he showed that this submanifold is symplectic if G_{μ_e} is a maximal torus and an additional non-resonance condition holds at p_e.

One of the main results of this paper, theorem 1, is a generalization of Patrick's theorem to relative equilibria p_e with arbitrary momentum-generator isotropy subgroup $G_{(\mu_e,\xi_e)}$. We define the *type* of a pair $(\mu, \xi) \in \mathfrak{g}^* \oplus \mathfrak{g}$ to be the conjugacy class in G of $G_\mu \cap G_\xi$ and the type of a relative equilibrium p_e to be the type of its momentum-generator pair (μ_e, ξ_e). Generally we denote the conjugacy class of a subgroup $K \subset G$ by (K), and so we say that the type of p_e is (K_e), where $K_e = G_{\mu_e} \cap G_{\xi_e}$. Let $\mathcal{E} \subset P$ denote the set of relative equilibria of a G-invariant Hamiltonian system on P and let $\mathcal{E}/G \subset P/G$ denote its quotient by the action of G. Then we show that near a transversal relative equilibrium both $\mathcal{E}$ and $\mathcal{E}/G$ are stratified by the types of the relative equilibria and that the quotient of a non-empty stratum consisting of relative equilibria of type (K) has dimension equal to $2\dim Z(K) - \dim K$, where $Z(K)$ is the centre of K. Moreover, transverse to this stratum the set $\mathcal{E}/G$ is locally isomorphic to the set of commuting pairs of the Lie algebra of $K/Z(K)$. See theorems 1 and 2 for precise statements.

These results are complemented by theorem 3 which states that the set of Hamiltonian functions on P for which *all* the relative equilibria are transversal is open and dense in the space of smooth G-invariant functions on P (with an appropriate Whitney topology). Simple examples show that non-degeneracy can fail in a structurally stable way in families of relative equilibria parametrized by conserved quantities. It follows from theorems 1 and 3 that, for generic Hamiltonians H, the sets $\mathcal{E}$ and $\mathcal{E}/G$ are globally stratified by the types of the relative equilibria. We note that this is not true if 'transversal' is replaced by 'non-degenerate'. Furthermore, generic Hamiltonians can only have relative equilibria of type (K) with $2\dim Z(K) \geqslant \dim K$.

Putting the genericity of transversality together with our local analysis of transversal relative equilibria, our results prove that, generically, the singularity types found in the set of relative equilibria are Lie-theoretic and independent of, for example, system parameters. In a sense we have classified the singularities that can occur generically in the set of relative equilibra, in the regular case.

For a simple, and perhaps rather surprising, illustration of these results consider Hamiltonian systems which are invariant under actions of $SO(3)$. Examples include systems of particles or rigid bodies in $\mathbb{R}^3$. A pair $(\mu_e, \xi_e) \in \mathfrak{so}(3)^* \oplus \mathfrak{so}(3)$ can be the momentum-generator pair of a relative equilibrium if and only if μ_e and ξ_e are parallel (under the standard identification of $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$). The isotropy subgroups of such pairs are conjugate to $SO(2)$. For a generic $SO(3)$-invariant Hamiltonian, if the set of relative equilibria with $(\mu_e, \xi_e) \neq (0, 0)$ is non-empty, then its quotient is one dimensional. If μ_e and ξ_e are both zero then the isotropy subgroup K is $SO(3)$ and $2\dim Z(K) - \dim K$ is negative. *Thus equilibrium solutions with zero 'angular momentum' do not occur in generic $SO(3)$-invariant systems.*

For another example consider a rigid body with distinct principal moments of inertia and with three rotors aligned with its principal axes. This has symmetry group $SO(3) \times SO(2)^3$. Theorems 1–3 imply that, for generic Hamiltonian systems with symmetry group $SO(3) \times SO(2)^3$, the relative equilibria of type $SO(3) \times SO(2)^3$ form isolated group orbits that are the singularities of $\mathcal{E}/G$ and that near each of these singularities $\mathcal{E}/G$ is four dimensional and is diffeomorphic to the set of commuting pairs of the Lie algebra $\mathfrak{so}(3)$. The rigid body with three generically placed rotors has precisely one group orbit of relative equilibria of type $SO(3) \times SO(2)^3$, namely the equilibria where neither the body nor the rotors move, and the set of all relative equilibria has the structure described for generic $SO(3) \times SO(2)^3$-invariant systems.

The discussion of $SO(3)$-invariant systems above shows that many systems of particles and rigid bodies in $\mathbb{R}^3$ which we are familiar with are not generic in the sense of this paper, since they have equilibrium points with zero angular momentum. We believe that this can be attributed to the fact that these systems are invariant under actions of groups which are larger than $SO(3)$ and have non-trivial isotropy subgroups on P. In particular, many well known systems are 'simple mechanical systems' defined on the cotangent bundles of their configuration spaces and such systems are always invariant under the action of an antisymplectic time-reversal symmetry. It will therefore be important to extend the results of this paper to actions of groups on symplectic manifolds P which are not free and which combine both symplectic and antisymplectic operators. It will also be of interest to extend the results to groups G which are not connected and not compact. Connectedness and compactness are used crucially in this paper to construct and describe a (Whitney regular) stratification of the set of possible momentum-generator pairs of relative equilibria (see section 1).

A number of recent works have extended Patrick's theorem to non-free actions of Lie groups and to non-compact groups. Ortega and Ratiu (1997) combine Patrick's result with singular reduction techniques to prove the existence of manifolds of relative equilibria through non-degenerate relative equilibria with non-trivial isotropy in phase space. Roberts and de Sousa Dias (1997), Lerman and Singer (1998) and Chossat *et al* (1999) all use the local normal form for symplectic group actions obtained by Guillemin and Sternberg (1984a, b), Marle (1985) and Bates and Lerman (1997) to again obtain smooth manifolds of relative equilibria through non-degenerate relative equilibria with non-trivial phase space isotropy subgroups.

Our results differ from those cited in the previous paragraph in several respects. First, we note that they generalize Patrick's theorem in a completely different direction and that the intersection is precisely Patrick's result. Secondly, our emphasis is on transversal, rather than non-degenerate relative equilibria. If the momentum-generator isotropy subgroup $G_{(\mu_e,\xi_e)}$ is a maximal torus then transversality is a strictly weaker condition than non-degeneracy, but, in general, a relative equilibrium may be transversal without being non-degenerate, or vice versa, (see corollary 3). However, as noted above, for free group actions it is a generic property of Hamiltonians to have only transversal relative equilibria. A third difference is that we describe the set of *all* relative equilibria near a given transversal relative equilibrium and show that, in general, this set is *not* a smooth manifold. Instead it is stratified by type. An important message of this paper is that it is the isotropy subgroup of the momentum-generator pair (μ_e, ξ_e) which determines the local structure of the set of relative equilibria, and not the isotropy subgroup of either μ_e or ξ_e taken singly.

Another difference between our results and those of Ortega and Ratiu (1997), Lerman and Singer (1998) and Chossat *et al* (1999) is that the latter show that their smooth manifolds of relative equilibria are symplectic submanifolds of P. Theorem 6 of this paper shows that for free actions a constant-type stratum of $\mathcal{E}$ is symplectic near a transversal relative equilibrium p_e if and only if p_e is non-degenerate and G_{μ_e} is a maximal torus in G, the situation covered by Patrick's original theorem.

A different direction for future work is suggested by comparing theorems 1 and 2 of this paper with theorem 3.2 of Montaldi (1997). Montaldi's result is again only for free compact group actions and exactly counts the number of (group orbits of) relative equilibria in each fibre of the momentum map J near a non-degenerate relative equilibrium p_e for which G_{ξ_e} is a maximal torus in G. It can therefore be interpreted as a result on how the momentum map restricts to the set $\mathcal{E}$ near such points. It would be of interest to extend this description to the whole of $\mathcal{E}$ and to obtain analogous results for non-free actions of possibly non-compact groups.

We end this introduction with a summary of the contents of this paper. Let $(\mathfrak{g}^* \oplus \mathfrak{g})^c \subseteq \mathfrak{g}^* \oplus \mathfrak{g}$ be the set of pairs (μ, ξ) such that $\mathrm{coad}_\xi\, \mu = 0$, where 'coad' denotes the coadjoint representation of $\mathfrak{g}$ on $\mathfrak{g}^*$. Any G-invariant inner product on $\mathfrak{g}$ defines an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ and between the set of commuting pairs of $\mathfrak{g}$ and the subset $(\mathfrak{g}^* \oplus \mathfrak{g})^c$. In section 1 we describe a Whitney regular stratification of $(\mathfrak{g}^* \oplus \mathfrak{g})^c$, and in section 2 we use this to construct stratifications of certain subsets $\mathcal{T}^c$ of the tangent bundle and $\mathcal{K}^{oc}$ of the cotangent bundle of P. A relative equilibrium is defined to be *transversal* if at that point the vector field is transversal to the stratification of $\mathcal{T}^c$ or, equivalently, the derivative of the Hamiltonian is transverse to the stratification of $\mathcal{K}^{oc}$. Some more or less standard results from transversality and stratification theory are then used to deduce theorems 1–3. In section 3 we describe a normal form for the linearization of a vector field at a transversal relative equilibrium and use this to give a characterization of transversal relative equilibria (theorem 4) and to determine necessary and sufficient conditions for the strata of the set of relative equilibria to be symplectic (theorem 6).

1. The stratified structure of $(\mathfrak{g}^* \oplus \mathfrak{g})^c$

The momentum-generator pair $(\mu_e, \xi_e) \in \mathfrak{g}^* \oplus \mathfrak{g}$ of a relative equilibrium of a G-invariant Hamiltonian system satisfies the relationship $\mathrm{coad}_{\xi_e}\, \mu_e = 0$. In this section we describe the structure of the set of all pairs satisfying this relationship:

$$(\mathfrak{g}^* \oplus \mathfrak{g})^c = \left\{ (\mu, \xi) \in \mathfrak{g}^* \oplus \mathfrak{g} : \mathrm{coad}_\xi\, \mu = 0 \right\}.$$

Note that $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ is a G-invariant subvariety of $\mathfrak{g}^* \oplus \mathfrak{g}$ under the product of the coadjoint and adjoint actions. For any $(\mu, \xi) \in \mathfrak{g}^* \oplus \mathfrak{g}$ let $G_{\mu,\xi}$ denote the isotropy subgroup of (μ, ξ) for the product of the coadjoint and adjoint actions. Clearly, $G_{\mu,\xi} = G_\mu \cap G_\xi$, where G_μ is the isotropy subgroup of μ for the coadjoint action of G and G_ξ is the isotropy subgroup of ξ for the adjoint action. Let $\mathfrak{g}_\mu$ and $\mathfrak{g}_\xi$ denote the Lie algebras of G_μ and G_ξ, respectively, and $\mathfrak{g}_\mu^*$ and $\mathfrak{g}_\xi^*$ their dual spaces.

Lemma 1.

(a) $(\mu, \xi) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c$ *if and only if* $\xi \in \mathfrak{g}_\mu \subset \mathfrak{g}$, *and* $G_{\mu,\xi} = (G_\mu)_\xi$, *the isotropy subgroup of* ξ *for the adjoint action of* G_μ *on* $\mathfrak{g}_\mu$.

(b) A point $(\mu, \xi) \in \mathfrak{g}^* \oplus \mathfrak{g}$ *lies in* $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ *if and only if* $G_{\mu,\xi}$ *contains a maximal torus of* G.

(c) If $(\mu, \xi) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c$ *then* $\mathfrak{g}_{\mu,\xi} = \mathfrak{g}_\mu \cap \mathfrak{g}_\xi$ *is Abelian if and only if* $G_{\mu,\xi}$ *is a maximal torus of* G.

Note that part (a) of the lemma implies that the torus generated by ξ lies in G_μ and that $G_{\mu,\xi}$ is the centralizer in G_μ of this torus.

Proof. We have

$$(\mu, \xi) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c \;\Leftrightarrow\; \mathrm{coad}_\xi\, \mu = 0 \;\Leftrightarrow\; \xi \in \mathfrak{g}_\mu.$$

Moreover,

$$G_{\mu,\xi} = G_\mu \cap G_\xi = \left\{ g \in G_\mu : \mathrm{Ad}_g\, \xi = \xi \right\}$$

and part (a) follows from the fact that the adjoint action of G on $\mathfrak{g}$ restricts to the adjoint action of G_μ on $\mathfrak{g}_\mu$.

For part (b), suppose first that $(\mu, \xi) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c$. Then, by part (a), $G_{\mu,\xi}$ is an isotropy subgroup of the adjoint action of G_μ and so contains a maximal torus of G_μ. However, G_μ is an isotropy subgroup of the coadjoint action of G and so maximal tori of G_μ are also maximal tori of G.

Conversely, suppose $G_{\mu,\xi}$ contains a maximal torus T of G. Then $(\mu, \xi) \in \mathrm{fix}(T; \mathfrak{g}^* \oplus \mathfrak{g}) = \mathfrak{t}^* \oplus \mathfrak{t}$ where $\mathfrak{t}$ is the Lie algebra of T. Since T is Abelian it follows that $\mathrm{coad}_\xi\, \mu = 0$ and so $(\mu, \xi) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c$.

For part (c), if $G_{\mu,\xi}$ is a maximal torus then its Lie algebra $\mathfrak{g}_{\mu,\xi}$ is Abelian. Conversely, suppose $\mathfrak{g}_{\mu,\xi}$ is Abelian. Then by proposition 4.25 of Adams (1969) and by part (a), $G_{\mu,\xi}$ is connected and so is also Abelian. Thus $G_{\mu,\xi}$ is a torus which contains a maximal torus, and so is a maximal torus. □

For any set X with an action of G we denote the subset consisting of points with isotropy subgroup conjugate to $K \subset G$ by $X_{(K)}$. We will denote the set of conjugacy classes of isotropy subgroups of the action of G on $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ by $\mathcal{I}^c$. By lemma 1, $\mathcal{I}^c$ is the set of conjugacy classes of isotropy subgroups of the action of G on $\mathfrak{g}^* \oplus \mathfrak{g}$, which contain a maximal torus of G, and

$$(\mathfrak{g}^* \oplus \mathfrak{g})^c = \bigsqcup_{(K) \in \mathcal{I}^c} (\mathfrak{g}^* \oplus \mathfrak{g})_{(K)}.$$

By the general theory of actions of a compact group, each set $(\mathfrak{g}^* \oplus \mathfrak{g})_{(K)}$ is a manifold and the collection of these manifolds is a Whitney regular stratification of $\mathfrak{g}^* \oplus \mathfrak{g}$. Since $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ is the union of all the strata of $\mathfrak{g}^* \oplus \mathfrak{g}$ which are smaller than the stratum of the maximal tori, it follows that $\{(\mathfrak{g}^* \oplus \mathfrak{g})_{(K)} : (K) \in \mathcal{I}^c\}$ is a Whitney regular stratification of $(\mathfrak{g}^* \oplus \mathfrak{g})^c$. The general theory also tells us that the quotients of the strata by the action of G are smooth manifolds and that the set of these quotients is a Whitney regular stratification of the orbit space $(\mathfrak{g}^* \oplus \mathfrak{g})/G$. It follows that the set of quotients of the strata $(\mathfrak{g}^* \oplus \mathfrak{g})_{(K)}$ with $(K) \in \mathcal{I}^c$ is a Whitney regular stratification of $(\mathfrak{g}^* \oplus \mathfrak{g})^c/G$.

The next proposition describes the local structure of these stratifications. We fix a G-invariant inner product on $\mathfrak{g}$, which determines a G-equivariant isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ and hence also a G-invariant inner product on $\mathfrak{g}^*$. The G-invariant inner product also defines an isomorphism between $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ and the set of commuting pairs of $\mathfrak{g}$:

$$(\mathfrak{g} \oplus \mathfrak{g})^c = \{ (\eta, \xi) \in \mathfrak{g} \oplus \mathfrak{g} : \mathrm{ad}_\xi\, \eta = [\xi, \eta] = 0 \}.$$

If $\mathfrak{k}$ is a subspace of $\mathfrak{g}$, the inner product on $\mathfrak{g}^*$ is used to identify $\mathfrak{k}^*$ with the subspace $\mathrm{ann}(\mathfrak{k})^\perp$ of $\mathfrak{g}^*$. Here $\mathrm{ann}(\mathfrak{k})$ is the annihilator of $\mathfrak{k}$ in $\mathfrak{g}^*$ and '$\perp$' denotes the orthogonal complement with respect to the G-invariant inner product on $\mathfrak{g}^*$.

Suppose $(\mu, \xi) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c$ and let $K = G_\mu \cap G_\xi$. Denote the Lie algebra of K by $\mathfrak{k}$. Let $\chi : U \to G$ be a section of the natural projection $G \to G/K$, defined on an open neighbourhood U of K in G/K and such that $\chi(K)$ is the identity in G. Give $\mathfrak{g}^* \oplus \mathfrak{g}$ and $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ the stratifications obtained by taking the connected components of the orbit-type strata for the actions of G. Similarly, give $\mathfrak{k}^* \oplus \mathfrak{k}$ and $(\mathfrak{k}^* \oplus \mathfrak{k})^c$ the corresponding stratifications obtained from the actions of K. Extend these stratifications to $U \times (\mathfrak{k}^* \oplus \mathfrak{k})$ and $U \times (\mathfrak{k}^* \oplus \mathfrak{k})^c$ by taking the product of each stratum with U. We will always consider any open subset of a stratified set to be automatically endowed with the stratification obtained by taking the intersections of the strata with the open subset.

Proposition 1. *Under the above assumptions, there exists a G-invariant open neighbourhood W of (μ, ξ) in $\mathfrak{g}^* \oplus \mathfrak{g}$ and a K-invariant open neighbourhood V of the origin in $\mathfrak{k}^* \oplus \mathfrak{k}$ such* *that the map*

$$\Sigma : U \times V \to \mathfrak{g}^* \oplus \mathfrak{g}$$
$$\big(u, (\nu, \eta)\big) \mapsto \chi(u) \cdot \big((\mu, \xi) + (\nu, \eta)\big)$$

is an embedding of $U \times V$ into W that restricts to an isomorphism of smoothly stratified spaces between $U \times \big(V \cap (\mathfrak{k}^* \oplus \mathfrak{k})^c\big)$ and $W \cap (\mathfrak{g}^* \oplus \mathfrak{g})^c$.

Proof. By the theory of compact group actions there is a K-invariant neighbourhood W of (μ, ξ) in $\mathfrak{g}^* \oplus \mathfrak{g}$ such that

$$S_{\mu,\xi} = W \cap \big((\mu, \xi) + (\mathfrak{g}.(\mu, \xi))^\perp\big)$$

is a K-invariant slice to the action of G on $\mathfrak{g}^* \oplus \mathfrak{g}$ at (μ, ξ). A straightforward calculation shows that $\mathfrak{k}^* \oplus \mathfrak{k}$ is contained in $(\mathfrak{g} \cdot \big(\mu, \xi)\big)^\perp$. Let V be the intersection of $\mathfrak{k}^* \oplus \mathfrak{k}$ with $S_{\mu,\xi} - (\mu, \xi)$. Then it follows from the slice theorem that Σ is an embedding of $U \times V$ into W.

It remains to be proved that

$$S_{\mu,\xi} \cap (\mathfrak{g}^* \oplus \mathfrak{g})^c = (\mu, \xi) + V \cap (\mathfrak{k}^* \oplus \mathfrak{k})^c.$$

The right-hand side of this equation is clearly contained in the left-hand side, so it is sufficient to prove that the left-hand side is contained in the right-hand side. Note that for any $(\nu, \eta) \in (\mathfrak{g}^* \oplus \mathfrak{g})$ we always have $\nu \in \mathfrak{g}^*_\nu$ and $\eta \in \mathfrak{g}_\eta$. If $(\nu, \eta) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c$ then we also have $\nu \in \mathfrak{g}^*_\eta$ and $\eta \in \mathfrak{g}_\nu$, and hence $\nu \in \mathfrak{g}^*_\eta \cap \mathfrak{g}^*_\nu$ and $\eta \in \mathfrak{g}_\eta \cap \mathfrak{g}_\nu$. If (ν, η) lies in the slice $S_{\mu,\xi}$ then the isotropy subgroup $G_{\nu,\eta}$ must be contained in K, and so $\mathfrak{g}_\eta \cap \mathfrak{g}_\nu \subset \mathfrak{k}$. It follows that if $(\nu, \eta) \in S_{\mu,\xi} \cap (\mathfrak{g}^* \oplus \mathfrak{g})^c$ then (ν, η) must lie in $\mathfrak{k}^* \oplus \mathfrak{k}$, and hence in $(\mathfrak{k}^* \oplus \mathfrak{k})^c$, as required. □

This proposition shows that the local structure of the stratification of $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ near a point with isotropy subgroup K can be reduced to that of the set $(\mathfrak{k}^* \oplus \mathfrak{k})^c$. For any $(K) \in \mathcal{I}^c$ let $\mathcal{I}^c(K)$ denote the subset of $\mathcal{I}^c$ consisting of orbit types (K') for which the closure of the stratum $(\mathfrak{g}^* \oplus \mathfrak{g})_{(K')}$ contains $(\mathfrak{g}^* \oplus \mathfrak{g})_{(K)}$. Then the proof of the proposition implies that $\mathcal{I}^c(K)$ is the set of conjugacy classes in G of the isotropy subgroups of the action of K on $(\mathfrak{k}^* \oplus \mathfrak{k})^c$.

The next result is valid for any connected compact Lie group K and gives a further reduction. Let $Z(K)$ denote the centre of K. Denote the quotient $K/Z(K)$ by L, its Lie algebra by $\mathfrak{l}$ and the Lie algebra of $Z(K)$ by $\mathfrak{z}$. Identify $\mathfrak{z}^*$ with the subspace $\mathrm{ann}(\mathfrak{z})^\perp$ of $\mathfrak{k}^*$.

Proposition 2. *A K-invariant inner product on $\mathfrak{k}$ determines a K-equivariant linear isomorphism*

$$\sigma : \mathfrak{k}^* \oplus \mathfrak{k} \cong (\mathfrak{l}^* \oplus \mathfrak{l}) \oplus (\mathfrak{z}^* \oplus \mathfrak{z})$$

where the action of K on $\mathfrak{k}^* \oplus \mathfrak{k}$ is the product of coadjoint and adjoint actions, the action on $\mathfrak{l}^* \oplus \mathfrak{l}$ factors through the product of the coadjoint and adjoint actions of L and the action on $\mathfrak{z}^* \oplus \mathfrak{z}$ is trivial. Moreover, σ maps $\mathrm{fix}(K; \mathfrak{k}^* \oplus \mathfrak{k})$ isomorphically to $\mathfrak{z}^* \oplus \mathfrak{z}$ and $(\mathfrak{k}^* \oplus \mathfrak{k})^c$ to $(\mathfrak{l}^* \oplus \mathfrak{l})^c \oplus (\mathfrak{z}^* \oplus \mathfrak{z})$.

Proof. Since K acts by the adjoint action on $\mathfrak{k}$, $\mathfrak{z} = \mathrm{fix}(K; \mathfrak{k})$. The natural projection from $\mathfrak{k}$ to $\mathfrak{k}/\mathfrak{z} \cong \mathfrak{l}$ is equivariant with respect to the natural actions of K on $\mathfrak{k}$ and $\mathfrak{l}$ and so induces an isomorphism of representations between $\mathrm{fix}(K; \mathfrak{k})^\perp$ and $\mathfrak{l}$. Hence $\mathfrak{k}$ is isomorphic as a representation of K to $\mathfrak{l} \oplus \mathfrak{z}$. The invariant inner product translates this into an isomorphism $\mathfrak{k}^* \cong \mathfrak{l}^* \oplus \mathfrak{z}^*$ and putting the two together gives σ. Since the adjoint action of $\mathfrak{z}$ on $\mathfrak{k}$ is trivial σ maps $(\mathfrak{k}^* \oplus \mathfrak{k})^c$ to $(\mathfrak{l}^* \oplus \mathfrak{l})^c \oplus (\mathfrak{z}^* \oplus \mathfrak{z})$. □

Thus the orbit-type stratification of $(\mathfrak{k}^* \oplus \mathfrak{k})^c$ is isomorphic to the stratification of $(\mathfrak{l}^* \oplus \mathfrak{l})^c \oplus (\mathfrak{z}^* \oplus \mathfrak{z})$ obtained by taking the products of the orbit-type strata of $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ with $(\mathfrak{z}^* \oplus \mathfrak{z})$.

Corollary 1. *If* $(K) \in \mathcal{I}^c$ *then the dimension of* $(\mathfrak{g}^* \oplus \mathfrak{g})^c_{(K)}$ *is equal to* $\dim G + 2\dim Z(K) - \dim K$.

Proof. By proposition 1

$$\dim(\mathfrak{g}^* \oplus \mathfrak{g})^c_{(K)} = \dim U + \dim(\mathfrak{k}^* \oplus \mathfrak{k})^c_{(K)}.$$

The dimension of U is $\dim G - \dim K$, while by proposition 2 the dimension of

$$(\mathfrak{k}^* \oplus \mathfrak{k})^c_{(K)} = \mathrm{fix}(K; \mathfrak{k}^* \oplus \mathfrak{k}) = \mathfrak{z}^* \oplus \mathfrak{z}$$

is $2\dim \mathfrak{z}$. This gives the result. □

We note that these structure results also follow from the results of Arms *et al* (1981) and Sjamaar and Lerman (1991) on the structure of level sets of momentum maps. Indeed, the product of the coadjoint and adjoint actions of G on $\mathfrak{g}^* \oplus \mathfrak{g}$ is symplectic with respect to the natural 'cotangent bundle' symplectic structure, the map $(\nu, \eta) \mapsto \mathrm{coad}_\eta\, \nu$ is an equivariant momentum map for this action, and $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ is its zero-level set. Moreover, the symplectic normal space to the group orbit through any point $(\mu, \xi) \in (\mathfrak{g}^* \oplus \mathfrak{g})^c$ with isotropy subgroup K can be identified with $\mathfrak{k}^* \oplus \mathfrak{k}$. This approach also shows that the strata $(\mathfrak{g}^* \oplus \mathfrak{g})^c_{(K)}$ are symplectic submanifolds of $\mathfrak{g}^* \oplus \mathfrak{g}$. We note that our approach gives a simple and explicit realization of the singularity type which might be useful for future work.

2. Transversal relative equilibria

In this section we first define what we mean by a transversal relative equilibrium (definition 1) and then give the main results on the structure of the space of relative equilibria near a transversal relative equilibrium (theorems 1 and 2) and on the genericity of the set of Hamiltonians for which all the relative equilibria are transversal (theorem 3).

Let (P, ω) be a symplectic manifold with a free symplectic action of a compact Lie group G. Let $J : P \to \mathfrak{g}^*$ be a G-equivariant momentum map for this action. Define the map $\widetilde{J} : P \times \mathfrak{g} \to \mathfrak{g}^* \oplus \mathfrak{g}$ by $(p, \xi) \mapsto \big(J(p), \xi\big)$. Since G is acting freely the map J is a G-equivariant submersion and hence so also is $\widetilde{J}$. Define

$$(P \times \mathfrak{g})^c = \widetilde{J}^{-1}(\mathfrak{g}^* \oplus \mathfrak{g})^c = \big\{ (p, \xi) \in P \times \mathfrak{g} : \mathrm{coad}_\xi\, J(p) = 0 \big\}.$$

If p_e is a relative equilibrium of a G-invariant Hamiltonian system on P with generator ξ_e then $(p_e, \xi_e) \in (P \times \mathfrak{g})^c$. The fact that $\widetilde{J}$ is a submersion implies that the Whitney regular stratification of $(\mathfrak{g}^* \oplus \mathfrak{g})^c$ pulls back to a Whitney regular stratification of $(P \times \mathfrak{g})^c$. The strata of $(P \times \mathfrak{g})^c$ are the non-empty submanifolds

$$\begin{aligned}(P \times \mathfrak{g})_{(K)} &= \widetilde{J}^{-1}\big((\mathfrak{g}^* \oplus \mathfrak{g})_{(K)}\big) \\ &= \big\{ (p, \xi) \in P \times \mathfrak{g} : G_{J(p),\xi} \text{ is conjugate to } K \big\}\end{aligned}$$

where $(K) \in \mathcal{I}^c$.

Let $\mathcal{T}$ denote the G-invariant sub-bundle of the vector bundle TP with fibre over $p \in P$ equal to the tangent space at p to the orbit of G through p, that is $\mathcal{T}_p = \mathfrak{g} \cdot p$. The sub-bundle $\mathcal{T}$ is the image of the map $I : P \times \mathfrak{g} \to TP$ defined by $(p, \xi) \mapsto \xi \cdot p$. Since G is acting freely on P, the map I is a G-equivariant vector bundle isomorphism between $P \times \mathfrak{g}$ (considered as a bundle over P) and $\mathcal{T}$.

Let $\mathcal{K}$ denote the G-invariant sub-bundle of the vector bundle TP with fibre over $p \in P$ given by $\mathcal{K}_p = \ker \mathrm{d}J(p)$. Note that the Hamiltonian vector field of any G-invariant Hamiltonian on P takes values in $\mathcal{K}$. Let $\mathcal{T}^c = \mathcal{T} \cap \mathcal{K}$, a G-invariant subset of $\mathcal{K}$. For each $(K) \in \mathcal{I}^c$ define the set $\mathcal{T}^c_{(K)}$ by

$$\mathcal{T}^c_{(K)} = \left\{ \xi \cdot p \in \mathcal{T}^c : \big(J(p), \xi\big) \text{ has type } (K) \right\}.$$

Let $\mathcal{T}^o$ and $\mathcal{K}^o$ denote the G-invariant vector sub-bundles of T^*P with fibres

$$\mathcal{T}^o_p = \mathrm{ann}(\mathcal{T}_p) \qquad \mathcal{K}^o_p = \mathrm{ann}(\mathcal{K}_p).$$

The sub-bundle $\mathcal{K}^o$ is the image of the map $I^o : P \times \mathfrak{g} \to T^*P$ defined by $(p, \xi) \mapsto \mathrm{d}J_\xi(p)$. Moreover, since G is acting freely on P, the map I^o is a G-equivariant vector bundle isomorphism between $P \times \mathfrak{g}$ (considered as a bundle over P) and $\mathcal{K}^o$. Note that any G-invariant Hamiltonian P is such that $\mathrm{d}H$ has values in $\mathcal{T}^o$. Let $\mathcal{K}^{oc} = \mathcal{T}^o \cap \mathcal{K}^o$, a G-invariant subset of $\mathcal{T}^o$. For each $(K) \in \mathcal{I}^c$ define the set $\mathcal{K}^{oc}_{(K)}$ by

$$\mathcal{K}^{oc}_{(K)} = \left\{ \mathrm{d}J_\xi(p) \in \mathcal{K}^{oc} : \big(J(p), \xi\big) \text{ has type } (K) \right\}.$$

Proposition 3.

(a) The set $\left\{ \mathcal{T}^c_{(K)} : (K) \in \mathcal{I}^c \right\}$ *is a Whitney regular stratification of* $\mathcal{T}^c$ *and the diffeomorphism* $I : P \times \mathfrak{g} \to \mathcal{T}$ *restricts to a stratum-preserving bijection of* $(P \times \mathfrak{g})^c$ *to* $\mathcal{T}^c$.

(b) The set $\left\{ \mathcal{K}^{oc}_{(K)} : (K) \in \mathcal{I}^c \right\}$ *is a Whitney regular stratification of* K^{oc} *and the diffeomorphism* $I^o : (P \times \mathfrak{g}) \to \mathcal{K}^o$ *restricts to a stratum-preserving bijection of* $(P \times \mathfrak{g})^c$ *to* $\mathcal{K}^{oc}$.

(c) The symplectic form ω *on* P *defines a vector bundle isomorphism* $\omega^\flat : TP \to T^*P$ *that maps* $\mathcal{T}^c$ *onto* $\mathcal{K}^{oc}$ *and the stratum* $\mathcal{T}^c_{(K)}$ *onto* $\mathcal{K}^{oc}_{(K)}$.

(d) The dimension of $\mathcal{T}^c_{(K)}$ *and* $\mathcal{K}^{oc}_{(K)}$ *is* $\dim P + 2 \dim Z(K) - \dim K$.

Proof. For part (a), equivariance of J implies that $\mathrm{d}J(p)(\xi \cdot p) = -\,\mathrm{coad}_\xi\, J(p)$ and so

$$I(p, \xi) \in \mathcal{T}^c \;\Leftrightarrow\; \xi \cdot p \in \mathcal{K} \;\Leftrightarrow\; 0 = -\,\mathrm{coad}_\xi\, J(p) \;\Leftrightarrow\; (p, \xi) \in (P \times \mathfrak{g})^c.$$

Hence I is a diffeomorphism that maps $(P \times \mathfrak{g})^c$ bijectively to $\mathcal{T}^c$. The strata $\mathcal{T}^c_{(K)}$ are the images under I of the strata of $(P \times \mathfrak{g})^c$ and so define a Whitney regular stratification of T^c. The map I is stratum preserving by construction. Part (b) is similar. Part (c) follows from unravelling the definitions of $\mathcal{T}^c$, $\mathcal{K}^{oc}$ and their strata.

For part (d) we note that the dimensions of $\mathcal{T}^c_{(K)}$ and $\mathcal{K}^{oc}_{(K)}$ are equal to that of $(P \times \mathfrak{g})_{(K)} = \widetilde{J}^{-1}(\mathfrak{g}^* \oplus \mathfrak{g})^c_{(K)}$. The map $\widetilde{J}$ is a submersion and so, by corollary 1, the stratum $(P \times \mathfrak{g})_{(K)}$ has codimension $\dim G - 2 \dim Z(K) + \dim K$ in $P \times \mathfrak{g}$ and hence dimension equal to $\dim P + 2 \dim Z(K) - \dim K$. □

Let $H : P \to \mathbb{R}$ be a G-invariant Hamiltonian function on P, $\mathrm{d}H : P \to \mathcal{T}^o \subset T^*P$ its derivative and $X_H : P \to \mathcal{K} \subset TP$ the corresponding G-equivariant Hamiltonian vector field. Denote by $\psi : P \times \mathfrak{g} \to TP$ the map $(p, \xi) \mapsto X_H(p) - \xi \cdot p$, and let $\psi_{(K)}$ denote the restriction of ψ to the submanifold $(P \times \mathfrak{g})^c_{(K)}$ of $P \times \mathfrak{g}$. Denote by $\psi^o : P \times \mathfrak{g} \to T^*P$ the map $(p, \xi) \mapsto \mathrm{d}H(p) - \mathrm{d}J_\xi(p)$ and let $\psi^o_{(K)}$ denote its restriction to $(P \times \mathfrak{g})^c_{(K)}$. Note that $\psi_{(K)}$ maps into $\mathcal{K}$ and $\psi^o_{(K)}$ into $\mathcal{T}^o$. The following lemma is a direct consequence of the definitions.

Lemma 2. *The vector field X_H has a relative equilibrium at p_e if and only if the following equivalent conditions hold:*

(a) $X_H(p_e) \in \mathcal{T}^c$;
(b) $\mathrm{d}H(p_e) \in \mathcal{K}^{oc}$;
(c) There exists $\xi_e \in \mathfrak{g}$ *such that* $(p_e, \xi_e) \in (P \times \mathfrak{g})^c$ *and* $\psi(p_e, \xi_e) = 0$;
(d) There exists $\xi_e \in \mathfrak{g}$ *such that* $(p_e, \xi_e) \in (P \times \mathfrak{g})^c$ *and* $\psi^o(p_e, \xi_e) = 0$.

In statements (c) and (d) the element ξ_e is the generator of the relative equilibrium. Moreover, if p_e is a relative equilibrium of X_H with generator ξ_e, then

$$p_e \text{ has type } (K) \Leftrightarrow (p_e, \xi_e) \in (P \times \mathfrak{g})^c_{(K)}$$
$$\Leftrightarrow X_H(p_e) \in \mathcal{T}^c_{(K)} \Leftrightarrow \mathrm{d}H(p_e) \in \mathcal{K}^{oc}_{(K)}.$$

The following lemma and the accompanying definition identify what we mean by a transversal relative equilibrium. As we shall show, transversality is generic in the class of symmetric Hamiltonian systems.

Lemma 3. *Let p_e be a relative equilibrium of X_H with generator ξ_e and momentum $\mu_e = J(p_e)$, and let $K_e = G_{\mu_e} \cap G_{\xi_e}$. Then the following are equivalent:*

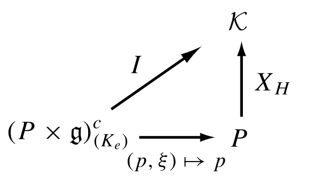
(a) $X_H : P \to \mathcal{K}$ *is transversal to* $\mathcal{T}^c_{(K_e)}$ *at* p_e;
(b) $\mathrm{d}H : P \to \mathcal{T}^o$ *is transversal to* $\mathcal{K}^{oc}_{(K_e)}$ *at* p_e;
(c) $\psi_{(K_e)} : (P \times \mathfrak{g})^c_{(K_e)} \to \mathcal{K}$ *is transversal to the zero section of* $\mathcal{K}$ *at* (p_e, ξ_e);
(d) $\psi^o_{(K_e)} : (P \times \mathfrak{g})^c_{(K_e)} \to \mathcal{T}^o$ *is transversal to the zero section of* $\mathcal{T}^o$ *at* (p_e, ξ_e).

Proof. By part (c) of proposition 3 the symplectic form ω on P defines a vector bundle isomorphism $\omega^\flat : TP \to T^*P$ that converts part (a) into part (b), since

$$\omega^\flat \circ X_H = \mathrm{d}H \qquad \omega^\flat(\mathcal{K}_e) = \mathcal{T}^o \qquad \omega^\flat(\mathcal{T}^c_{(K_e)}) = \mathcal{K}^{oc}_{(K_e)}.$$

Thus (a) is equivalent to (b) and similarly (c) is equivalent to (d), so we need only show the equivalence of (a) and (c).

Generally, as is easily verified, given a section X of a vector bundle $\pi : E \to P$ and a mapping f from a manifold M to E over $f_0 : M \to P$, the condition that X and f are transversal is equivalent to the condition that $f - X \circ f_0$ is transversal to the zero section $Z(E)$ of E. When applied to the diagram



this shows that (c) is equivalent to the statement that $I|(P \times \mathfrak{g})^c_{(K_e)} = \mathcal{T}^c_{(K_e)}$ is transversal to X_H, and so is equivalent to (a). □

Definition 1. *A relative equilibrium p_e is said to be transversal if the equivalent conditions in lemma 3 hold.*

Let $\mathcal{E} \subset P$ denote the set of all relative equilibria of X_H and $\mathcal{E}_{(K)}$ the subset consisting of those of type K.

Theorem 1. *Let p_e be a relative equilibrium of X_H with generator ξ_e and momentum $\mu_e = J(p_e)$, and let $K_e = G_{\mu_e} \cap G_{\xi_e}$. If p_e is transversal then there exists a G-invariant open neighbourhood U of p_e in P such that*

(a) every relative equilibrium of X_H in $\mathcal{E} \cap U$ is transversal;
(b) for each $(K) \in \mathcal{I}^c(K_e)$ the subset $\mathcal{E}_{(K)} \cap U$ is a submanifold of dimension $\dim G + 2\dim Z(K) - \dim K$ and its quotient by the action of G, $(\mathcal{E}_{(K)} \cap U)/G$ is a manifold of dimension $2\dim Z(K) - \dim K$;
(c) the sets $\left\{ \mathcal{E}_{(K)} \cap U : (K) \in \mathcal{I}^c(K_e) \right\}$ and $\left\{ (\mathcal{E}_{(K)} \cap U)/G : (K) \in \mathcal{I}^c(K_e) \right\}$ are Whitney regular stratifications of $\mathcal{E} \cap U$ and $(\mathcal{E} \cap U)/G$, respectively.

Proof. If p_e is a transversal relative equilibrium then, by definition, the map $X_H : P \to \mathcal{K}$ is transversal to $\mathcal{T}^c_{(K_e)}$ at p_e. The fact that $\{\mathcal{T}^c_{(K)} : (K) \in \mathcal{I}^c\}$ is a Whitney regular stratification of $\mathcal{T}^c \subset \mathcal{K}$ implies that there exists a neighbourhood U of p_e in P such that X_H is transversal to the stratification at all points in U (this follows immediately from the outline below of the local structure of Whitney stratified sets). This proves part (a).

The dimension of $\mathcal{K}$ is $2\dim P - \dim G$. By part (d) of proposition 3 the codimension of $\mathcal{T}^c_{(K)}$ in $\mathcal{K}$ is $\dim P - \dim G - 2\dim Z(K) + \dim K$ and hence so is the codimension of $\mathcal{E}_{(K)} \cap U$ in P. This gives the dimension of $\mathcal{E}_{(K)} \cap U$ in part (b).

Since $\{\mathcal{E}_{(K)} \cap U : (K) \in \mathcal{I}^c(\mathcal{K}_e)\}$ is the pull-back of a Whitney regular stratification by a transversal mapping it is also a Whitney regular stratification (Gibson *et al* 1976). The results on $\mathcal{E}/G$ follow from those on $\mathcal{E}$ and the fact that G acts freely on P. □

The possible types of a transversal relative equilibrium are severely restricted by the following corollary. For example, if G has no centre then no transversal relative equilibrium can have type (G).

Corollary 2. *If p_e is a transversal relative equilibrium then*

$$\dim Z(K_e) \geqslant \frac{1}{2} \dim K_e.$$

Proof. This is immediate since $\dim \mathcal{E}_{(K_e)}/G = 2\dim Z(K_e) - \dim K_e$ is non-negative. □

We next give a more detailed description of the local structure of the set $\mathcal{E}$. First, we recall some properties of Whitney stratifications, as described in Goresky and MacPherson (1998). Let M be a manifold, $Z \subset M$ a Whitney stratified space, $p \in Z$, and S the stratum through p. Let N' be a submanifold of M transverse to S at p, meaning that $T_pM = T_pS \oplus T_pN'$. Choose a Riemannian metric on N'. Then for small enough δ the *normal slice* $N = N' \cap Z \cap B_\delta(p)$ and the *link* $L = N' \cap Z \cap \partial B_\delta(p)$ have topological types independent of δ, the Riemannian metric, and N'. Moreover, N is a cone over L and locally Z is the product of N and S. These results follow from the Thom isotopy lemma.

Suppose that near p the stratified set Z is the image of the product of a fixed stratified cone C in a vector space $\mathbb{E}$ and an open subset U of Euclidean space under a smooth embedding of $\mathbb{E} \times U$ into M. Proposition 1 shows that this applies to points in $(\mathfrak{g}^* \oplus \mathfrak{g})^c$. Without loss of generality we may assume that the cone linearly spans $\mathbb{E}$. Then we will say that Z has (smooth) singularity type C at p. We make the following elementary observations, all based on more or less transparent applications of the implicit function theorem.

(a) Given any transverse submanifold N', there is an embedding of $\mathbb{E}$ into N' which maps C onto the normal space N. Conversely, if N is such for some such N' then Z has singularity type C at p.

(b) Suppose M' is a submanifold of M and $Z \subseteq M'$. Take any transverse submanifold N'' in M' to the stratum S and extend it to a transverse submanifold N' to S within M. By item (a) just above, the normal space N is the image of C by a smooth embedding $\iota : \mathbb{E} \to N'$, say. Then $\iota(\mathbb{E})$ is a submanifold with tangent space contained in N'', and consequently ι may be deformed so that its image is contained in N'', all the while fixing its values on C. The point of this is that Z also has singularity type C at p when it is regarded as a subset of M'.

(c) If N is some manifold and $f : N \to M$ is a smooth map such that $f(n) = p$ and f is transversal to the stratum S at n, then $f^{-1}(Z)$ is a stratified space with the same singularity type at n as that of Z at p. For this we invoke smooth coordinates at z such that Z locally becomes $C \times S \times \{0\} \subset \mathbb{E} \times S \times \mathbb{F}$, where $\mathbb{F}$ is some vector space. Then f is transversal to $\mathbb{E} \times S \times \{0\}$, so we may replace f by its restriction to $f^{-1}(\mathbb{E} \times S \times \{0\})$, thereby removing $\mathbb{F}$. Let $\pi_1 : \mathbb{E} \times S \to \mathbb{E}$ be the projection. Again by transversality, $\pi_1 \circ f$ is a submersion at n and so, by local diffeomorphism of N near n, $\pi_1 \circ f$ becomes the projection $(x, y) \mapsto x$, and consequently $f^{-1}(Z)$ also has singularity type C at n.

Our point is just that these singularity theory results are obtainable in the smooth category without the use of the Thom isotopy lemma provided the smooth structure of the stratified spaces is *a priori* known, as in proposition 1.

Theorem 2. *The stratified spaces $\mathcal{E} \cap U$ and $(\mathcal{E} \cap U)/G$ of theorem 1 both have singularity type $\left(\mathfrak{l}_e^* \oplus \mathfrak{l}_e\right)^c$ at p_e and $\bar{p}_e = G \cdot p_e$, respectively, where $\mathfrak{l}_e$ is the Lie algebra of $K_e/Z(K_e)$.*

Proof. By propositions 1 and 2, the set $(\mathfrak{g} \oplus \mathfrak{g})^c$ has singularity type $\left(\mathfrak{l}_e^* \oplus \mathfrak{l}_e\right)^c$ at $(J(p_e), \xi_e)$. Since $J \times \mathrm{Id} : P \times \mathfrak{g} \to \mathfrak{g}^* \times \mathfrak{g}$ is a submersion, the singularity type of $(P \times \mathfrak{g})^c = (J \times \mathrm{Id})^{-1}(\mathfrak{g}^* \oplus \mathfrak{g})^c$ at (p_e, ξ_e) is also $\left(\mathfrak{l}_e^* \oplus \mathfrak{l}_e\right)^c$. The set $\mathcal{T}^c$ also has this singularity type at the point $(p_e, \xi_e \cdot p_e)$, since it is mapped diffeomorphically to $(P \times \mathfrak{g})^c$ by I. Regarding $\mathcal{T}^c$ as a subset of $\mathcal{K}$ leaves its singularity type unchanged. By transversality, and since $\mathcal{E} = X_H^{-1}(\mathcal{T}^c)$, this is also the singularity type of $\mathcal{E}$ at p_e. Since G acts freely on P the set $\mathcal{E}$ is locally isomorphic to the product of $\mathcal{E}/G$ and G and so $\mathcal{E}/G$ has the same singularity type at $\bar{p}_e$ as $\mathcal{E}$ does at p_e. □

We end this section by showing that all the relative equilibria of 'most' G-invariant Hamiltonians are transversal. Recall that G is a connected, compact Lie group acting freely and symplectically on the symplectic manifold (P, ω). Let $C_G^\infty(P)$ denote the set of all G-invariant C^∞ functions on P and $C^\infty(P/G)$ the set of all C^∞ functions on the orbit space P/G. Pulling back functions by the orbit map $\pi : P \to P/G$ defines a bijection $\pi^* : C^\infty(P/G) \cong C_G^\infty(P/G)$. We give $C^\infty(P/G)$ the Whitney C^∞ topology and $C_G^\infty(P)$ the isomorphic topology induced by π^*.

Theorem 3. *The set $\mathcal{H}$ of G-invariant functions H which have only transversal relative equilibria is open and dense in $C_G^\infty(P)$.*

Proof. For any G-invariant function H on P let $\bar{H}$ denote the quotient function on P/G. The derivative $\mathrm{d}H : P \to \mathcal{T}^o \subset T^*P$ is G-equivariant and so descends to a mapping from P/G to $\mathcal{T}^o/G$ which we will denote by $(\mathrm{d}H)^-$. Since G is acting freely on P the orbit space P/G is a smooth manifold and $\mathcal{T}^o/G$ is a smooth vector bundle over P/G that can be identified with the cotangent bundle $T^*(P/G)$. Under this identification $(\mathrm{d}H)^-$ becomes equal to $\mathrm{d}\bar{H}$.

The Hamiltonian H has a relative equilibrium at $p_e \in P$ if and only if $\mathrm{d}H(p_e) \in \mathcal{K}^{oc}$ and so if and only if $\mathrm{d}\bar{H}(\bar{p}_e) \in \mathcal{K}^{oc}/G$. The set $\left\{\mathcal{K}_{(K)}^{oc}/G : (K) \in \mathcal{I}^c\right\}$ is a Whitney regular stratification of $\mathcal{K}^{oc}/G \subset T^*(P/G)$ and p_e is a transversal relative equilibrium if and only if $d\bar{H} : P/G \to T^*(P/G)$ is transversal to this stratification at $\bar{p}_e$. So the set $\mathcal{H}$ as a subset of smooth G-invariant functions on P can be identified which the set of smooth functions $\bar{H} \in C^\infty(P/G)$ for which $d\bar{H}$ is transversal to K^{oc}.

If K^{oc}/G is a submanifold of T^*P then $\mathcal{H}$ is residual (and hence dense) directly from the jet transversality theorem (theorem 2.8 of Hirsch (1976)). If K^{oc}/G is not a submanifold, then P, and hence P/G, can be covered by neighbourhoods which meet only finitely many strata of K^{oc}/G. By taking a countable subcover, and applying the jet transversality theorem to each stratum of each neighbourhood therein, one realizes $\mathcal{H}$ as a countable intersection of residual sets, which is residual and therefore dense.

Moreover, it follows from proposition 3.6 of Feldman (1965) (see also Trotman (1979)), coupled with the continuity of prolongations as a map of function spaces (proposition 3.4, chapter 2 of Golubitsky and Guillemin (1976)), that $\mathcal{H}$ is open. □

Thus for H in the set $\mathcal{H}$ given by the theorem the set of all relative equilibria $\mathcal{E} \subset P$ is Whitney stratified by the types of the relative equilibria, the manifold $\mathcal{E}_{(K)}$ of relative equilibria of type (K) has dimension $\dim G + 2\dim Z(K) - \dim K$, and the singularity type of $\mathcal{E}$ at a point in $\mathcal{E}_{(K)}$ is $(\mathfrak{l}^* \oplus \mathfrak{l})^c$ where $\mathfrak{l}$ is the Lie algebra of $K/Z(K)$.

3. Linearization of a transversal relative equilibrium

In this section we give an alternative form of the transversality condition using a normal form for the linearization of a vector field at a relative equilibrium due to Patrick (1999). This is then used to describe the tangent spaces of the strata of $\mathcal{E}/G$ and to prove a generalization and partial converse of a result of Patrick (1999) on the symplectic properties of these strata.

Recall that $\mathcal{T}_{p_e} = \mathfrak{g} \cdot p_e$ and $\mathcal{K}_{p_e} = \ker\, dJ(p_e)$, and decompose $T_{p_e}P$ as a direct sum

$$T_{p_e}P = T_0 \oplus N_1 \oplus N_0 \oplus T_1 \tag{1}$$

as follows. Let $T_0 = \mathfrak{g}_{\mu_e} \cdot p_e$. Let $\mathfrak{g}_{\mu_e}^\perp$ be a G-invariant complement to $\mathfrak{g}_{\mu_e}$ in $\mathfrak{g}$ and set $T_1 = \mathfrak{g}_{\mu_e}^\perp \cdot p_e$. Choose a complement N_1 to T_0 within $\mathcal{K}_{p_e}$ and a complement N_0 to $\mathcal{T}_{p_e} \oplus \mathcal{K}_{p_e}$ within $T_{p_e}P$. The subspace N_1 is a symplectic normal space at p_e and can be identified with the tangent space to the reduced phase space P_{μ_e} at $\bar{p}_e = G \cdot p_e$. The subspace N_0 can be identified with $\mathfrak{g}_{\mu_e}^*$, regarded as a subspace of $\mathfrak{g}^*$, via the map $dJ(p_e)|N_0$. Since p_e is regular, we have $T_0 \cong \mathfrak{g}_{\mu_e}$ and $T_1 \cong \mathfrak{g}_{\mu_e}^\perp$, and so (1) becomes

$$T_{p_e}P \cong \mathfrak{g}_{\mu_e} \oplus T_{\bar{p}_e}P_{\mu_e} \oplus \mathfrak{g}_{\mu_e}^* \oplus \mathfrak{g}_{\mu_e}^\perp. \tag{2}$$

The symplectic form with respect to this decomposition is the product of the reduced form on $T_{\bar{p}_e}P_{\mu_e}$, the canonical form on $\mathfrak{g}_{\mu_e} \oplus \mathfrak{g}_{\mu_e}^*$ and the Kostant–Souriau form on $\mathfrak{g}_{\mu_e}^\perp = T_{\mu_e}(G \cdot \mu_e)$. The map $dJ(p_e)$ has the explicit form

$$dJ(p_e)(\xi_0 \oplus w \oplus \mu_0 \oplus \xi_1) = \mu_0 + \mathrm{coad}_{\xi_1}\, \mu_e. \tag{3}$$

A relative equilibrium p_e of H is an equilibrium point of $X_{H_{\xi_e}}$ where $H_{\xi_e} = H - J_{\xi_e}$. Patrick (1999) shows that the complements N_0 and N_1 can be chosen so that, with respect to the decomposition (2), the linearization of $X_{H_{\xi_e}}$ at p_e has the form

$$dX_{H_{\xi_e}}(p_e) = \begin{bmatrix} -\mathrm{ad}_{\xi_e}|\mathfrak{g}_{\mu_e} & C^* & D & 0 \\ 0 & dX_{H_{\mu_e}}(\bar{p}_e) & C & 0 \\ 0 & 0 & -\mathrm{coad}_{\xi_e}|\mathfrak{g}_{\mu_e}^* & 0 \\ 0 & 0 & 0 & -\mathrm{ad}_{\xi_e}|\mathfrak{g}_{\mu_e}^\perp \end{bmatrix} \tag{4}$$

where H_{μ_e} is the induced Hamiltonian on the reduced phase space P_{μ_e}, $X_{H_{\mu_e}}$ is the associated Hamiltonian vector field and $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ is its linearization at the equilibrium point $\bar{p}_e$. The operator $D : \mathfrak{g}^*_{\mu_e} \cong N_0 \to T_0 \cong \mathfrak{g}_{\mu_e}$ describes the *drift* along the group orbit and $C : \mathfrak{g}^*_{\mu_e} \cong N_0 \to N_1$ describes interactions between the reduced dynamics and the motion along group orbits due to possible 1:1 resonances between these two motions. If the spectrum of ad_{ξ_e} is distinct from the spectrum of $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ then such interactions do not occur to first order and N_0 and N_1 can be chosen so that $C = 0$. In general, the dual operator C^* is regarded as a map from N_1 (instead of N_1^*) by using the symplectic form to identify N_1 with its dual. The operators C, C^* and D enjoy the following properties:

$$\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)C = -C\,\mathrm{coad}_{\xi_e} \tag{5}$$

$$C^*\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e) = -\,\mathrm{ad}_{\xi_e} C^* \tag{6}$$

$$\mathrm{ad}_{\xi_e} D = D\,\mathrm{coad}_{\xi_e} \tag{7}$$

and the operator D is symmetric.

As in the theorems of the previous section we will set $K_e = G_{\mu_e} \cap G_{\xi_e}$, denote by $\mathfrak{k}_e$ the Lie algebra of K_e, $\mathfrak{z}_e$ the centre of $\mathfrak{k}_e$ and $\mathfrak{l}_e = \mathfrak{k}_e/\mathfrak{z}_e$. As a consequence of (5), C maps $\mathfrak{k}_e^*$ into $\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$, since coad_{ξ_e} is zero on $\mathfrak{k}_e^*$. Equation (6) similarly implies that C^* maps $\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ into $\mathfrak{k}_e$, while equation (7) implies that D maps $\mathfrak{k}_e^*$ into $\mathfrak{k}_e$. The next theorem characterizes transversal relative equilibria in terms of the above normal form for $\mathrm{d}X_{H_{\xi_e}}(p_e)$. Recall that p_e is said to be *non-degenerate* if $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ is invertible.

Theorem 4. *The relative equilibrium p_e is transversal if and only if all the following three conditions hold:*

(a) either p_e is non-degenerate or 0 is a semisimple eigenvalue of $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$*;*
(b) C maps $\mathfrak{z}_e^* \subset \mathfrak{k}_e^* \subset \mathfrak{g}^*_{\mu_e}$ onto $\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$*;*
(c) $C^*\big(\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)\big) + D\big(\ker C \cap \mathfrak{z}_e^*\big) + \mathfrak{z}_e = \mathfrak{k}_e$.

Proof. From Patrick (1995), p 409, there is the following formula for the derivative of $\psi : P \times \mathfrak{g} \to TP$ at (p_e, ξ_e):

$$T_{(p_e,\xi_e)}\psi(v,\eta) = \big(v, \mathrm{d}X_{H_{\xi_e}}(p_e)v - \eta \cdot p_e\big) \tag{8}$$

where on the right-hand side the tangent space $T_{0_{p_e}}(TP)$ of the vector bundle TP at the zero 0_{p_e} at p_e has been decomposed into horizontal and vertical parts. By lemma 3, p_e is transversal if and only if $\psi_{(K_e)} : (P \times \mathfrak{g})^c_{(K_e)} \to \mathcal{K}$ is transversal to the zero section of $\mathcal{K}$ at (p_e, ξ_e), which is equivalent to

$$\ker\, \mathrm{d}J(p_e) \subseteq \big\{\, \mathrm{d}X_{H_{\xi_e}}(p_e)v - \eta \cdot p_e : (v,\eta) \in T_{(p_e,\xi_e)}(P \times \mathfrak{g})^c_{(K_e)} \,\big\}$$

while from propositions 1 and 2,

$$\begin{aligned} T_{(p_e,\xi_e)}(P \times \mathfrak{g})^c_{(K_e)} &= \big\{\, (v,\eta) \in T_{p_e}P \times \mathfrak{g} : \big(\mathrm{d}J(p_e)v, \eta\big) \in T_{(\mu_e,\xi_e)}(\mathfrak{g}^* \oplus \mathfrak{g})^c_{(K_e)} \,\big\} \\ &= \big\{\, (v,\eta) \in T_{p_e}P \times \mathfrak{g} : \big(\mathrm{d}J(p_e)v, \eta\big) \in (\mathfrak{z}_e^* \oplus \mathfrak{z}_e) \oplus \mathfrak{g}.(\mu_e, \xi_e) \,\big\}. \end{aligned}$$

Using (3) and (4) transversality is equivalent to

$$\begin{aligned} \mathfrak{g}_{\mu_e} \oplus N_1 = \big\{\, &\big(-\mathrm{ad}_{\xi_e}\xi_0 + C^*w + D\mu_0 - \eta_0, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)w + C\mu_0\big) : \\ &\xi_0,\ \eta_0 \in \mathfrak{g}_{\mu_e},\ w \in T_{\bar{p}_e}P_{\mu_e},\ \mu_0 \in \mathfrak{g}^*_{\mu_e} \text{ and conditions (10)–(13) hold} \,\big\} \end{aligned} \tag{9}$$

the conditions (10)–(13) being that there exist $\xi_1, \eta_1 \in \mathfrak{g}_{\mu_e}^{\perp}$, $\zeta \in \mathfrak{z}_e^*$, $z \in \mathfrak{z}_e$ and $\tilde{\xi} \in \mathfrak{g}$ such that

$$\mathrm{coad}_{\xi_e}\, \mu_0 = 0, \tag{10}$$

$$\eta_1 = -\,\mathrm{ad}_{\xi_e}\, \xi_1, \tag{11}$$

$$\mu_0 + \mathrm{coad}_{\xi_1}\, \mu_e = \zeta + \mathrm{coad}_{\tilde{\xi}}\, \mu_e, \tag{12}$$

$$\eta_0 + \eta_1 = z + \mathrm{ad}_{\tilde{\xi}}\, \xi_e. \tag{13}$$

Condition (10) gives $\mu_0 \in \mathrm{ann}\big(\mathfrak{g}_{\mu_e} \cdot \xi_e\big) = \mathfrak{k}_e^*$. Inserting (11) into (13), one sees that (10)–(13) are equivalent to the conditions that $\mu_0 \in \mathfrak{k}_e^*$ and there exist $\zeta \in \mathfrak{z}_e^*$, $z \in \mathfrak{z}_e$ and $\tilde{\xi} \in \mathfrak{g}$ such that

$$\mu_0 = \zeta + \mathrm{coad}_{\tilde{\xi}-\xi_1}\, \mu_e, \tag{14}$$

$$\eta_0 = z + \mathrm{ad}_{\tilde{\xi}-\xi_1}\, \xi_e. \tag{15}$$

In (14) μ_0 and ζ lie in $\mathfrak{k}_e^*$, which is orthogonal to the image of coad, and so (14) is equivalent to $\mu_0 \in \mathfrak{z}_e^*$ and $\tilde{\xi} - \xi_1 \in \mathfrak{g}_{\mu_e}$. Since the right-hand side of (15) is then clearly in $\mathfrak{g}_{\mu_e}$, after substitution of (15) into (9), transversality becomes equivalent to

$$\mathfrak{g}_{\mu_e} \oplus N_1 = \Big\{ \big(C^* w + D\mu_0 + g_{\mu_e} \cdot \xi_e + \mathfrak{z}_e, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)w + C\mu_0\big) : w \in T_{\bar{p}_e} P_{\mu_e},\ \mu_0 \in \mathfrak{z}_e^* \Big\}. \tag{16}$$

Let E_0 be the generalized eigenspace of $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ associated with the eigenvalue 0 and let E_1 be the sum of the other generalized eigenspaces, so that $N_1 = E_0 \oplus E_1$. This decomposition is preserved by $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ and for $j = 0, 1$ we denote the restriction of $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ to E_j by L_j. It follows from (16) that for p_e to be transversal we must have image L_0 + image $C \supseteq E_0$. Since image $C \subset \ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e) = \ker L_0 \subset E_0$ this implies image $L_0 + \ker L_0 = E_0$. For a nilpotent operator, such as L_0, an application of the Jordan normal form shows that this is only possible if $L_0 = 0$. It follows that if p_e is transversal then either it is non-degenerate or 0 is a semisimple eigenvalue of $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$.

So in the proving of either direction of the theorem we may assume $L_0 = 0$. However, then the transversality condition (16) reduces to

$$\mathfrak{k}_e \oplus \ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e) = \Big\{ \big(C^* w + D\mu_0 + \mathfrak{g}_{\mu_e} \cdot \xi_e + \mathfrak{z}_e, C\mu_0\big) : w \in \ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e),\ \mu_0 \in \mathfrak{z}_e^* \Big\}. \tag{17}$$

As C^* maps $\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ into $\mathfrak{k}_e$, while D maps $\mathfrak{z}_e^*$ into $\mathfrak{k}_e$, $\mathfrak{z}_e \subseteq \mathfrak{k}_e$, and $\mathfrak{g}_{\mu_e} \cdot \xi_e = \mathfrak{k}_e^{\perp} \cap \mathfrak{g}_{\mu_e}$, (17) is equivalent to the pair of statements (b) and (c) in the theorem. □

Let $\bar{D} : \mathfrak{z}_e^* \to \mathfrak{l}_e \cong \mathfrak{k}_e/\mathfrak{z}_e$ denote the mapping obtained by the restricting the drift operator D to $\mathfrak{z}_e^*$ and then composing with the projection from $\mathfrak{k}_e$ to $\mathfrak{l}_e \cong \mathfrak{k}_e/\mathfrak{z}_e$.

Corollary 3.

(a) If p_e is non-degenerate then p_e is transversal if and only if $\bar{D}$ is surjective.
(b) If p_e is transversal then

$$\dim \ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e) \leqslant 2 \dim Z(K_e) - \dim K_e.$$

Proof. If p_e is non-degenerate then $\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e) = \{0\}$, $\ker C = \mathfrak{g}_{\mu_e}^*$ and the transversality conditions reduce to $D(\mathfrak{z}_e^*) + \mathfrak{z}_e = \mathfrak{k}_e$, as required.

The second statement follows immediately from the third transversality condition in theorem 4. □

We note that if K_e is a maximal torus then $\mathfrak{z}_e = \mathfrak{k}_e$ and so $\bar{D}$ is trivially surjective and non-degeneracy implies transversality. However, in general, a non-trivial non-degeneracy condition must be satisfied by the drift operator for a relative equilibrium to be transversal.

The normal form (4) also enables us to give a description of the tangent spaces to the manifolds $\mathcal{E}_{(K_e)}$ at transversal relative equilibria.

Theorem 5. *If the relative equilibrium p_e is transversal then, with respect to the decomposition (2),*

$$T_{p_e}\mathcal{E}_{(K_e)} = \Big\{ \xi_0 \oplus w \oplus \mu_0 \oplus \xi_1 : \xi_0 \in \mathfrak{g}_{\mu_e},\ \xi_1 \in \mathfrak{g}_{\mu_e}^{\perp},\ \mu_0 \in \mathfrak{z}_e^* \cap \ker C,$$
$$w \in \ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e),\ C^*w + D\mu_0 \in \mathfrak{z}_e \Big\}.$$

Proof. Since $\mathcal{E}_{(K_e)}$ is the projection to P of $\psi_{(K_e)}^{-1}\big(Z(TP)\big) \subset (P \times \mathfrak{g})_{(K_e)}^c$, and in view of (8), we have

$$T_{p_e}\mathcal{E}_{(K_e)} = \big\{ v \in T_{p_e}P : \mathrm{d}X_{H_{\xi_e}}(p_e)v - \eta \cdot p_e = 0$$
$$\text{for some } \eta \in \mathfrak{g} \text{ such that } (v, \eta) \in T_{(p_e,\xi_e)}(P \times \mathfrak{g})_{(K_e)}^c \big\}$$

which, in terms of the local normal form (4), becomes

$$T_{p_e}\mathcal{E}_{(K_e)} = \Big\{ \xi_0 \oplus w \oplus \mu_0 \oplus \xi_1 : \text{the conditions directly below hold} \Big\}$$

the conditions being that there exists $\eta_0 \in \mathfrak{g}_{\mu_e}$ and $\eta_1 \in \mathfrak{g}_{\mu_e}^{\perp}$ such that

$$-\mathrm{ad}_{\xi_e}\xi_0 + C^*w + D\mu_0 - \eta_0 = 0 \qquad \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)w + C\mu_0 = 0$$
$$\mathrm{coad}_{\xi_e}\,\mu_0 = 0 \qquad -\mathrm{ad}_{\xi_e}\xi_1 - \eta_1 = 0$$

subject to the constraints that there exist $\zeta \in \mathfrak{z}_e^*$, $z \in \mathfrak{k}_e$ and $\tilde{\xi} \in \mathfrak{g}$ such that

$$\mu_0 + \mathrm{coad}_{\xi_1}\,\mu_e = \zeta + \mathrm{coad}_{\tilde{\xi}}\,\mu_e \qquad \eta_0 + \eta_1 = z + \mathrm{ad}_{\tilde{\xi}}\,\xi_e.$$

As in the proof of theorem (4), these conditions are equivalent to the conditions that there exist $\tilde{\xi} \in \mathfrak{g}$ and $z \in \mathfrak{z}_e$ such that

$$\mu_0 \in \mathfrak{z}_e \qquad C\mu_0 = 0 \qquad \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)w = 0 \tag{18}$$
$$-\mathrm{ad}_{\xi_e}\xi_0 + C^*w + D\mu_0 - z - \mathrm{ad}_{\tilde{\xi}-\xi_1}\,\xi_e = 0 \tag{19}$$

subject only to the constraint that $\tilde{\xi} - \xi_1 \in \mathfrak{g}_{\mu_e}$, as long as one takes

$$\eta_1 = -\mathrm{ad}_{\xi_e}\xi_0 \qquad \eta_0 = z + \mathrm{ad}_{\tilde{\xi}-\xi_1}\,\xi_e. \tag{20}$$

Again as in the proof of theorem (4), (18) and (19) are equivalent to (18) and

$$C^*w + D\mu_0 - z = 0 \qquad \mathrm{ad}_{\tilde{\xi}-\xi_1-\xi_0}\,\xi_e = 0 \tag{21}$$

or equivalently (18), $C^*w + D\mu_0 \in \mathfrak{z}_e$ and $\tilde{\xi} - \xi_1 - \xi_0 \in \mathfrak{k}_e$. This proves the theorem since for any ξ_0 and ξ_1 we can find $\tilde{\xi}$ such that $\tilde{\xi} - \xi_1 - \xi_0 \in \mathfrak{k}_e$. □

One can calculate the linearized dependence at p_e of the generators of the relative equilibria on the relative equilibria themselves by taking, at the end of the proof of theorem 5, $\tilde{\xi} - \xi_1 - \xi_0 = k \in \mathfrak{k}$, and substituting this and z from (21) into (20), which gives

$$\eta_0 = C^*w + D\mu_0 - \mathrm{ad}_{\xi_e}\,\xi_0 \qquad \eta_1 = -\,\mathrm{ad}_{\xi_e}\,\xi_1. \tag{22}$$

If we define

$$\hat{\mathcal{E}}_{(K_e)} = \psi_{(K_e)}^{-1}\big(Z(TP)\big)$$

that is $\hat{\mathcal{E}}_{(K_e)}$ is the set of relative equilibria paired with their generators, then

$$T_{p_e}\hat{\mathcal{E}}_{(K_e)} = \left\{ (v, \eta) \in T_{p_e}P \times \mathfrak{g} : \mathrm{d}X_{H_{\xi_e}}(p_e)v - \eta \cdot p_e = 0 \right\}$$

and so $T_{p_e}\hat{\mathcal{E}}_{(K_e)}$ is the graph of (22) over the tangent space $T_{p_e}\mathcal{E}_{(K_e)}$ calculated in theorem 5.

The following result uses theorem 5 to give a generalization and partial converse of a result of Patrick (1995).

Theorem 6. *If the relative equilibrium p_e is transversal then $\mathcal{E}_{(K_e)}$ is a symplectic submanifold of P in a neighbourhood of p_e if and only if p_e is non-degenerate and G_{μ_e} is a maximal torus of G.*

Proof. The manifold $\mathcal{E}_{(K_e)}$ is symplectic near p_e if and only if the tangent space $T_{p_e}\mathcal{E}_{(K_e)}$ is a symplectic subspace of $T_{p_e}P$. From theorem 5 and the description of the symplectic form on $T_{p_e}P$ in terms of the decomposition (2) it is clear that the restriction of the symplectic form to $T_{p_e}\mathcal{E}_{(K_e)}$ is degenerate on the subspace $\mathrm{ann}_{\mathfrak{g}_{\mu_e}}\left(\mathfrak{z}_e^*\right) \subset \mathfrak{g}_{\mu_e} \cong T_0$. Thus $T_{p_e}\mathcal{E}_{(K_e)}$ can only be a symplectic subspace of $T_{p_e}P$ if $\mathrm{ann}_{\mathfrak{g}_{\mu_e}}\left(\mathfrak{z}_e^*\right) = \{0\}$, that is $\mathfrak{g}_{\mu_e} = \mathfrak{z}_e$. This implies that $\mathfrak{g}_{\mu_e} = \mathfrak{k}_e$ and is Abelian, and so G_{μ_e} is a maximal torus of G by lemma 1.

When $\mathfrak{z}_e = \mathfrak{k}_e = \mathfrak{g}_{\mu_e}$ the condition $C^*w + D\mu_0 \in \mathfrak{z}_e$ in the description of $T_{p_e}\mathcal{E}_{(K_e)}$ in theorem 5 is automatically satisfied. With respect to the decomposition (2) we therefore have

$$T_{p_e}\mathcal{E}_{(K_e)} = \mathfrak{g}_{\mu_e} \oplus \ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e) \oplus \ker C \oplus \mathfrak{g}_{\mu_e}^{\perp}. \qquad (23)$$

The subspace $\mathfrak{g}_{\mu_e}^{\perp}$ is always symplectic. Since p_e is transversal any 0 eigenvalue of $\mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ is semisimple and so $\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ is symplectic. It follows that $T_{p_e}\mathcal{E}_{(K_e)}$ is a symplectic subspace of $T_{p_e}P$ if and only if $\ker C = \mathfrak{g}_{\mu_e}^*$ and so $C = 0$. However, for a transversal relative equilibrium C must map onto $\ker\, \mathrm{d}X_{H_{\mu_e}}(\bar{p}_e)$ and so $T_{p_e}\mathcal{E}_{(K_e)}$ can only be a symplectic subspace of $T_{p_e}P$ if p_e is non-degenerate.

Conversely, suppose p_e is non-degenerate and G_{μ_e} is a maximal torus. Then $K_e \subseteq G_{\mu_e}$, K_e contains a maximal torus and so $K = G_{\mu_e}$ and $\mathfrak{z}_e = \mathfrak{k}_e = \mathfrak{g}_{\mu_e}$. It follows that $C = 0$, equation (23) holds, and $T_{p_e}\mathcal{E}_{(K_e)}$ is a symplectic subspace of $T_{p_e}P$. □

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