

Dynamics near relative equilibria: Nongeneric momenta at a 1:1 group-reduced resonance

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Abstract. An interesting situation occurs when the linearized dynamics of the shape of a formally stable Hamiltonian relative equilibrium at nongeneric momentum 1:1 resonates with a frequency of the relative equilibrium's generator. In this case some of the shape variables couple to the group variables to first order in the momentum perturbation, and the first order perturbation theory implies that the relative equilibrium slowly changes orientation in the same way that a charged particle *with magnetic moment* moves on a sphere under the influence of a radial magnetic monopole. In the course of showing this a normal form is constructed for linearizations of relative equilibria and for Hamiltonians near group orbits of relative equilibria.

1 Introduction

Suppose we are given a symplectic manifold (P, ω) and a compact Lie group G acting freely and symplectically on P with Ad^* -equivariant momentum mapping $J : P \rightarrow \mathfrak{g}^*$. Let the Lie algebra of G be \mathfrak{g} and $H : P \rightarrow \mathbb{R}$ be a G -invariant Hamiltonian. Let us fix our attention on a specific relative equilibrium p_e , with generator ξ_e , so that the integral curve of the Hamiltonian vector field X_H at p_e is $\exp(\xi_e t)p_e$. Let $J(p_e) = \mu_e$, let G_{μ_e} be the isotropy group of μ_e under the coadjoint action of G on \mathfrak{g}^* , and let \mathfrak{g}_{μ_e} be the Lie algebra of G_{μ_e} . For more information on these basic definitions, see, for example, Abraham and Marsden [1978], Marsden [1992], or Marsden and Ratiu [1994].

Suppose that $G = SO(3)$, although the results of this article are much more general than that. Since the action is free, a neighborhood of the group orbit Gp_e may be (with respect to the left action) equivariantly projected to

$SO(3)$ itself. Let us realize $SO(3)$ as the unit circle bundle of the 2-sphere in \mathbb{R}^3 , arranging that the orbit $\exp(\xi t)p_e$ be projected to the unit circle in the tangent space of the vertical vector \mathbf{k} . The system, when at the relative equilibrium p_e , and when viewed through this projection, appears merely as a point, say \mathcal{P} , rotating uniformly on the unit circle in the tangent space of S^2 at \mathbf{k} , and the other relative equilibria gp_e , $g \in SO(3)$, appear as this same motion but reoriented by the rotation g . Suppose p_e is formally stable, and suppose the system is then perturbed from the relative equilibrium p_e . If $\mu_e \neq 0$ then the motion of the system is bound to remain near $\exp(\xi_e \mathbb{R})p_e$, as shown in Patrick [1992] (extensions to the nonfree case may be found in Ortega and Ratiu [1999] and Lerman and Singer [1997]). The perturbation affects the motion of the point \mathcal{P} by superimposing on its original motion some small vibration; however the motion of \mathcal{P} does not ever carry that point far from its original circular path.

The situation is different when $\mu_e = 0$, for then $G_{\mu_e} = SO(3)$, and under perturbation the system is bound only to remain near Gp_e . The effect is that, after a perturbation from p_e , the point \mathcal{P} still moves nearly circularly in tangent spaces of S^2 , but the base point of the tangent space slowly changes location on S^2 . In fact, as shown in Patrick [1995], the base point \mathcal{B} moves as a point charge under the influence of a radial magnetic monopole. This result is obtained by constructing coordinates which make an open neighborhood of Gp_e in P into an open neighborhood of $Z(T^*SO(3)) \times \{\bar{p}_e\}$ in $T^*SO(3) \times P_{\mu_e}$, where $P_{\mu_e} = J^{-1}(\mu_e)/G_{\mu_e}$ is the symplectic reduced space, $\bar{p}_e \in P_{\mu_e}$ is the equilibrium of the reduced space corresponding to p_e , and $Z(T^*SO(3))$ is the zero section of T^*SO . In the new coordinates, to first order in the momentum perturbation, the “group variables” in $T^*SO(3)$ and the “reduced variables” in P_{μ_e} become decoupled; truncating the higher order interaction terms gives a new Hamiltonian system on $T^*SO(3) \times P_{\mu_e}$ called the *drift system*. The drift system has an additional S^1 *normal form symmetry*. The Hamiltonian system for the slow motion of the base point \mathcal{B} is obtained from the drift system by ignoring the variables on P_{μ_e} and reducing the resulting Hamiltonian on $T^*SO(3)$ by the normal form symmetry. The approximating, truncated Hamiltonian of the drift system may be obtained merely by calculating the nilpotent part of the linearization of the vector field $X_{H_{\xi_e}}(p) \equiv X_H(p) - \xi_e p$ at p_e , where $\xi p \equiv (\xi)_P(p)$ denotes the infinitesimal generator of ξ_e at p_e .

Actually, the results in Patrick [1995] depend on a nonresonance condition that the frequency $|\xi_e|$ cannot equal any linearized frequency of the reduced system at \bar{p}_e . This article considers the problem of constructing a Hamiltonian system for the motion of the base point \mathcal{B} in the case where there is such a *1:1 group-reduced resonance*. The main result of this article is that, in the presence of resonance, some of the reduced and group variables

couple to first order in the momentum perturbation, with the result that \mathcal{B} moves as a charged particle *with magnetic moment* on a sphere under the influence of a radial magnetic monopole. However, this “particle” regularly exchanges its charge with its magnetic moment.

The construction of the Hamiltonian system which models the motion of \mathcal{B} in the 1:1 resonant case generalizes and parallels the nonresonant case. In the course of the construction the linearization of $X_{H_{\xi_e}}$ at p_e is put into a certain block normal form through a Witt or Moncreif decomposition of the tangent space $T_{p_e}P$. The objective of this normal form is the separation of the group and reduced motions and it is not the full normal form of the linearization as a infinitesimal symplectic operator. The linear normal form is then extended to a neighborhood of the group orbit Gp_e using the isotropic embedding/equivariant Darboux theorem. These normal forms—for linearizations of relative equilibria and for Hamiltonian systems near group orbits of relative equilibria—are the second main contribution of this article. In Patrick and Roberts [1999] these normal forms have been used to study the structure of the set of relative equilibria.

A new aspect that is absent in the nonresonant case but emerges in the resonant case is the appearance of a *gauge group*. There is too much freedom inherent in the normal forms to exactly fit the needs of first order agreement between the original and drift systems. This slack is taken up by the group $(\mathbb{R}, +)$ and presents itself as an inherent freedom in the choice of the normal forms. This *gauge freedom* can be used to simplify the drift system, a motif which is quite useful in this work.

The Hamiltonian system modeling the motion of \mathcal{B} is approximate since it is the result of a truncation, so there arises the question of how well this model Hamiltonian reflects the behavior of the real one. To check this I have numerically simulated the system of two (identical) axially symmetric rods which are joined by a frictionless ball-and-socket joint, but otherwise move freely (Patrick [1989, 1991]). The relative equilibria of this system correspond to motions such that the two rods spin on their axes while otherwise maintaining constant mutual orientations, while the whole assemblage rotates about some fixed axis. There exist, due to the system’s multiple rotating parts, relative equilibria which are formally stable and which have zero (hence nongeneric) total-angular-momentum. In fact there is a continuum of relative equilibria with zero total-angular-momentum, and it happens that parameters may be chosen so that there are relative equilibria at zero total-angular-momentum and at the 1:1 group-reduced resonance. The third main contribution of this article is the actual calculation of the drift system for the coupled rod system near one such relative equilibria, and the verification of the drift approximation by comparison of simulations of the coupled rod system itself and certain predictions of its drift system.

Here is an overview of this work. I begin in Sect. 2.1 with an analysis of the linearization $dX_{H_{\xi_e}}(p_e)$ of $X_{H_{\xi_e}}$ at p_e , by splitting the linearization into semisimple and nilpotent parts: $dX_{H_{\xi_e}}(p_e) = S_{p_e} + N_{p_e}$. The key tool in the analysis is a certain Moncrief or Witt decomposition of $T_{p_e}P$:

$$T_{p_e}P \cong W_{\text{red}} \oplus (\mathfrak{g}_{\mu_e} \oplus \mathfrak{g}_{\mu_e}^*) \oplus T_{\mu_e}(G\mu_e).$$

Whereas this decomposition is obtained in Patrick [1995] for the nonresonant case with the aid of the generalized eigenspaces of $dX_{H_{\xi_e}}(p_e)$, and so was $dX_{H_{\xi_e}}(p_e)$ -invariant, here the decomposition is only S_{p_e} -invariant. Using this decomposition, the nilpotent part N_{p_e} acquires a block form which gives rise to certain operators:

$$N_{p_e}^{23} : \mathfrak{g}_{\mu_e}^* \rightarrow \mathfrak{g}_{\mu_e}, \quad N_{p_e}^{21} : T_{\bar{p}_e}P_{\mu_e} \rightarrow \mathfrak{g}_{\mu_e}, \quad N_{p_e}^{13} : \mathfrak{g}_{\mu_e}^* \rightarrow T_{\bar{p}_e}P_{\mu_e}.$$

In the nonresonant case $N_{p_e}^{21} = 0$ and $N_{p_e}^{13} = 0$ necessarily. I expose, in Sect. 2.2, the special properties of these operators, the most notable of which are that $N_{p_e}^{23}$ is symmetric, $N_{p_e}^{13}$ and $N_{p_e}^{21}$ are dual, and that these operators have certain commutation relations with operators ad_{ξ_e} and coad_{ξ_e} .

Next, in Sect. 3, I focus on the case where the reduced and group spectra have intersection $\{\pm i\lambda_{p_e}\}$, a purely imaginary eigenvalue and its conjugate, each of which occur with multiplicity 1 in both the reduced and group spectrum. The generalized eigenspace of the reduced linearization corresponding to $\{\pm i\lambda_{p_e}\}$ is symplectic and has dimension 2, and so is linearly symplectomorphic to $(\mathbb{R}^2 = \{x_1, x_2\}, dx_1 \wedge dx_2)$, and the operator $N_{p_e}^{21}$ is zero on the sum of the complimentary generalized eigenspaces. Consequently, the splitting of $T_{p_e}P$ refines and the operator $N_{p_e}^{21}$ may be replaced by another operator $N_{p_e}^{211} : \mathbb{R}^2 \rightarrow \mathfrak{g}_{\mu_e}$. After this simplification, the splitting of $T_{p_e}P$ and the isotropic embedding theorem (Marsden [1981], Weinstein [1977]) together give a map which transforms the Hamiltonian H , to first order near $G_{\mu_e}p_e$, to the Hamiltonian

$$H_{\text{drift}}(x, \alpha_g) \equiv \langle g^{-1}\alpha_g, \xi_e \rangle + \frac{1}{2}N_{p_e}^{23}(\alpha_g, \alpha_g) + \langle g^{-1}\alpha_g, N_{p_e}^{211}x \rangle$$

near $\{0\} \times Z(T^*G_{\mu_e})$ in the phase space $\mathbb{R}^2 \times T^*G_{\mu_e}$, where $Z(T^*G_{\mu_e})$ denotes the zero section. In addition to the expected invariance under the left action of G_{μ_e} , H_{drift} is invariant under a diagonal action of the toral subgroup generated by ξ_e ; this is the normal form symmetry.

In Sect. 4 I further assume that G is the largest compact continuous symmetry group for an ordinary mechanical system, namely $SO(3) \times (S^1)^n$. After various manipulations, including an Abelian reduction, the drift Hamiltonian becomes

$$H_{\text{drift}} \equiv \frac{1}{2}I_1(\pi_1^2 + \pi_2^2) + \frac{1}{2}I_2\pi_3^2 + \kappa(\pi_1x_1 + \pi_2x_2) + a\pi_3$$

on the phase space $\mathbb{R}^2 \times T^*SO(3) = \{(x, A, \pi)\}$. Here I_1 and I_2 come from $N_{p_e}^{23}$, κ , a single coupling constant, is all that remains of $N_{p_e}^{211}$, a is a constant, and there are two possibilities: the relative equilibrium can be either a “+” type or a “-” type. In this context the gauge freedom has the effect of making I_1 arbitrary and interest is focused on the dynamics near to $x = 0, \pi = 0$. The Hamiltonian H_{drift} is defined on a phase space of dimension 8 and has symmetry $SO(3) \times S^1$, and so defines a system which is completely integrable. Although a complete analysis in the general case seems difficult, I analyze this system in Sect. 4.1, where I show that the system has, for example, some singular reduced phase spaces and spectrally unstable relative equilibria with homoclinic connections. It is in Sect. 4.2 that I show that the drift system reduced by the normal form symmetry can be cast as a charged particle with magnetic moment moving on the sphere while under the influence of a magnetic monopole.

Finally, in Sect. 5, I numerically investigate the dynamics of the coupled rod system near one particular resonant relative equilibrium and compare this dynamics with that of the drift system. I explore three distinct regions of phase space: 1) within zero total-angular-momentum, 2) near a stable relative equilibria of the drift system, and 3) near a spectrally unstable relative equilibrium at a singularity of the drift system. The comparison of the two systems is hampered by the implicit nature of the coordinates relating them, but in the first two comparisons agreement between the two systems is obtained uneventfully. However, the singular points of the third comparison involve an unexpected *reconstruction phase jump*; the situation is delicate and small perturbations are required to elicit quantitative agreement between the two systems. Nevertheless it becomes clear that many elements of the dynamics of the coupled rod system near the resonant relative equilibrium are indeed captured by the drift system.

2 The linearization

Here are the basic notations:

1. p_e is a relative equilibrium, ξ_e is the generator of p_e , and the momentum of p_e is $\mu_e \equiv J(p_e)$.
2. G_{μ_e} is the isotropy group of μ_e under the coadjoint action of G on \mathfrak{g}^* , and \mathfrak{g}_{μ_e} is the Lie algebra of G_{μ_e} . $\text{CoAd}_g \equiv (\text{Ad}_{g^{-1}})^*$, $g \in G$, and $\text{coad}_\xi \equiv -(\text{ad}_\xi)^*$, $\xi \in \mathfrak{g}$.
3. $H_{\xi_e} \equiv H - J_{\xi_e}$, so that H_{ξ_e} has a critical point at p_e . The Hessian of H_{ξ_e} at p_e will be denoted by $d^2H_{\xi_e}(p_e)$ and the linearization of $X_{H_{\xi_e}}$ at p_e will be denoted $dX_{H_{\xi_e}}(p_e)$.
4. Without loss of generality, since everything is local near to p_e and p_e is regular, the Marsden-Weinstein symplectic reduced space $(P_{\mu_e}, \omega_{\mu_e})$

exists; let $\pi_{\mu_e} : J^{-1}(\mu_e) \rightarrow P_{\mu_e}$ be the projection. The reduced system has this phase space with Hamiltonian H_{μ_e} defined by $H_{\mu_e}\pi_{\mu_e} = H|_{J^{-1}(\mu_e)}$. Also, $\bar{p}_e \equiv \pi_{\mu_e}(p_e)$, and the linearization of the reduced system at \bar{p}_e is denoted by $dX_{H_{\mu_e}}(\bar{p}_e)$. Suppose that p_e is regular, which means $\xi p_e \neq 0$ for all $\xi \in \mathfrak{g}$.

5. The *reduced spectrum* is the spectrum of the linearization $dX_{H_{\mu_e}}(\bar{p}_e)$ at \bar{p}_e of the reduced system. The *group spectrum* is the the spectrum of $\text{ad}_{\xi_e} : \mathfrak{g} \rightarrow \mathfrak{g}$.

Let $dX_{H_{\xi_e}}(p_e) = N_{p_e} + S_{p_e}$ be the Jordan decomposition of $dX_{H_{\xi_e}}(p_e)$ into its semisimple part S_{p_e} and nilpotent part N_{p_e} . The aim of this section is an analysis of $dX_{H_{\xi_e}}(p_e)$, focusing on its nilpotent part N_{p_e} . Do not assume that the reduced spectrum and the group spectrum are disjoint.

2.1 The splitting of $T_{p_e}P$

I begin by deriving a ‘‘Moncreif’’ splitting (Marsden [1981]) of $T_{p_e}P$ which is slightly weaker than its analogue in Patrick [1995], in that the ‘‘reduced’’ part (below W_{red}) of the splitting is not (and cannot, in general) be constructed to be $dX_{H_{\xi_e}}(p_e)$ invariant. The details are similar to those in Patrick [1995]; I will not belabor them here. The subspace $\ker dJ(p_e)$ is S_{p_e} -invariant, so one can choose an S_{p_e} -invariant subspace W_{red} such that

$$\ker dJ(p_e) = W_{\text{red}} \oplus \mathfrak{g}_{\mu_e} p_e.$$

The subspace W_{red} is symplectic, and $T\pi_{\mu_e}|_{W_{\text{red}}} : W_{\text{red}} \rightarrow T_{\bar{p}_e}P_{\mu_e}$ is a linear symplectomorphism. Choose an Ad-invariant complement \mathfrak{b} to \mathfrak{g}_{μ_e} , so that

$$\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{g}_{\mu_e}.$$

The subspace $\mathfrak{b}p_e$ is symplectic, and $\mathfrak{g}_{\mu_e} \subseteq (W_{\text{red}} \oplus \mathfrak{b}p_e)^{\omega\perp}$ as an S_{p_e} -invariant Lagrangian subspace. Choose an S_{p_e} -invariant Lagrangian complement¹ Z to \mathfrak{g}_{μ_e} in $(W_{\text{red}} \oplus \mathfrak{b}p_e)^{\omega\perp}$, giving the S_{p_e} -invariant splitting

$$(1) \quad T_{p_e}P = W_{\text{red}} \oplus \mathfrak{g}_{\mu_e} p_e \oplus Z \oplus \mathfrak{b}p_e.$$

As already stated, $T\pi_{\mu_e}$ is a symplectomorphism between W_{red} and $T_{\bar{p}_e}P_{\mu_e}$. Also, $dJ(p_e)$ is a linear isomorphism between Z and $\mathfrak{g}_{\mu_e}^*$, and a symplectomorphism between $\mathfrak{b}p_e$ and the tangent space $T_{\mu_e}(G\mu_e)$ at μ_e to the coadjoint

¹ Since S_{p_e} is semisimple, every S_{p_e} -invariant Lagrangian subspace has an S_{p_e} -invariant Lagrangian complement. This is a general fact; the proof is a simple modification of the argument found at the bottom of page 401 of Patrick [1995].

orbit $G\mu_e$. Thus (1) becomes, through these identifications, the S_{p_e} -invariant splitting,

$$(2) \quad T_{p_e}P = T_{\bar{p}_e}P_{\mu_e} \oplus \mathfrak{g}_{\mu_e} \oplus \mathfrak{g}_{\mu_e}^* \oplus T_{\mu_e}(G\mu_e).$$

With respect to the decomposition (2), let $dX_{H_{\xi_e}}(p_e)$, S_{p_e} , and N_{p_e} have blocks $[A^{ij}]$, $[S^{ij}]$, and $[N_{p_e}^{ij}]$, respectively; the block form of $\omega(p_e)$ becomes

$$(3) \quad \omega(p_e) = \begin{bmatrix} \omega_{\mu_e}(\bar{p}_e) & 0 & 0 & 0 \\ 0 & 0 & \text{Id} & 0 \\ 0 & -\text{Id} & 0 & 0 \\ 0 & 0 & 0 & \check{\omega}_{\mu_e}(\mu_e) \end{bmatrix},$$

where ω_{μ_e} is the reduced symplectic form of P_{μ_e} and $\check{\omega}_{\mu_e}$ is the Kostant-Souriau form on $G\mu_e$. In (1), the subspaces $\mathfrak{g}_{\mu_e}p_e$, $\mathfrak{b}p_e$, and $\ker dJ(p_e) = W_{\text{red}} \oplus \mathfrak{g}_{\mu_e}p_e$ are $dX_{H_{\xi_e}}(p_e)$ -invariant, and hence are N_{p_e} -invariant, and this implies certain of the A^{ij} and $N_{p_e}^{ij}$ vanish. Using the identities (Proposition 5 of Patrick [1995])

$$\begin{aligned} X_{H_{\xi_e}}(p_e) &= -\text{coad}_{\xi_e} dJ(p_e) \\ dX_{H_{\xi_e}}(p_e)\eta(p_e) &= -(\text{ad}_{\xi_e} \eta)_P(p_e), \quad \eta \in \mathfrak{g} \end{aligned}$$

to calculate the diagonal blocks of $dX_{H_{\xi_e}}(p_e)$, the Jordan decomposition becomes

$$\begin{aligned} dX_{H_{\xi_e}}(p_e) &= \begin{bmatrix} dX_{H_{\mu_e}}(\bar{p}_e) & 0 & A^{13} & 0 \\ A^{21} & -\text{ad}_{\xi_e} & A^{23} & 0 \\ 0 & 0 & \text{ad}_{\xi_e}^* & 0 \\ 0 & 0 & 0 & \text{ad}_{\xi_e}^* \end{bmatrix} \\ &= \begin{bmatrix} S^{11} & 0 & 0 & 0 \\ 0 & S^{22} & 0 & 0 \\ 0 & 0 & S^{33} & 0 \\ 0 & 0 & 0 & S^{44} \end{bmatrix} + \begin{bmatrix} N_{p_e}^{11} & 0 & N_{p_e}^{13} & 0 \\ N_{p_e}^{21} & N_{p_e}^{22} & N_{p_e}^{23} & 0 \\ 0 & 0 & N_{p_e}^{33} & 0 \\ 0 & 0 & 0 & N_{p_e}^{44} \end{bmatrix}. \end{aligned}$$

Since S_{p_e} and N_{p_e} commute, so do S^{11} and $N_{p_e}^{11}$, and S^{11} is semisimple and $N_{p_e}^{11}$ is nilpotent since S_{p_e} and N_{p_e} are. Thus, $dX_{H_{\mu_e}}(\bar{p}_e) = S^{11} + N_{p_e}^{11}$ is the Jordan decomposition of the reduced linearization $dX_{H_{\mu_e}}(\bar{p}_e)$, which is semisimple, since p_e is formally stable. Consequently, $S^{11} = dX_{H_{\mu_e}}(\bar{p}_e)$ and $N_{p_e}^{11} = 0$. Similarly, $N_{p_e}^{22} = N_{p_e}^{33} = N_{p_e}^{44} = 0$, and

$$S^{22} = -\text{ad}_{\xi_e}, \quad S^{33} = (\text{ad}_{\xi_e})^*, \quad S^{44} = (\text{ad}_{\xi_e})^*.$$

Thus,

$$(4) \quad S_{p_e} = \begin{bmatrix} dX_{H_{\mu_e}}(\bar{p}_e) & 0 & 0 & 0 \\ 0 & -\text{ad}_{\xi_e} & 0 & 0 \\ 0 & 0 & (\text{ad}_{\xi_e})^* & 0 \\ 0 & 0 & 0 & (\text{ad}_{\xi_e})^* \end{bmatrix}$$

and

$$(5) \quad N_{p_e} = \begin{bmatrix} 0 & 0 & N_{p_e}^{13} & 0 \\ N_{p_e}^{21} & 0 & N_{p_e}^{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As will be seen immediately below, a large amount of information can be discerned from structure of N_{p_e} and S_{p_e} visible in (4) and (5).

2.2 Properties of N_{p_e} and its subblocks

Directly from (5), $(N_{p_e})^3 = 0$ so that N_{p_e} has nilpotent order at most 3. This upper bound on the nilpotent order occurs because $dX_{H_{\xi_e}}(p_e)$ is semisimple when suitably restricted and projected to the summands of (1), so that N_{p_e} must “transport” between those summands. Moreover, the “direction” of this transport is “one way”, since $\ker dJ(p_e)$ and $\mathfrak{g}_{\mu_e p_e}$ are invariant subspaces, and the “transport” stops at $\mathfrak{g}_{\mu_e p_e}$, on which N_{p_e} is zero:

$$(6) \quad \begin{aligned} \text{Image } N_{p_e} &\subseteq \ker dJ(p_e), \\ N_{p_e}(\ker dJ(p_e)) &\subseteq \mathfrak{g}_{\mu_e p_e}, \end{aligned}$$

$$(7) \quad N_{p_e}|_{\mathfrak{g}_{\mu_e p_e}} = 0.$$

When iteratively acted upon by N_{p_e} , a vector can make just 2 steps before annihilation: $\ker dJ(p_e)$ and $\mathfrak{g}_{\mu_e p_e}$; hence the nilpotent order of N_{p_e} is a most 3.

By (7),

$$\ker T_{p_e} \pi_{\mu_e} = \mathfrak{g}_{\mu_e p_e} \subseteq \ker(N_{p_e}|_{\ker dJ(p_e)}),$$

so $N_{p_e}^{21} : T_{\bar{p}_e} P_{\mu_e} \rightarrow \mathfrak{g}$ is the unique linear map such that

$$(8) \quad \begin{array}{ccccc} \ker dJ(p_e) & \xrightarrow{N_{p_e}} & \mathfrak{g}_{\mu_e p_e} & \xrightarrow{\xi p_e \mapsto \xi} & \mathfrak{g}_{\mu_e} \\ \downarrow T_{p_e} \pi_{\mu_e} & & \nearrow N_{p_e}^{21} & & \\ T_{\bar{p}_e} P_{\mu_e} & & & & \end{array}$$

Since this diagram does not depend on choices made in the construction of the splitting (1), neither does N^{21} . Also, $N_{p_e}^{21}$ may be calculated merely by reference to the diagram; it is not necessary to calculate all summands of (1). Similarly, by (6), and since, by (5),

$$\ker dJ(p_e) \subseteq \ker(T\pi_{\mu_e} N_{p_e}),$$

$N_{p_e}^{13} : \mathfrak{g}_{\mu_e}^* \rightarrow T_{\bar{p}_e} P_{\mu_e}$ is the unique linear map such that

$$\begin{array}{ccccc} T_{p_e} P & \xrightarrow{dJ(p_e)} & \mathfrak{g}^* & \longrightarrow & \mathfrak{g}_{\mu_e}^* \\ \downarrow N_{p_e} & & & \nearrow N_{p_e}^{13} & \\ \ker dJ(p_e) & & & & \\ \downarrow T_{p_e} \pi_{\mu_e} & & & & \\ T_{\bar{p}_e} P_{\mu_e} & & & & \end{array}$$

Since $dX_{H_{\xi_e}}(p_e)$ is infinitesimally symplectic, so is N_{p_e} , so that

$$(9) \quad (N_{p_e})^t \omega(p_e) + \omega(p_e) N_{p_e} = 0,$$

and inserting (3) and (5) into (9) gives

$$(10) \quad N_{p_e}^{21} = -(N_{p_e}^{13})^* \omega_{\mu_e}^b, \quad (N_{p_e}^{23})^* = N_{p_e}^{23},$$

so $N_{p_e}^{23}$ is symmetric, and in a certain sense, $N_{p_e}^{13}$ and $N_{p_e}^{21}$ are dual. Similarly, using (4) and (5) to write out what it means for S_{p_e} and N_{p_e} to commute,

$$(11) \quad dX_{H_{\mu_e}}(\bar{p}_e) N_{p_e}^{13} = -N_{p_e}^{13} \text{coad}_{\xi_e},$$

$$(12) \quad N_{p_e}^{21} dX_{H_{\mu_e}}(\bar{p}_e) = -\text{ad}_{\xi_e} N_{p_e}^{21},$$

$$(13) \quad \text{ad}_{\xi_e} N_{p_e}^{23} = N_{p_e}^{23} \text{coad}_{\xi_e}.$$

These properties will yield the important *normal form symmetry* of the drift system.

Temporarily let pr_2 be the projection onto the second factor of $\ker dJ(p_e) = W_{\text{red}} \oplus \mathfrak{g}_{\mu_e} p_e$. Then

$$\ker(dJ(p_e)|(W_{\text{red}} + \mathfrak{b}p_e)^{\omega^\perp}) = \mathfrak{g}_{\mu_e} p_e \subseteq \ker(\text{pr}_2 N_{p_e}|(W_{\text{red}} + \mathfrak{b}p_e)^{\omega^\perp}),$$

so $N_{p_e}^{23} : \mathfrak{g}_{\mu_e} \times \mathfrak{g}_{\mu_e} \rightarrow \mathbb{R}$ is the unique bilinear from such that

$$(14) \quad \begin{array}{ccc} (W_{\text{red}} + \mathfrak{b}p_e)^{\omega^\perp} & \xrightarrow{dJ(p_e)} & \mathfrak{g}_{\mu_e}^* \\ \downarrow N_{p_e} & \nearrow & \\ \ker dJ(p_e) & & (N_{p_e}^{23})^\flat \\ \downarrow \text{pr}_2 & \nearrow & \\ \mathfrak{g}_{\mu_e} & & \end{array}$$

Thus, N^{23} may be calculated without calculating the Lagrangian complement Z of (1).

However, $N_{p_e}^{23}$ does depend on the particular choice of W_{red} , even if W_{red} is an S_{p_e} -invariant complement of $\mathfrak{g}_{\mu_e}p_e$ in $\ker dJ(p_e)$. To see this dependence, choose a linear map $A : W_{\text{red}} \rightarrow \mathfrak{g}_{\mu_e}$, which commutes with S_{p_e} , and then choose a new W'_{red} where

$$(15) \quad W'_{\text{red}} = \{ w + (Aw)p_e \mid w \in W_{\text{red}} \}.$$

Also, choose a map $B : \mathfrak{b} \rightarrow \mathfrak{g}_{\mu_e}$ which intertwines the adjoint action and then choose a new \mathfrak{b}' by

$$(16) \quad \mathfrak{b}' = \{ \eta + B\eta \mid \eta \in \mathfrak{b} \}.$$

Given $\nu \in \mathfrak{g}_{\mu_e}^*$, the process of calculating $(N_{p_e}^{23'})^\flat \nu$, given by the analog of (14) for the new choices, is

1. pick $z' \in (W'_{\text{red}} + \mathfrak{b}'p_e)^{\omega^\perp}$ such that $dJ(p_e)z' = \nu$;
2. calculate $N_{p_e} z'$;
3. project $N_{p_e} z'$ to $\mathfrak{g}_{\mu_e}p_e \equiv \mathfrak{g}_{\mu_e}$ using the splitting of $\ker dJ(p_e)$ defined by W'_{red} , with result $(N_{p_e}^{23'})^\flat \nu$.

It is straightforward from (3), (15), and (16), that

$$(W'_{\text{red}} + \mathfrak{b}'p_e)^{\omega^\perp} = \{ \omega_{\mu_e}^\sharp A^* \mu \oplus \xi \oplus \mu \oplus \check{\omega}_{\mu_e}^\sharp B^* \mu \mid \xi \in \mathfrak{g}_{\mu_e}, \mu \in \mathfrak{g}_{\mu_e}^* \},$$

so given $\nu \in \mathfrak{g}_{\mu_e}^*$, let

$$z' = \omega_{\mu_e}^\sharp A^* \nu \oplus 0 \oplus \nu \oplus \check{\omega}_{\mu_e}^\sharp B^* \nu.$$

Then

$$\begin{aligned} N_{p_e} z' &= N_{p_e}^{13} \nu \oplus (N_{p_e}^{21} \omega_{\mu_e}^\sharp A^* \nu + (N_{p_e}^{23})^\flat \nu) \oplus 0 \oplus 0 \\ &= N_{p_e}^{13} \nu \oplus AN_{p_e}^{13} \nu \oplus 0 \oplus 0 \\ &\quad + 0 \oplus (N_{p_e}^{21} \omega_{\mu_e}^\sharp A^* \nu - AN_{p_e}^{13} \nu + (N_{p_e}^{23})^\flat \nu) \oplus 0 \oplus 0, \end{aligned}$$

which by (15) is the appropriate decomposition of $N_{p_e} z'$. Thus, using (10),

$$\begin{aligned}
 (17) \quad (N_{p_e}^{23'})^b \nu &= N_{p_e}^{21} \omega_{\mu_e}^\# A^* \nu - AN_{p_e}^{13} \nu + (N_{p_e}^{23})^b \nu \\
 &= -((N_{p_e}^{13})^* A^* + AN_{p_e}^{13}) \nu + (N_{p_e}^{23})^b \nu.
 \end{aligned}$$

This freedom to adjust $N_{p_e}^{23}$ by adjusting W_{red} will give the important *gauge freedom* of the drift system.

This gauge freedom can be encoded as a group, as follows. Temporarily set $\mathbb{E}_{p_e} = T_{\bar{p}_e} P_{\mu_e} \times \mathfrak{g}_{\mu_e} \times \mathfrak{g}_{\mu_e}^* \times T_{\mu_e} G_{\mu_e}$, and fix one particular splitting of type (1), giving a symplectomorphism, say $\phi_0 : T_{p_e} P \rightarrow \mathbb{E}_{p_e}$. Given $A : W_{\text{red}} \rightarrow \mathfrak{g}_{\mu_e}$ and $B : \mathfrak{b} \rightarrow \mathfrak{g}_{\mu_e}$ as above, there is another symplectomorphism $\phi_{A,B} : T_{p_e} P \rightarrow \mathbb{E}_{p_e}$, and hence a unique $\Delta_{A,B}$ in the symplectic group $Sp(\mathbb{E}_{p_e})$ such that

$$\begin{array}{ccc}
 & T_{p_e} P & \\
 \phi_0 \swarrow & & \searrow \phi_{A,B} \\
 \mathbb{E}_{p_e} & \xrightarrow{\Delta_{A,B}} & \mathbb{E}_{p_e}
 \end{array}$$

Conversely, A is determined by $\Delta_{A,B}$, since if $x \in T_{\bar{p}_e} P_{\mu_e}$, $\xi \in \mathfrak{g}_{\mu_e}$, and $w \in W_{\text{red}}$ is such that $\phi_0(w) = x$, then

$$\begin{aligned}
 \Delta_{A,b}(x) &= \phi_{A,B}(w) = \phi_{A,B}(w + (Aw)p_e - (Aw)p_e) \\
 &= T\pi_{\mu_e}(w + (Aw)p_e) \oplus (-Aw) = x \oplus (-Aw).
 \end{aligned}$$

Similarly B is determined by $\Delta_{A,B}$, so the pairs (A, B) have a natural group structure given by the injection $(A, B) \mapsto \phi_{A,B}$, and a simple calculation gives

$$\phi_{A+A', B+B'} = \phi_{A,B} \circ \phi_{A',B'}.$$

Thus, at this level, the *gauge group* is the additive Abelian group of pairs (A, B) such that A commutes with S_{p_e} and B commutes with ad_{ξ_e} .

3 The simplest resonant case

All the above has been established without any particular presumption on the way that the group and reduced spectra intersect. There are some apriori features of the spectra: the spectra are purely imaginary (or zero) and are invariant under change of sign. Since p_e is presumed to be formally stable,

zero cannot occur in the reduced spectrum, and hence not in the intersection of the spectra. Thus, the following *resonance condition* is the simplest possible case beyond an empty intersection:

The intersection of the spectrum of ad_{ξ_e} and the spectrum of $dX_{H_{\mu_e}}(\bar{p}_e)$ is $\{\pm i\lambda_{p_e}\}$, $\lambda_{p_e} > 0$, and each of the eigenvalues $\pm i\lambda_{p_e}$ occur in each of these operators with multiplicity one.

While many of the results below remain true or have analogues in a more relaxed environment, for simplicity, this resonance condition will be assumed for the remainder of this article.

3.1 The linearization

Given the resonance assumption, it is natural to consider certain spectral splittings of the factors W_{red} and \mathfrak{g}_{μ_e} in (2). In particular, let W_{red}^1 be the generalized eigenspace of $\pm i\lambda_{p_e}$ for $dX_{H_{\mu_e}}(\bar{p}_e)$, and W_{red}^0 be the sum of the other generalized eigenspaces, so that W_{red}^1 is two dimensional, symplectic, and there is the symplectic splitting

$$(18) \quad T_{\bar{p}_e}P_{\mu_e} = W_{\text{red}}^0 \oplus W_{\text{red}}^1.$$

Similarly, let $\mathfrak{g}_{\mu_e}^1$ be the generalized eigenspace of $\pm i\lambda_{p_e}$ for the linear map ad_{ξ_e} , and $\mathfrak{g}_{\mu_e}^0$ be the sum of the other generalized eigenspaces, so $\mathfrak{g}_{\mu_e}^1$ is two dimensional, and

$$(19) \quad \mathfrak{g}_{\mu_e} = \mathfrak{g}_{\mu_e}^0 \oplus \mathfrak{g}_{\mu_e}^1.$$

Of course, there is also then the dual splitting $\mathfrak{g}_{\mu_e}^* = \mathfrak{g}_{\mu_e}^{0*} \oplus \mathfrak{g}_{\mu_e}^{1*}$.

So we have a refinement of the splitting (2) into “nonresonant” (with superscript 0) and “resonant” (superscript 1, dimension 2) parts:

$$(20) \quad \begin{aligned} T_{p_e}P &= W_{\text{red}}^0 \oplus W_{\text{red}}^1 \oplus \mathfrak{g}_{\mu_e} \oplus \mathfrak{g}_{\mu_e}^* \oplus T_{\mu_e}(G\mu_e) \\ &= W_{\text{red}}^0 \oplus W_{\text{red}}^1 \oplus \mathfrak{g}_{\mu_e}^0 \oplus \mathfrak{g}_{\mu_e}^1 \oplus \mathfrak{g}_{\mu_e}^{0*} \oplus \mathfrak{g}_{\mu_e}^{1*} \oplus T_{\mu_e}(G\mu_e). \end{aligned}$$

As one might expect, $N_{p_e}^{13}$ and $N_{p_e}^{21}$ localize to the resonant parts:

$$(21) \quad W_{\text{red}}^0 \subseteq \ker N_{p_e}^{21}, \quad \text{Image } N_{p_e}^{21} \subseteq \mathfrak{g}_{\mu_e}^1,$$

$$(22) \quad \mathfrak{g}_{\mu_e}^0 \subseteq \ker N_{p_e}^{13}, \quad \text{Image } N_{p_e}^{13} \subseteq W_{\text{red}}^1.$$

Indeed, to show the first of (21), it suffices to find a subspace, say $W_{\text{red}}^{0'}$ of $\ker dJ(p_e)$ such that

$$(23) \quad T_{p_e}\pi_{\mu_e}W_{\text{red}}^{0'} = W_{\text{red}}^0 \quad \text{and} \quad N_{p_e}|_{W_{\text{red}}^{0'}} = 0.$$

The result then follows by using $W_{\text{red}}^{0'}$ to reverse the vertical arrow of (8). The first of (23) can be assured by setting $W_{\text{red}}^{0'}$ to be the sum of the of nonresonant (i.e. not $\pm i\lambda_{p_e}$) generalized eigenspaces of $dX_{H_{\xi_e}}(p_e)$, and the second of (23) follows since $W_{\text{red}}^{0'}$ is $dX_{H_{\xi_e}}(p_e)$ -invariant and that $dX_{H_{\xi_e}}(p_e)|_{W_{\text{red}}^{0'}}$ is semisimple. For the second of (21), N_{p_e} maps the $\pm i\lambda_{p_e}$ generalized eigenspace of $dX_{H_{\xi_e}}(p_e)$ to itself, since $dX_{H_{\xi_e}}(p_e)$ does that. Thus the image of N_{p_e} is contained in the intersection of that generalized eigenspace with $\mathfrak{g}_{\mu_e} p_e$, which is exactly $\mathfrak{g}_{\mu_e}^1 p_e$. From (21) one gets (22) by using the duality (10) between $N_{p_e}^{21}$ and $N_{p_e}^{13}$.

With respect to (20), and in view of (5), (21) and (22), the linear map N_{p_e} has the form

$$N_{p_e} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_{p_e}^{131} & 0 \\ 0 & N_{p_e}^{211} & 0 & N_{p_e}^{23} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $N_{p_e}^{211} \equiv N_{p_e}^{21}|_{W_{\text{red}}^1}$ and where $N_{p_e}^{131}$ is $N_{p_e}^{13}$ regarded as a map into W_{red}^1 . Let the quadratic Hamiltonian for the infinitesimally linear map N_{p_e} be H_{nil} . With respect to (20), the array for H_{nil} is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_{p_e}^{131} & 0 \\ 0 & N_{p_e}^{211} & 0 & N_{p_e}^{23} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} \omega_{\mu_e}^0(\bar{p}_e) & 0 & 0 & 0 & 0 \\ 0 & \omega_{\mu_e}^1(\bar{p}_e) & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id} & 0 \\ 0 & 0 & -\text{Id} & 0 & 0 \\ 0 & 0 & 0 & 0 & \check{\omega}_{\mu_e}(p_e) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (N_{p_e}^{211})^* & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & (N_{p_e}^{131})^* \omega_{\mu_e}^1(\bar{p}_e) & 0 & N_{p_e}^{23} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\omega_{\mu_e}^i(\bar{p}_e)$ denotes the symplectic form on W_{red}^i , $i = 1, 2$. Using (13), and the fact that the projections from the splitting $\mathfrak{g}_{\mu_e}^* = \mathfrak{g}_{\mu_e}^{0*} \oplus \mathfrak{g}_{\mu_e}^{1*}$ are certain polynomials in coad_{ξ_e} , the bilinear form $N_{p_e}^{23}$ block diagonalizes over that

splitting. Denote the blocks by $N_{p_e}^{230}$ and $N_{p_e}^{231}$. Then, using the variables

$$(x, \mu) = (x, \mu^0, \mu^1) \in W_{\text{red}}^1 \times \mathfrak{g}_{\mu_e}^* = W_{\text{red}}^1 \times \mathfrak{g}_{\mu_e}^{0*} \times \mathfrak{g}_{\mu_e}^{1*},$$

we have

$$(24) \quad \begin{aligned} H_{\text{nil}} &= \frac{1}{2} N_{p_e}^{23}(\mu, \mu) + \langle \mu, N_{p_e}^{211} x \rangle \\ &= \frac{1}{2} N_{p_e}^{230}(\mu^0, \mu^0) + \frac{1}{2} N_{p_e}^{231}(\mu^1, \mu^1) + \langle \mu^1, N_{p_e}^{211} x \rangle. \end{aligned}$$

I now construct the normal form symmetry for H_{nil} . Let T_{ξ_e} be the closure of $\exp(\mathbb{R}\xi_e)$, so T_{ξ_e} is Abelian, is generically a maximal torus of G_{μ_e} , and T_{ξ_e} naturally acts on both \mathfrak{g}_{μ_e} and $\mathfrak{g}_{\mu_e}^*$, with invariant subspaces $\mathfrak{g}_{\mu_e}^i$ and $\mathfrak{g}_{\mu_e}^{i*}$, $i = 1, 2$. By (13), $N_{p_e}^{23}$ is T_{ξ_e} -invariant, and so (24) is invariant under T_{ξ_e} restricted to the μ^0 variables alone. After this, it is the last two terms of (24) that will dictate its further symmetries. Now S^1 also acts² on $\mathfrak{g}_{\mu_e}^1$ by

$$\exp(\theta^\wedge)\eta \equiv \exp\left(\frac{\theta}{\lambda_{p_e}}\xi_e\right)\eta, \quad \theta \in \mathbb{R},$$

which is merely a renormalization of the action of T_{ξ_e} on $\mathfrak{g}_{\mu_e}^1$, and by duality there is a corresponding action of S^1 on $\mathfrak{g}_{\mu_e}^{1*}$. Also there is the following symplectic action of S^1 on W_{red}^1 :

$$(25) \quad \exp(\theta^\wedge)x \equiv \exp\left(\frac{\theta}{\lambda_{p_e}}dX_{H_{\mu_e}}(\bar{p}_e)\right)x, \quad \theta \in \mathbb{R}.$$

By (11) and (12), $N_{p_e}^{131}$ and $N_{p_e}^{211}$ reverse-intertwine the two S^1 actions, so that

$$\begin{aligned} \langle \exp(\theta^\wedge)^{-1}\mu^1, N_{p_e}^{211} \exp(\theta^\wedge)x \rangle &= \langle \exp(\theta^\wedge)^{-1}\mu^1, \exp(\theta^\wedge)^{-1}N_{p_e}^{211}x \rangle \\ &= \langle \mu^1, x \rangle. \end{aligned}$$

Consequently, the *normal form symmetry* will be $T_{\xi_e} \times S^1$ where the first factor acts on the μ^0 variables via the coadjoint action and the second factor acts by

$$a \cdot (x, \mu^1) \equiv (a^{-1}x, a\mu), \quad a = \exp(\theta^\wedge).$$

The normal form symmetry restricts the quadratic Hamiltonian H_{nil} , as follows. There is a basis of the linear space $W_{\text{red}}^1 = \{(x_1, x_2) \in \mathbb{R}^2\}$ so that the symplectic form is $dx^1 \wedge dx^2$ and the S^1 action on W_{red}^1 is either $(\exp(\theta^\wedge), x) \mapsto \exp(\theta^\wedge)x$ or $(\exp(\theta^\wedge), x) \mapsto \exp(-\theta^\wedge)x$ (i.e. counter-clockwise or clockwise rotation). Using the Ad-invariant metric of \mathfrak{g} , choose

² Define $\theta^\wedge = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that $S^1 = SO(2)$ and \exp is the usual matrix exponential.

a orthonormal basis for the space $\mathfrak{g}_{\mu_e}^1$, so that $\mathfrak{g}_{\mu_e}^1 = \{(\pi_1, \pi_2) \in \mathbb{R}^2\}$ and the S^1 action on $\mathfrak{g}_{\mu_e}^1$ is a rotation in the opposite sense as the S^1 action on W_{red} . The map $N_{p_e}^{211}$ is highly restricted: since it reverse-intertwines these two reverse S^1 actions, there is a κ such that

$$(26) \quad N_{p_e}^{211} = \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad N_{p_e}^{211} = \kappa \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Equation (26) defines κ , so that the map N^{211} has the effect of giving rise to this single ‘‘coupling constant’’ and one of the two possible choices in (26). Without loss of generality, by the symplectic isomorphisms $(x_1, x_2) \mapsto (-x_1, -x_2)$ and $(x_1, x_2) \mapsto (-x_2, x_1)$ on W_{red}^1 , $\kappa \geq 0$ and the second choice in (26) may be discarded, respectively. Then, since $N_{p_e}^{23}$ is S^1 -symmetric, a constant I_1 may be obtained from $N_{p_e}^{231}$ such that

$$(27) \quad H_{\text{nil}} = \frac{1}{2} N_{p_e}^{230}(\mu^0, \mu^0) + \frac{1}{2} I_1(\pi_1^2 + \pi_2^2) + \kappa(\pi_1 x_1 + \pi_2 x_2).$$

According to the above conventions, N^{211} reverse intertwines reverse-sense S^1 actions on the (x_1, x_2) variables and the (π_1, π_2) variables, and the second action has the same sense as the action of ad_{ξ_e} . Consequently, *there are two distinct possibilities*: the S^1 action on the (x_1, x_2) variables can be counterclockwise or clockwise. To avoid unnecessary signs that would otherwise appear later, I will redefine the S^1 actions in the following way: if the S^1 action on the (x_1, x_2) variables is clockwise then reverse it, and both actions are then counterclockwise and the S^1 action on the (π_1, π_2) variables is in the same sense as the action of ad_{ξ_e} . If on the other hand the S^1 action on the (x_1, x_2) variables is counterclockwise then the action of the (π_1, π_2) variables is clockwise, and is to be reversed, so that it has the opposite sense to the action of ad_{ξ_e} . Thus, by these conventions, both actions are counterclockwise, N^{211} is intertwining, and the distinctness of the two cases appears as one of two possibilities: the S^1 action on the (π_1, π_2) variables might be in the same sense as the action of ad_{ξ_e} (I call this the ‘‘+’’ case, and it occurs when the original action on the (x_1, x_2) variables is clockwise) or the opposite sense as the action of ad_{ξ_e} (the ‘‘-’’ case, occurring when the original action on the (x_1, x_2) variables is counterclockwise). Although it may appear that same-sense vs. opposite-sense with respect to ad_{ξ_e} holds some significance, similar to, for example, the difference between a $1 : 1$ and $1 : -1$ resonance as in Kummer [1976, 1978], this is not so: changing the the symplectic form $dx_1 \wedge dx_2$ to $dx_2 \wedge dx_1$ will exchange my ‘‘+’’ and ‘‘-’’ cases. These two cases are qualitatively identical and the subsequent theory is ambidextrous with respect to them.

Now I will determine the effect on (27) of choosing various subspaces W_{red} , or equivalently, various operators A commuting with S_{p_e} in (15). With

respect to the decompositions (18) and (19) let A have the block form

$$A = \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}.$$

To determine which A commute with S_{p_e} , note that S_{p_e} has a block diagonal form on $W_{\text{red}}^0 \oplus W_{\text{red}}^1 \oplus \mathfrak{g}_{\mu_e}^0 \oplus \mathfrak{g}_{\mu_e}^1$ since (18) and (19) are generalized eigenspace decompositions for S_{p_e} ; let the blocks be S^{110} , S^{111} , S^{220} , and S^{221} , so $AS = SA$ becomes

$$(28) \quad \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix} \begin{bmatrix} S^{110} & 0 \\ 0 & S^{111} \end{bmatrix} = \begin{bmatrix} S^{220} & 0 \\ 0 & S^{221} \end{bmatrix} \begin{bmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{bmatrix}.$$

When multiplied, the top left blocks of (28) give $A^{11}S^{110} = S^{220}A^{11}$, which is possible only if A^{11} is zero, since S^{110} and S^{220} have no common spectrum, and similarly $A^{12} = 0$ and $A^{21} = 0$. However, S^{111} and S^{221} both have spectrum $\{\pm\lambda_e\}$, so one cannot conclude $A^{22} = 0$, but only that $A^{22}S^{111} = S^{221}A^{22}$. Since S^{111} is exactly the linearization $dX_{H_{\mu_e}}(\bar{p}_e)$ restricted to W_{red}^1 , which by (25) defined the S^1 action on W_{red}^1 , which is counterclockwise or clockwise, we have

$$S^{111} = \pm\sqrt{\lambda_e} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since S^{221} is $-\text{ad}_{\xi_e}$ on $\mathfrak{g}_{\mu_e}^1$, and ad_{ξ_e} generates the S^1 action of opposite sense to that generated by S^{111} , we have

$$S^{221} = \pm\sqrt{\lambda_e} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

By letting

$$A^{22} = \begin{bmatrix} a^{11} & a^{12} \\ a^{21} & a^{22} \end{bmatrix},$$

$A^{22}S^{111} = S^{221}A^{22}$ becomes $a^{12} = -a^{21}$ and $a^{11} = a^{22}$. By (26) and the duality (10),

$$N_{p_e}^{13} = \begin{bmatrix} 0 & 0 \\ 0 & \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \end{bmatrix},$$

so that

$$(N_{p_e}^{13})^* A^* + AN_{p_e}^{13} = \begin{bmatrix} 0 & 0 \\ 0 & -\kappa \begin{bmatrix} 2a^{12} & 0 \\ 0 & 2a^{12} \end{bmatrix} \end{bmatrix},$$

so by (17), the Hamiltonian generated by the new choice of W_{red} has the same form as (27) but with I_1 replaced by $I_1 + 2\kappa a^{12}$. Thus, *the entire effect of the freedom to choose W_{red} is exactly that the constant I_1 may be arbitrarily manipulated, as long as $\kappa \neq 0$. The effective action of the gauge group as it acts on the normal form is isomorphic to the additive group of real numbers.*

3.2 The normal form

The splitting of $T_{p_e}P$ and subsequent analysis of the linearization $dX_{H_{\xi_e}}(p_e)$ amounts to the construction of a normal form for linearizations of relative equilibria. When equivariantly moved through $G_{\mu_e}p_e$, this normal form can be combined with the equivariant isotropic embedding theorem (Marsden [1981], Weinstein [1977]), to provide a G_{μ_e} -equivariant normal form for the entire Hamiltonian system near $G_{\mu_e}p_e$. I now construct this normal form, closely following Patrick [1995].

Let \tilde{P} be the symplectic manifold

$$\tilde{P} \equiv W_{\text{red}}^0 \times W_{\text{red}}^1 \times T^*G_{\mu_e} \times G_{\mu_e},$$

and define

$$\tilde{p}_e \equiv 0 \oplus 0 \oplus 0_{\text{Id}} \oplus \mu_e.$$

Let the group G_{μ_e} act on \tilde{P} by left translation on the third factor. At the zero section of $T^*G_{\mu_e}$, there is the splitting of the tangent space of $T^*G_{\mu_e}$ into vertical and horizontal parts, and hence

$$(29) \quad T_{\tilde{p}_e}\tilde{P} = W_{\text{red}}^0 \oplus W_{\text{red}}^1 \oplus \mathfrak{g}_{\mu_e} \oplus \mathfrak{g}_{\mu_e}^* \oplus T_{\mu_e}(G_{\mu_e}).$$

Together, (20) and (29) yield a linear symplectomorphism between $T_{p_e}P$ and $T_{\tilde{p}_e}\tilde{P}$, which extends by G_{μ_e} -equivariance to a vector bundle isomorphism, say $\Lambda' : TP|(G_{\mu_e}p_e) \rightarrow T\tilde{P}|(G_{\mu_e}\tilde{p}_e)$, which is symplectic on each fiber. The equivariant isotropic embedding theorem then extends Λ' to an equivariant symplectomorphism Λ from a neighborhood of $G_{\mu_e}p_e$ to a neighborhood of $G_{\mu_e}\tilde{p}_e$. The relationship between Λ' and Λ is $\Lambda' = T\Lambda$ on the domain of Λ' .

The linearization $dX_{H_{\xi_e}}(p_e)$ is in a sense a first order approximation of $X_{H_{\xi_e}}$ at p_e , and the quadratic Hamiltonian associated to $dX_{H_{\xi_e}}(p_e)$ is a second order approximation to H_{ξ_e} at p_e . Together with the map Λ , this yields a second order approximation to the Hamiltonian \tilde{H} in a neighborhood of $G_{\mu_e}\tilde{p}_e$. Indeed, let \tilde{J}^ξ be the momentum mapping for the left action of right translations $(g, h) \mapsto hg^{-1}$, so that

$$\tilde{J}^\xi(\alpha_g) \equiv -\langle \alpha_g, X_\xi(g) \rangle,$$

where X_ξ is the left invariant vector field generated by ξ . Then

$$(30) \quad \tilde{H} = H_{\text{nil}} - (\tilde{J}^{\xi_e} - H_{\text{red}}^1) + H_{\text{red}}^0 + H_{\text{rsd}} + H_{\text{rmd}},$$

where

1. $H_{\text{nil}} : W_{\text{red}}^1 \times T^*G_{\mu_e} \rightarrow \mathbb{R}$ is the extension of (24) to $T^*G_{\mu_e}$ by left invariance,
2. $H_{\text{red}}^i : W_{\text{red}}^i \rightarrow \mathbb{R}$ is $d^2H_{\mu_e}(\tilde{p}_e)|W_{\text{red}}^i, i = 1, 2,$
3. $H_{\text{rsd}} : G_{\mu_e} \rightarrow \mathbb{R}$ by $H_{\text{rsd}}(g\mu_e) \equiv -\langle g\mu_e, \xi_e \rangle,$ and
4. $dH_{\text{rmd}} = 0$ and $d^2H_{\text{rmd}} = 0$ on $G_{\mu_e}\tilde{p}_e.$

Items (1)–(3) are definitions, while the content is in item (4), which follows since the two Hamiltonians

$$\tilde{H}, \quad H_{\text{nil}} - (\tilde{J}^{\xi_e} - H_{\text{red}}^1) + H_{\text{red}}^0 + H_{\text{rsd}}$$

both have relative equilibria at \tilde{p}_e and have been manipulated to have matched linearizations (and consequently matched Hessians) at each point of $G_{\mu_e}\tilde{p}_e.$ The point is that a second order approximation to \tilde{H} may be obtained by dropping the remainder term H_{rmd} of (30).

Now, consider \tilde{H} as defined by (30). The second term (the one grouped within brackets) generates the one parameter group of symplectomorphisms generated by ξ_e through the action (31). Therefore, the first term H_{nil} and the second term Poisson commute. Obviously, then, the first four terms of (30) pairwise Poisson commute. The effect of the term $-\tilde{J}^{\xi_e}$ is merely to generate the flow of the right invariant vector field generated by $\xi_e,$ and this corresponds to the “fast” evolution of the relative equilibrium $\tilde{p}_e,$ while the flows of the terms H_{red}^0 and H_{rsd} cannot cause evolution along $G_{\mu_e}\tilde{p}_e.$ So the slow evolution near $G_{\mu_e}\tilde{p}_e$ is to first order dictated by the flow of $H_{\text{nil}}.$

As for the symmetries of $\tilde{H},$ since the Hamiltonian H is G invariant and A' is G_{μ_e} -equivariant, the Hamiltonian (30) is G_{μ_e} -invariant, including the remainder term. Also, when H_{nil} is extended to be a left invariant function of $W_{\text{red}}^1 \times T^*G_{\mu_e},$ invariance under the normal form action given by (16) becomes right invariance, i.e., for the S^1 part, invariance under the action

$$(31) \quad a \cdot (x, \alpha_g) = (a^{-1}x, (TR_a)^*\alpha_g),$$

where R denotes right translation of $G_{\mu_e},$ and the functions H_{red}^0 and H_{rsd} are also invariant under (31). However this normal form symmetry does not in general extend to the remainder term $H_{\text{rmd}}.$ The action (31) is counterclockwise or clockwise diagonal rotation on the variables (π_1, π_2) and (x_1, x_2) in a way that is a sense-preserving renormalization of the coadjoint action of $\exp(\mathbb{R}\xi_e)$ on $(\pi_1, \pi_2).$ Since (31) is the action of the Abelian $S^1,$

this action can be reversed and thus assumed to be counterclockwise; however, if this is done then the resulting action may acquire the *opposite* sense to the coadjoint action of $\exp(\mathbb{R}\xi_e)$ on (π_1, π_2) .

In concept the map Λ is like a coordinate system near $G_{\mu_e}p_e$; it is only that its range is a manifold rather than an open subset of Euclidean space. Through its dependence on the splitting of $T_{p_e}P$, Λ depends on the choice of W_{red} , so the effect of varying W_{red} like varying a coordinate system near $G_{\mu_e}p_e$. Thus, the freedom to vary W_{red} is like a gauge freedom. I will call a particular choice of W_{red} a *choice of gauge*, and say that, in view of Sect. (3.1), the gauge group of p_e is $(\mathbb{R}, +)$. As shown in Sect. (3.1), varying W_{red} itself is a linear process, the effect of which on H_{nil} is easily found and described. Yet, through the isotropic embedding theorem, Λ depends on W_{red} in a nontrivial way.

This ability to cause very nontrivial “coordinate changes” while easily calculating the concomitant effect on H_{nil} will enter powerfully in the subsequent analysis. Particularly, one can imagine choosing a gauge that “simplifies” H_{nil} , making the analysis of H_{nil} tractable and yielding information about the flow of the original Hamiltonian system near $G_{\mu_e}p_e$. On the other hand, a simple H_{nil} will in general be nongeneric, so that higher order terms in H_{rmd} might destroy whatever delicate nongeneric structures arise from H_{nil} . In that case, one can focus on the identification and perturbation of these nongeneric structures, which will occupy smaller regions of phase space as ones attention is restricted more nearly to $G_{\mu_e}\bar{p}_e$.

Summary: The phase space P_{drift} on which some Hamiltonian H_{drift} approximates to first order the evolution of the system, when that system is perturbed from p_e , is

$$P_{\text{drift}} = \mathbb{R}^2 \times T^*G_{\mu_e},$$

$$H_{\text{drift}}(x, \alpha_g) = \langle g^{-1}\alpha_g, \xi_e \rangle + \frac{1}{2}N_{p_e}^{23}(\alpha_g, \alpha_g) + \langle g^{-1}\alpha_g, N_{p_e}^{211}x \rangle.$$

The equations of motion for this system³ are

$$g^{-1} \frac{dg}{dt} = \Omega, \quad \frac{d\pi}{dt} = \text{coad}_\Omega \pi, \quad \frac{dx}{dt} = N_{p_e}^{131} \pi.$$

where

$$\Omega \equiv \xi_e + (N^{23})^b \pi + N_{p_e}^{211} x.$$

³ For a Hamiltonian system on the cotangent bundle of a Lie group G of the form $\alpha_g \mapsto 1/2h(\alpha_g, \alpha_g) + \langle \alpha_g, X \rangle$ where h is a left invariant metric on G and X is a left invariant vector field on G , one has the equations of motion

$$\Omega \equiv h^\sharp \pi + X(e), \quad \pi \equiv g^{-1} \alpha_g, \quad g^{-1} \frac{dg}{dt} = \Omega, \quad \frac{d\pi}{dt} = -\text{coad}_\Omega \pi.$$

There is a splitting $\mathfrak{g}_{\mu_e} = \{(\pi_1, \pi_2) \in \mathbb{R}^2\} \oplus \{\mu^0\}$ such that H_{drift} becomes, after Poisson reduction by left translation of G_{μ_e} ,

$$(32) \quad H_{\text{drift}} = \langle \mu^0, \xi_e \rangle + \frac{1}{2} N_{p_e}^{230}(\mu^0, \mu^0) + \frac{1}{2} I_1(\pi_1^2 + \pi_2^2) + \kappa(\pi_1 x_1 + \pi_2 x_2).$$

Here N^{230} is a constant quadratic form, and I_1 and κ are constants, and all of these are calculable from the nilpotent part of the linearization of the relative equilibrium. The Hamiltonian (32) is invariant under counterclockwise diagonal rotation on the variables (π_1, π_2) and (x_1, x_2) and the coadjoint action of T_{ξ_e} on the variables μ^0 . The gauge group of the approximation is $(\mathbb{R}, +)$, and the effect of the gauge freedom is to exactly to undetermine I_1 .

4 The mechanical case

Here I consider the special case $G = SO(3) \times (S^1)^n$. Standard identifications give $\mathfrak{g} = \mathbb{R}^3 \times \mathbb{R}^n = \mathfrak{g}^*$. In order that the momentum be nongeneric, assume that the $SO(3)$ part of the momentum is zero. Reorient the system (i.e. left translate it) so that $\mathfrak{g}_{\mu_e}^1 = \{(\pi_1, \pi_2, 0)\} \subseteq \mathfrak{so}(3)$. Since the action of ad_{ξ_e} fixes $\mathfrak{g}_{\mu_e}^1$, the $SO(3)$ part of ξ_e is either parallel or antiparallel to to $\mathbf{k} \in \mathfrak{so}(3)$, where $\mathbf{k} = (0, 0, 1)$. I drop the first term of (32), since the background motion of the relative equilibrium p_e itself will not be of interest, after which (32) becomes

$$H_{\text{drift}} = \frac{1}{2} I_1(\pi_1^2 + \pi_2^2) + \frac{1}{2} I_2 \pi_3^2 + \kappa(\pi_1 x_1 + \pi_2 x_2) + \pi_3 \sum_{i=1}^n a_i p_i + \sum_{i,j=1}^n a_{ij} p_i p_j,$$

where the variable labeling is

$$(\pi_1, \pi_2, \pi_3, p_1, \dots, p_n) \in \mathfrak{g}^* = \mathbb{R}^3 \times \mathbb{R}^n,$$

and a_i and a_{ij} are constants.

The whole of $(S^1)^n$ is ignorable; an Abelian reduction yields the phase space and Hamiltonian (which by abuse of notation will have the same names)

$$(33) \quad P_{\text{drift}} = \mathbb{R}^2 \times T^*SO(3) = \{((x_1, x_2) \in \mathbb{R}^2, A \in SO(3), (\pi_1, \pi_2, \pi_3) \in \mathbb{R}^3)\} \\ H_{\text{drift}} = \frac{1}{2} I_1(\pi_1^2 + \pi_2^2) + \frac{1}{2} I_2 \pi_3^2 + \kappa(\pi_1 x_1 + \pi_2 x_2) + a \pi_3,$$

where a is constant. The normal form symmetry is counterclockwise diagonal action of S^1 on the variables $(x_1, x_2), (\pi_1, \pi_2)$, and the momentum mapping for the normal form symmetry is

$$J^{nf} = -\frac{1}{2}(x_1^2 + x_2^2) - \pi_3$$

An easy verification (see the footnote on page 765) gives the equations of motion for (33) as

$$(34) \quad \frac{d}{dt} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I\pi_2\pi_3 - \kappa x_2\pi_3 + a\pi_2 \\ -I\pi_1\pi_3 + \kappa x_1\pi_3 - a\pi_1 \\ \kappa(\pi_1x_2 - \pi_2x_1) \\ \kappa\pi_2 \\ -\kappa\pi_1 \end{bmatrix},$$

and

$$(35) \quad A^{-1} \frac{dA}{dt} = \begin{bmatrix} I_1\pi_1 + \kappa x_1 \\ I_1\pi_2 + \kappa x_2 \\ I_2\pi_3 + a \end{bmatrix}^\wedge,$$

where $I \equiv I_2 - I_1$ and as usual, for $v \in \mathbb{R}^3$,

$$v^\wedge \equiv \begin{bmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{bmatrix}.$$

To obtain a first impression of (33) one can contrast it with the nonresonant case. On the one hand, the phase space P_{drift} of (33) has dimension 8; the symmetry group is $SO(3) \times S^1$. Hence, the Marsden-Weinstein reduced systems are integrable, since they have dimension $8 - 4 - 2 = 2$, and energy is a conserved quantity. We shall see that homoclinic connections appear in the reduced spaces of (33), which are spheres in general, and that singular reduced spaces (pinched spheres) also occur for some momentum values. On the other hand, the *nonresonant* case (obtained from (33) by setting $\kappa = 0$ and dropping the variables x_1 and x_2) has the same symmetry group but dimension 6 phase space; it is integrable by its symmetry group alone, without the use of energy. Dynamically, the nonresonant case is simpler, since every motion is a relative equilibrium and the compact symmetry group rules out homoclinic connections.

4.1 Full reduction

The Poisson manifold $\mathbb{R}^2 \times \mathbb{R}^3 = \{(x, \pi)\}$ may be reduced by the S^1 normal form symmetry by use of the Hopf variables (see, for example, Cushman and Bates [1997], page 14; also page 407 has a summary of references about singular reduction):

$$\begin{aligned} w_1 &\equiv 2(x_1\pi_2 - x_2\pi_1), & w_2 &\equiv 2(x_1\pi_1 + x_2\pi_2), \\ w_3 &\equiv (x_1^2 + x_2^2) - (\pi_1^2 + \pi_2^2), & w_4 &\equiv (x_1^2 + x_2^2) + (\pi_1^2 + \pi_2^2). \end{aligned}$$

This map is a quotient map for the S^1 normal form symmetry and it has the semialgebraic image given by the subset of $\mathbb{R}^5 = \{(w_1, w_2, w_3, w_4, \pi_3)\}$ satisfying

$$w_1^2 + w_2^2 + w_3^2 - w_4^2 = 0, \quad w_4 \geq 0.$$

The symplectic reduced spaces are the level sets of the two Casimirs j_1 and j_2 given by

$$(36) \quad j_1 \equiv (\pi_1^2 + \pi_2^2 + \pi_3^2)^{\frac{1}{2}} = \left(\pi_3^2 + \frac{1}{2}(w_4 - w_3) \right)^{\frac{1}{2}},$$

$$(37) \quad j_2 \equiv J^{nf} = -\frac{1}{2}(x_1^2 + x_2^2) - \pi_3 = -\frac{1}{4}(w_3 + w_4) - \pi_3.$$

Given j_1 and j_2 , (36) and (37) can be used to eliminate w_3 and w_4 , thereby obtaining the symplectic reduced spaces as the subsets of $\mathbb{R}^3 = \{(w_1, w_2, \pi_3)\}$ given by

$$(38) \quad w_1^2 + w_2^2 = 8(\pi_3^2 - j_1^2)(\pi_3 + j_2)$$

$$(39) \quad w_4 = (j_1^2 - 2j_2) - (\pi_3^2 + 2\pi_3) \geq 0.$$

Various symplectic reduced spaces are obtained by fixing various values of j_1 and j_2 , but the values of j_1 and j_2 are not arbitrary, since $j_1 \geq 0$ and

$$j_1 = (\pi_1^2 + \pi_2^2 + \pi_3^2)^{\frac{1}{2}} \geq -\pi_3 = j_2 + \frac{1}{2}(x_1^2 + x_2^2) \geq j_2.$$

Moreover, the singularities in the symplectic reduced spaces occur where $x_1 = x_2 = \pi_1 = \pi_2 = 0$, and here $j_1 = |\pi_3| = \pm j_2$. Equation (38) has solutions only over the intervals where the cubic on the right hand side is nonnegative, and moreover, by (37), $\pi_3 \geq -j_2$. Putting all this together gives the bifurcation diagram shown in Fig. (1). As shown there, the symplectic reduced spaces are the surfaces of revolution of $(8(\pi_3^2 - j_1^2)(\pi_3 + j_2))^{\frac{1}{2}}$ over the *finite* interval where the cubic is positive. Thus, these spaces are points, spheres, or, in the case that $j_2 = -j_1$, they are topological spheres with one conical singularity.

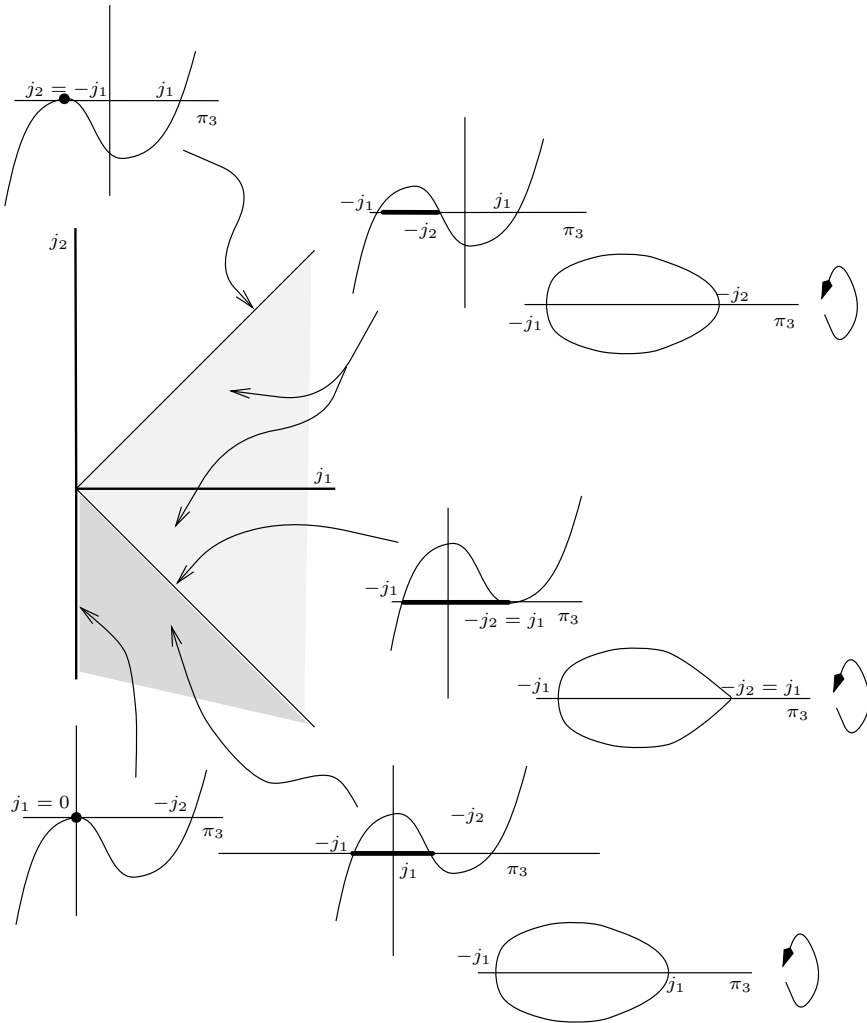


Fig. 1 The bifurcation diagram for the symplectic reduced spaces. Clockwise from the top, the reduced spaces are points, spheres, pinched spheres, again spheres and finally points. The finite intervals where the cubic is positive is the thicker black line on the π_3 axis. One verifies that Inequality (39) is respected by showing that the concave down quadratic $w_4 = -(\pi_3^2 + 2\pi_3) - j_1^2 + 2j_2$ is positive on those intervals by checking that it is positive at the various endpoints of the intervals.

In passing, I note that the symplectic volumes of the $-j_2 = j_1 > 0$ reduced phase spaces are infinite, and of course the infinity is concentrated at the singularity. In fact, by regarding two functions $f(w_1, w_2)$ and $g(w_1, w_2)$ as functions on one of those reduced phase spaces, lifting those functions to the Poisson phase space $\{(x, \pi)\} = \mathbb{R}^2 \times so(3)$, and then calculating the

Poisson bracket, one verifies that

$$\{f, g\} = -4(\pi_3^2 + 2j_2\pi_3 + j_1^2) \left(\frac{\partial f}{\partial w_2} \frac{\partial g}{\partial w_1} - \frac{\partial f}{\partial w_1} \frac{\partial g}{\partial w_2} \right).$$

Consequently, the symplectic form on the reduced spaces in the coordinates (w_1, w_2) is

$$\omega_{j_1, j_2} = \frac{1}{4(\pi_3^2 + 2j_2\pi_3 + j_1^2)} dw_1 \wedge dw_2.$$

Switching to polar coordinates $w_1 = r \cos \theta$ and $w_2 = r \sin \theta$, the symplectic volume of a small circle of radius δ about the origin is

$$A_\delta \equiv 2\pi \int_0^\delta \frac{r}{4(\pi_3^2 + 2j_2\pi_3 + j_1^2)} dr = \frac{\pi}{2} \int_0^\delta \frac{r}{(\pi_3 - j_1)^2} dr$$

and upon substituting

$$r^2 = 8(\pi_3^2 - j_1^2)(\pi_3 - j_1) = 8(\pi_3 + j_1)(\pi_3 - j_1)^2$$

that symplectic volume becomes

$$A_\delta = 4\pi \int_0^a \frac{\pi_3 + j_1}{r} dr.$$

Since the numerator tends to $2j_1$ as r becomes small, A_δ is infinite.

The Hamiltonian (33) on the reduced spaces is, with the aid of the Hopf variables and (36),

$$\begin{aligned} H_{\text{drift}} &= \frac{1}{4}I_1(w_4 - w_3) + \frac{1}{2}I_2\pi_3^2 + \frac{\kappa}{2}w_2 + a\pi_3 \\ &= \frac{1}{2}I\pi_3^2 + \frac{\kappa}{2}w_2 + a\pi_3 + \frac{1}{2}I_1j_1^2 \end{aligned}$$

Obviously this expression is simplified if $I_1 = I_2$, since then $I = I_2 - I_1 = 0$. This choice of I_1 gives the first part of (33) the same form as the kinetic energy of a spherical ball moving in 3-space, so I call this choice the *spherical gauge*.

In the spherical gauge the flow lines of the reduced systems are trivial to determine: they are the intersection of the planes $(\kappa/2)w_2 + a\pi_3 = \text{constant}$ with the surfaces of revolution in Fig. (4.1). In the case of a nonsingular reduced space of nonzero dimension the flow is that of the flow on a 2-sphere with two stable equilibrium points. For the singular reduced spaces $-j_2 = j_1 > 0$ the type of flow depends on whether or not the plane cuts

through the cone at the $\pi_3 = j_1$ singularity of $w_2^2 = 8(\pi_3 + j_1)(\pi_3 - j_1)^2$. That cone is

$$|w_2| = 4\sqrt{j_1}(j_1 - \pi_3) \iff \pi_3 = j_1 - \frac{1}{4\sqrt{j_1}}|w_2|,$$

and so there is an unstable equilibrium at the singularity if the line $(\kappa/2)w_2 + a\pi_3 = aj_1$ passes through that cone, which is when $a^2 < 4\kappa^2 j_1$, and a stable equilibrium otherwise. The former case is the flow on a 2-sphere with two stable equilibrium points and a single homoclinic connection to a singular point, while in the latter case this flow has two stable equilibrium points, one of which resides at a singularity.

For the remainder of this section I will assume the spherical gauge.

The equilibria of (34) correspond to relative equilibria for just the action of $SO(3)$, since the space $\{(\pi, x)\}$ is the Poisson phase space of P_{drift} reduced by the $SO(3)$ action. These have a greater significance than relative equilibria which also use the S^1 normal form symmetry, since the full unapproximated Hamiltonian has the $SO(3)$ symmetry but it is only the truncated approximation which has the $SO(3)$ symmetry *and* the normal form symmetry. As well, it is expedient to separate the equilibria of (34) at the outset of the analysis. Setting the right side of (34) to zero immediately gives $\pi_1 = \pi_2 = 0$ and then $x_1\pi_3 = x_2\pi_3 = 0$, so there are the following solutions:

$$(40) \quad x = 0, \quad \pi = 0$$

$$(41) \quad x = 0, \quad \pi_1 = \pi_2 = 0, \quad \pi_3 \neq 0$$

$$(42) \quad x \neq 0, \quad \pi = 0$$

The set of relative equilibria given by (42) contains relative equilibria equivalent under the S^1 normal form symmetry. The $SO(3)$ generator corresponding to (42) is, by substitution into (35), $(\kappa x_1, \kappa x_2, 0)$. The zero section of $T^*SO(3)$ (i.e. the set $\pi = 0$) corresponds to perturbations of the original relative equilibrium p_e such that the *perturbation* has zero total angular momentum, and in the nonresonant case such perturbations would imply no drift and no drift has been numerically observed for such. For the resonant case, however, one expects regular rotation around the axis $(\kappa x_1, \kappa x_2, 0)$, which is perpendicular to the generator ξ_e . Nonzero x corresponds to some “reduced excitation”. Consequently, *at zero total angular momentum the system drifts so that the generator of the relative equilibrium moves along a fixed great circle at a rate dictated by a reduced excitation.*

The relative equilibria (42) are the relative equilibria with nongeneric momenta *for the system* (33). Consequently, the methods of Patrick [1995] might be applied to them: the drifting relative equilibria could themselves drift. I will not, however, pursue this aspect here.

To obtain a list of nonequivalent relative equilibria (i.e. a list of relative equilibria, no two of which are in the same $SO(3) \times S^1$ orbit), assume $A = \text{Id}$ and discard Equation (35), and, using the S^1 normal form symmetry, assume $\pi_2 = 0$ and $\pi_1 \geq 0$, and when $\pi_1 = 0$ assume that $x_2 = 0$. Equating the first of (34) with the S^1 infinitesimal generator of $s_e \in \mathbb{R}$ gives

$$(43) \quad \begin{bmatrix} -\kappa x_2 \pi_3 + a \pi_2 \\ \kappa x_1 \pi_3 - a \pi_1 \\ \kappa(\pi_1 x_2 - \pi_2 x_1) \\ \kappa \pi_2 \\ -\kappa \pi_1 \end{bmatrix} = \begin{bmatrix} -s_e \pi_2 \\ s_e \pi_1 \\ 0 \\ -s_e x_2 \\ s_e x_1 \end{bmatrix}.$$

Putting $\pi_2 = 0$ into the fourth component gives $s_e x_2 = 0$. Now $s_e = 0$ corresponds to the $SO(3)$ relative equilibria (40)–(42), and in (40) and (41), we have $x_2 = 0$, while in (42) we can assume $x_2 = 0$ and $x_1 > 0$ by the normal form symmetry. So if $s_e = 0$ then we assume $x_2 = 0$, while if $s_e \neq 0$ then $x_2 = 0$ anyway by $s_e x_2 = 0$. Putting $x_2 = \pi_2 = 0$ into (43) then gives

$$-\kappa \pi_1 = s_e x_1, \quad \kappa x_1 \pi_3 - a \pi_1 = s_e \pi_1,$$

and these equations are easily solved to obtain the following list of nonequivalent relative equilibria:

$$(44) \quad \pi_1 = \pi_2 = x_1 = x_2 = 0, \eta_e = (a + s_e)\mathbf{k}, \pi_3 \in \mathbb{R};$$

$$(45) \quad \pi_2 = x_2 = 0, \pi_3 = \frac{\pi_1(a x_1 - \kappa \pi_1)}{\kappa x_1^2}, s_e = -\frac{\kappa \pi_1}{x_1},$$

$$\eta_e = \kappa x_1 \mathbf{i} + (a + s_e)\mathbf{k}, \pi_1 \in \mathbb{R}, x_1 > 0.$$

Here the $SO(3)$ generators η_e of these relative equilibria have been determined by comparison of (35) and the infinitesimal generator of the $SO(3)$ action of P_{drift} at the relative equilibrium. In (44) η_e and s_e are not unique due to the presence of isotropy in the phase space P_{drift} . For later use, the characteristic polynomials of the linearizations of these relative equilibria are, respectively,

$$p_1 \equiv x^2(x^2 + |\eta_e|^2) \times (x^4 + (a^2 + 2as_e + 2s_e^2 - 2\pi_3 \kappa^2)x^2 + (s_e^2 + as_e + \pi_3 \kappa^2)^2)$$

and

$$p_2 \equiv \frac{1}{x_1^2} x^4 (x^2 + |\eta_e|^2) (x_1^2 x^2 + \kappa^2 x_1^4 + (ax_1 - 2\kappa \pi_1)^2).$$

For an understanding of the relative equilibria and the flow in general it is important to understand which relative equilibria reside on which reduced

spaces. For (44), $j_1 = |\pi_3|$ and $j_2 = \pi_3$, so for $\pi_3 \leq 0$ one has $j_1 = j_2$, which corresponds to pointlike reduced spaces, while for $\pi_3 > 0$ one has $j_1 = -j_2$, which corresponds to the singular reduced spaces, and the relative equilibria occupy the singularities. As for (45), substituting (45) into j_1 and j_2 and eliminating π_1 gives

$$(46) \quad (3x_1^4 + 8j_2x_1^2 + 4j_2^2 - 4j_1^2)^2\kappa^2 + 4a^2x_1^2(x_1^2 + 2j_2 - 2j_1)(x_1^2 + 2j_2 + 2j_1) = 0$$

so it is a matter of solving this quartic in x_1^2 for its positive roots. The pointlike reduced spaces corresponding to $j_1 = 0$ are occupied by the relative equilibria of (45) after putting $\pi_1 = 0$, corresponding to the solutions $j_1 = 0, x_1 = \sqrt{-2j_2}$ of (46).

4.2 Partial reduction by the normal form symmetry

In this section I symplectically reduce the Hamiltonian system by the normal form symmetry at a value, say σ of the momentum mapping J^{nf} . My objective is an interpretation of the reduced system via the equations of motion on the reduced space.

Use the notation $r^2 = x_1^2 + x_2^2$. The σ -level of the momentum map J^{nf} is

$$\pi_3 = -\sigma - \frac{1}{2}r^2.$$

Below, when x appears in the context of a vector in \mathbb{R}^3 , it is as $(x_1, x_2, 0)$. The map

$$(47) \quad \Psi(A, \pi, x) \equiv (y, \alpha, z) \equiv (A\mathbf{k}, A(\pi - (\pi \cdot \mathbf{k})\mathbf{k}), Ax)$$

is a projection of the phase space $T^*SO(3) \times \mathbb{R}^2$ to the Whitney direct sum $T^*S^2 \oplus TS^2$. The restriction of this map to the σ level set of J^{nf} is clearly a quotient map for the normal form action, and thus the symplectic reduced space is $T^*S^2 \oplus TS^2$. The action of $SO(3)$ on $T^*S^2 \oplus TS^2$ becomes

$$A \cdot (y, \alpha, z) = (Ay, A\alpha, Az)$$

To find the reduced vector field at $(y, \alpha, z) \in T^*S^2 \oplus TS^2$, first evaluate the original Hamiltonian vector field, namely

$$(48) \quad \begin{aligned} \Omega &\equiv I_1\pi + I((\pi \cdot \mathbf{k}) + a)\mathbf{k}, & A^{-1}\frac{dg}{dt} &= \Omega^\wedge, \\ \frac{d\pi}{dt} &= \dot{\pi} = \pi \times \Omega + a\mathbf{k}, & \frac{dx}{dt} &= \dot{x} = -\kappa\mathbf{k} \times \pi \end{aligned}$$

at the point

$$(49) \quad \pi = A^{-1}\alpha - \left(\sigma + \frac{1}{2}r^2\right) \mathbf{k}, \quad x = A^{-1}z,$$

where A is chosen so that $A\mathbf{k} = y$, and then apply the derivative of (47), which is

$$(50) \quad \begin{aligned} & T\Psi(A, \pi, x, \Omega, \dot{\pi}, \dot{x}) \\ & \equiv \frac{d}{dt} \Big|_{t=0} \Psi(A \exp(\Omega \wedge t), \pi + t\dot{\pi}, x + t\dot{x}) \\ & = A \cdot (\Omega \times \mathbf{k}, \Omega \times (\pi - (\pi \cdot \mathbf{k})\mathbf{k}) + \dot{\pi} - (\dot{\pi} \cdot \mathbf{k})\mathbf{k}, \dot{x} + \Omega \times x). \end{aligned}$$

Thus it is a matter of substituting (48) and (49) into (50). Without care the calculation can be onerous; however, it is straightforward, and yields the equations of motion

$$(51) \quad \frac{dy}{dt} = (I_1\alpha + \kappa z) \times y,$$

$$(52) \quad \frac{d\alpha}{dt} = -\pi_3(I_1\alpha + \kappa z) \times y + \kappa z \times \alpha,$$

$$(53) \quad \frac{dz}{dt} = (\kappa\alpha - (I_2\pi_3 + a)z) \times y + I_1\alpha \times z,$$

where π_3 stands for $-(\sigma + |z|^2/2)$. To these equations must be added the constraints $|y| = 1, \alpha \cdot y = 0$ and $z \cdot y = 0$.

I want to impose the viewpoint that there is a ‘‘particle’’ at y having direction z . To do this, I use (51) to replace α with dy/dt in (52) and (53) while writing equation (52) as a second order equation in y . I also use the standard Levi-Cevita connection of S^2 for the time derivatives. Thus, Equation (52) becomes

$$\begin{aligned} \frac{d^2y}{dt^2} &= (I_1\alpha + \kappa z) \times \frac{dy}{dt} + \left(I_1 \frac{d\alpha}{dt} + \kappa \frac{dz}{dt} \right) \times y \\ &= (I_1\alpha + \kappa z) \times ((I_1\alpha + \kappa z) \times y) \\ &\quad + I_1\pi_3(I_1\alpha + \kappa z) - \kappa(\kappa\alpha - (I_2\pi_3 + a)z) \\ &= - \left| \frac{dy}{dt} \right|^2 y + I_1\pi_3 y \times \frac{dy}{dt} \\ &\quad - \kappa \left(\frac{\kappa}{I_1} \left(y \times \frac{dy}{dt} - \kappa z \right) - (I_2\pi_3 + a)z \right), \end{aligned}$$

and similarly with Equation (53), so that

$$(54) \quad \frac{\nabla^2 y}{dt^2} = \frac{I_1^2\pi_3 - \kappa^2}{I_1} y \times \frac{dy}{dt} + \kappa \left(\frac{\kappa^2}{I_1} + I_2\pi_3 + a \right) z$$

$$(55) \quad \frac{\nabla z}{dt} = \frac{\kappa}{I_1} \frac{dy}{dt} + \left(\frac{\kappa^2}{I_1} + I_2\pi_3 + a \right) y \times z.$$

Some aspects of the particle become apparent by replacing α with dy/dt in the total energy and angular momentum. Using (49), the total energy is

$$\begin{aligned}
 H &= \frac{1}{2}|\alpha|^2 + \frac{1}{2}\pi_3^2 + a\pi_3 + \kappa z \cdot \alpha \\
 (56) \quad &= \frac{1}{2I_1}|I_1\alpha + \kappa z| + \frac{1}{2}\pi_3^2 + a\pi_3 - \frac{\kappa^2}{2I_1}|z|^2 \\
 &= \frac{1}{2I_1}\left|\frac{dy}{dt}\right|^2 + \frac{1}{2I_2}\left(\frac{\kappa^2}{I_1} + I_2\pi_3 + a\right)^2 + \frac{\kappa^2\sigma}{I_1},
 \end{aligned}$$

so the internal energy of the particle is

$$E_{\text{int}} = \frac{1}{2I_2}\left(\frac{\kappa^2}{I_1} + I_2\pi_3 + a\right)^2.$$

The total angular momentum, or the $SO(3)$ -momentum-mapping is, using (49),

$$(57) \quad J = A\pi = \alpha - \pi_3\mathbf{k} = \frac{1}{I_1}y \times \frac{dy}{dt} - \frac{\kappa}{I_1}z - \pi_3y,$$

and below I replace z with the tangent to the sphere part of the angular momentum, which by (57) is

$$L \equiv -\frac{\kappa}{I_1}z,$$

while the total angular momentum attributed to the particle must be $L + \pi_3y$. By the first terms of both (56) and (57), the particle should be viewed as having mass $1/I_1$.

From Jackson [1975] the classical nonrelativistic equations of motion on $|y| = 1$ of a particle of mass m , charge Q , and gyromagnetic ratio Γ , and magnetic moment $\mathbf{m} = \Gamma L$, are

$$(58) \quad m\frac{\nabla^2 y}{dt^2} = \frac{Q}{c}B \times \mathbf{v} + \nabla(\mathbf{m} \cdot B) - (\nabla \cdot B)\mathbf{m},$$

$$(59) \quad \frac{\nabla L}{dt} = \mathcal{I} + \Gamma L \times B,$$

where c is the velocity of light and \mathbf{v} is the particle's velocity. Here \mathcal{I} is compensates for the inductive forces that would, were \mathbf{m} to be constant, be required to maintain the constant current that generates \mathbf{m} itself. One can fit equations (54) and (55) to these equations as follows. If one takes $B = y$, comparison of the first terms of (55) and (59) gives the inductive term

$$(60) \quad \mathcal{I} = -\frac{\kappa^2}{I_1^2}\frac{dy}{dt},$$

while the second term of (55) matches the second term of (59) if the gyro-magnetic ratio is

$$(61) \quad \Gamma = \frac{I_1^2}{\kappa^2} \left(\frac{\kappa^2}{I_1} + I_2\pi_3 + a \right).$$

Matching the second term of (54) with the third term of (58) implies Γ is $\kappa^2/3I_1$ times the left side of (61). So choose a gauge so that $I_1 = \kappa^2/3$. Then matching the first term of (54) with the first term of (58) yields

$$\frac{Q}{c} = \pi_3 - \frac{\kappa^2}{I_1^2}.$$

Finally, I note that the inductive term (60) is exactly what is required to balance the rate of change of internal energy with the work done on the particle through its magnetic moment via the last term of (54).

5 Example: Two coupled rods

In this section I illustrate and verify the above theory using numerical simulations of the system of two axially symmetric rods which are joined by a frictionless ball-and-socket joint. In the center of mass frame this system can be cast as a geodesic flow with configuration space $SO(3)^2 = \{(A_1, A_2)\}$ and kinetic energy metric

$$L = \frac{1}{2} [\Omega_1^t \ \Omega_2^t] J(A) \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix},$$

where $A = [A^{ij}] = A_1^t A_2$, Ω_1 and Ω_2 are body referenced (left translation) angular velocities, and

$$J(A) = \begin{bmatrix} 1 & 0 & 0 & -\beta A^{22} & \beta A^{21} & 0 \\ 0 & 1 & 0 & \beta A^{12} & -\beta A^{11} & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ -\beta A^{22} & \beta A^{12} & 0 & 1 & 0 & 0 \\ \beta A^{21} & -\beta A^{11} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix}.$$

Here $0 < \alpha < 2 - 2\beta$ is a parameter which increases with the diameter of the rods and $0 \leq \beta < 1$ is a parameter which measures the degree of coupling between the rods; β is zero if the joint lies at the mutual centers of mass of the rods. The continuous symmetry group of this system is $SO(3) \times (S^1)^2$, which acts on the configuration space $SO(3)^2$ by

$$(B, \theta_1, \theta_2) \cdot (A_1, A_2) = (BA_1 \exp(-\theta_1 \mathbf{k}^\wedge), BA_2 \exp(-\theta_2 \mathbf{k}^\wedge)).$$

All relative equilibria for this system are explicitly known, as are their formal stability; a complete list can be found in Patrick [1991]. The relative equilibria of interest here are the phase space points $(\text{Id}, \exp(\theta \mathbf{j}^\wedge), \Omega_1, \Omega_2)$ parameterized by $t_1, t_2, \theta \in \mathbb{R}$ such that $t_1 \neq 0, t_2 \neq 0, 0 < \theta < \pi$, where

$$\Omega_1 = t_1 \mathbf{i} - \kappa_1^{t_1 t_2 \theta} \mathbf{k}, \Omega_2 = t_2 \mathbf{i} + \kappa_1^{t_2 t_1 \theta} \mathbf{k}$$

and where

$$\kappa_\gamma^{t_1 t_2 \theta} \equiv \frac{(t_1 \cos \theta - t_2)(\beta t_2 - \gamma t_1)}{\alpha t_1 \sin \theta}.$$

The corresponding generators $(\Omega, \sigma_1, \sigma_2) \in \mathfrak{so}(3) \times \mathbb{R}^2$ are given by

$$\sigma_1 = \kappa_{1-\alpha}^{t_1 t_2 \theta}, \sigma_2 = -\kappa_{1-\alpha}^{t_2 t_1 \theta}, \Omega = t_1 \mathbf{i} + \frac{t_1 \cos \theta - t_2}{\sin \theta} \mathbf{k}.$$

By calculating the linearizations, one sees that a 1:1 group reduced resonance with zero total-angular-momentum can be arranged by setting

$$\beta = \frac{2t_1 t_2}{t_1^2 + t_2^2}, \quad 3t_1^4 + t_1^2 t_2^2 (4 \cos^2 \theta - 10) + 3t_2^4 = 0,$$

and for the work presented in this section I have chosen to perturb the particular relative equilibrium obtained by setting

$$(62) \quad \theta = \frac{\pi}{3}, t_1 = 1, t_2 = \frac{1 + \sqrt{5}}{2}, \alpha = \frac{1}{2}, \beta = \frac{2}{\sqrt{5}}.$$

5.1 Calculations for the coupled rod system

To compare the dynamics of the drift system to the dynamics of the coupled rod system near the relative equilibrium (62), it is necessary to calculate the splitting (1) or (2) in this special case. Here is one general way, inspired by the proof of Theorem 3.1.19 of Abraham and Marsden [1978]:

- (a) Calculate the semisimple part S_{p_e} and the nilpotent part N_{p_e} of the linearization $dX_{H_{\xi_e}}(p_e)$.
- (b) Since S_{p_e} is semisimple it has a basis of eigenvectors, some complex and some real, and the complex eigenvectors may be grouped into complex conjugate pairs. Taking the real and imaginary parts of one eigenvector in each pair, and then including the real eigenvectors yields a basis \mathcal{B} of $T_{p_e} P$ in which S_{p_e} is *skew symmetric*.

- (c) Let the matrix of $\omega(p_e)$ with respect to the basis \mathcal{B} be $-W$. Since S is infinitesimally symplectic and skew, $0 = S^t W + W S = -S W + W S$, so S and W commute. By going to a basis of eigenvectors of the positive matrix $-W^2$, find a symmetric, positive square-root B of $-W^2$ such that B and S commute, and set $J = W B^{-1}$. Using the basis \mathcal{B} , regard J as its corresponding operator on $T_{p_e} P$, and regard B as the bilinear form on $T_{p_e} P$ corresponding to $(x, y) \mapsto x^t B y$. Then

$$\omega(p_e)(v, w) = B(Jv, w), \quad J^2 = -\text{Id},$$

and J commutes with S_{p_e} .

- (d) Set $Z = J(\mathfrak{g}_{\mu_e})$, so Z is a S_{p_e} invariant Lagrangian complement to \mathfrak{g}_{μ_e} in $\mathfrak{g}_{\mu_e} \oplus Z$. Choose an Ad-invariant complement \mathfrak{b} to \mathfrak{g}_{μ_e} , and set $W_{\text{red}} = (\mathfrak{g}_{\mu_e} p_e \oplus Z \oplus \mathfrak{b} p_e)^{\omega^\perp}$. Then W_{red} is symplectic, S_{p_e} invariant and is contained in $\ker dJ(p_e)$, since $W_{\text{red}} \subseteq \mathfrak{g} p_e^{\omega^\perp}$. The subspace W_{red} complements \mathfrak{g}_{μ_e} in $\ker dJ(p_e)$ since $\mathfrak{g}_{\mu_e} \subseteq W_{\text{red}}^{\omega^\perp}$ and W_{red} is symplectic.

In the case of two coupled rods, further refinement of the splitting of W_{red} to resonant and nonresonant parts is not necessary, since the reduced spaces for zero total-angular-momentum are two dimensional; there is no “nonresonant” part of the phase space. Final adjustments to achieve the spherical gauge are easily arranged using (15) and (17). The end result of all this is a basis of the tangent space at the relative equilibrium which reflects the splitting (2), a basis of the Lie algebra $so(3) \times \mathbb{R}^3$, reflecting the splitting (19) into resonant and nonresonant parts, the information

$$(63) \quad \kappa = .9115064, \quad I_1 = I_2 = 4.321619,$$

and the information that (62) is a “+” type relative equilibrium.

Further to calculating the splitting (1) or (2), means must be found to translate initial conditions of the drift system to perturbations (i.e. initial conditions near (62)) of the coupled rod system. For this a map was used, say Ψ , of the tangent space $T_{p_e} P$ into phase space P such that $T_{p_e} \Psi$ is the identity at p_e . The map Ψ only injected initial conditions of the drift system into perturbations of (62) to first order; actual correspondence could not be easily achieved due to the implicit nature of the coordinates provided by the isotropic embedding theorem. Moreover, Ψ did not match the momentum of the drift system exactly to a momentum perturbation of the actual system, while practice indicated that an exact match of these momenta is important. For the comparisons just below, given an initial condition of the drift system with a particular momentum, an initial condition of the coupled rod system with matching momentum was obtained as Ψ of another nearby initial condition of the drift system. That nearby initial condition of the drift system

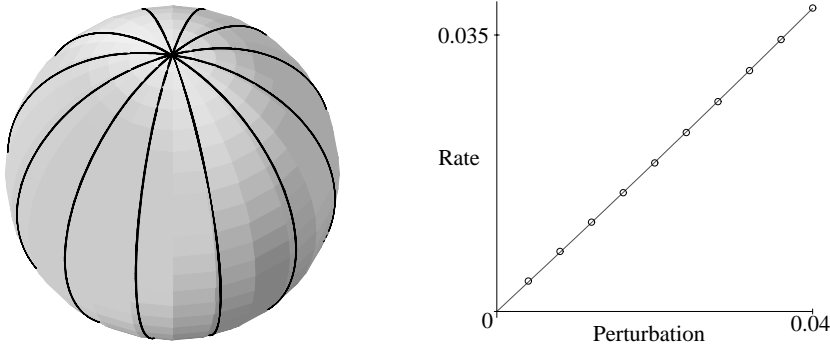


Fig. 2 Perturbations within zero-total-angular momentum. *Left*: the paths traced out by $\bar{\tau}$, resulting from 6 initial conditions within zero total-angular-momentum. *Right*: the rotation rates of τ vs. the magnitude of the perturbation $|x|$.

was obtained by a slight (second order) iterative refinement of the original initial condition of the drift system.

By the conventions of Sect. 4, and since the relative equilibrium (62) is of “+” type, the vector $(\pi_1, \pi_2, \pi_3) = \mathbf{k}$ is parallel to the normalized rotation vector ξ_e of the coupled rod system. Also, the quantity $A \in SO(3)$ of the drift system corresponds to the drifted orientation of the coupled rod system. Thus, the quantity $A\mathbf{k}$ of the drift system corresponds to the drifting rotation vector of the coupled rod system. Now, the rotation vector of the relative equilibrium (62) is a constant linear sum of the locations of the rods, exactly because the rods when in the relative equilibrium rotate around that vector: if one defines

$$\tau \equiv \frac{1}{\sin \theta} (t_1 A_1 \mathbf{k} - t_2 A_2 \mathbf{k})$$

then $\tau = \xi_e$ at the relative equilibrium (62) and also at any reorientation of (62). Thus, for a perturbation of (62), τ approximates the drifting rotation vector and the prediction of the preceding theory is that

$$\bar{\tau} \equiv \frac{\tau}{|\tau|} \approx A\mathbf{k}.$$

The numerical verifications I have undertaken consist of predictions of the drift system for the motion of $A\mathbf{k}$ and the comparison of these predictions with evolution of the the unit vector $\bar{\tau}$ calculated via a symplectic integration of the coupled rod system. The particular algorithm used was a (implicit) Riemannian leapfrog algorithm (Leimkuhler and Patrick [1996]).

5.2 Numerical results

5.2.1 *First comparison: zero total-angular-momentum* By the results in Sect. (4.1), the part of the drift phase space corresponding to perturbations with zero total-angular-momentum is occupied by $SO(3)$ relative equilibria, and the motion of $\mathbf{A}k$ from initial condition $\pi = 0$ and x arbitrary is that of uniform rotation at angular frequency $\kappa|x|$ along a great circle through \mathbf{k} and perpendicular to $(x_1, x_2, 0)$. Translated to the coupled rod system, the prediction is that $\bar{\tau}$ undergoes regular rotation in a great circle through $\bar{\tau}(0)$, the particular great circle regularly rotating as x is rotated. The left of Fig. (2) shows the motion of τ as x is varied from 0 to 0 through $5\pi/6$ radians by increments of $\pi/6$, so the coupled rod system conforms to this prediction. As the magnitude of x is varied the great circles should be traced out at an angular rotation rate $\kappa|x|$, so κ can be determined by plotting that rotation rate against $|x|$; this is done on the right of Fig. (2). The rotation rates fit well to the curve

$$\text{Rate} = .9284|x| + .7750|x|^2,$$

so from the simulation κ is .9284, which is 1.8% off the calculated value of .9115064 already displayed in (63).

5.2.2 *Second comparison: a stable relative equilibrium of the drift system*

For another comparison, I examined the coupled rod system for perturbations of (62) corresponding to being near one of the stable relative equilibria in the list (45). For perturbations of (62) corresponding to the values

$$(64) \quad \pi_1 = .001, \quad x_1 = .04$$

in (45) the predictions of the drift system are that τ moves as small periodic oscillation of the full reduction of the drift system superimposed on the periodic motion $\exp(\eta_e \wedge t)\mathbf{k}$, where η_e is the $SO(3)$ generator in the list (45). The two predicted frequencies are the linearized frequency of the corresponding equilibrium on the reduced space of the drift system and the rotation frequency of η_e , which are by substitution of (64) into (45) and/or (46), respectively,

$$(65) \quad .04477, \quad .05837.$$

Again by substitution of (64) into (45), the motion of τ should be such that its projection onto the unit vector $\eta_e/|\eta_e|$ is near $-.5300$.

Figure (3) shows the results of a simulation of the coupled rod system for the perturbation of (62) corresponding to the values (64). In the simulation the normalized rotation vector $\bar{\tau}$ moved on the unit sphere as shown in the top left of the Figure; the sense of the rotation is clockwise as seen from

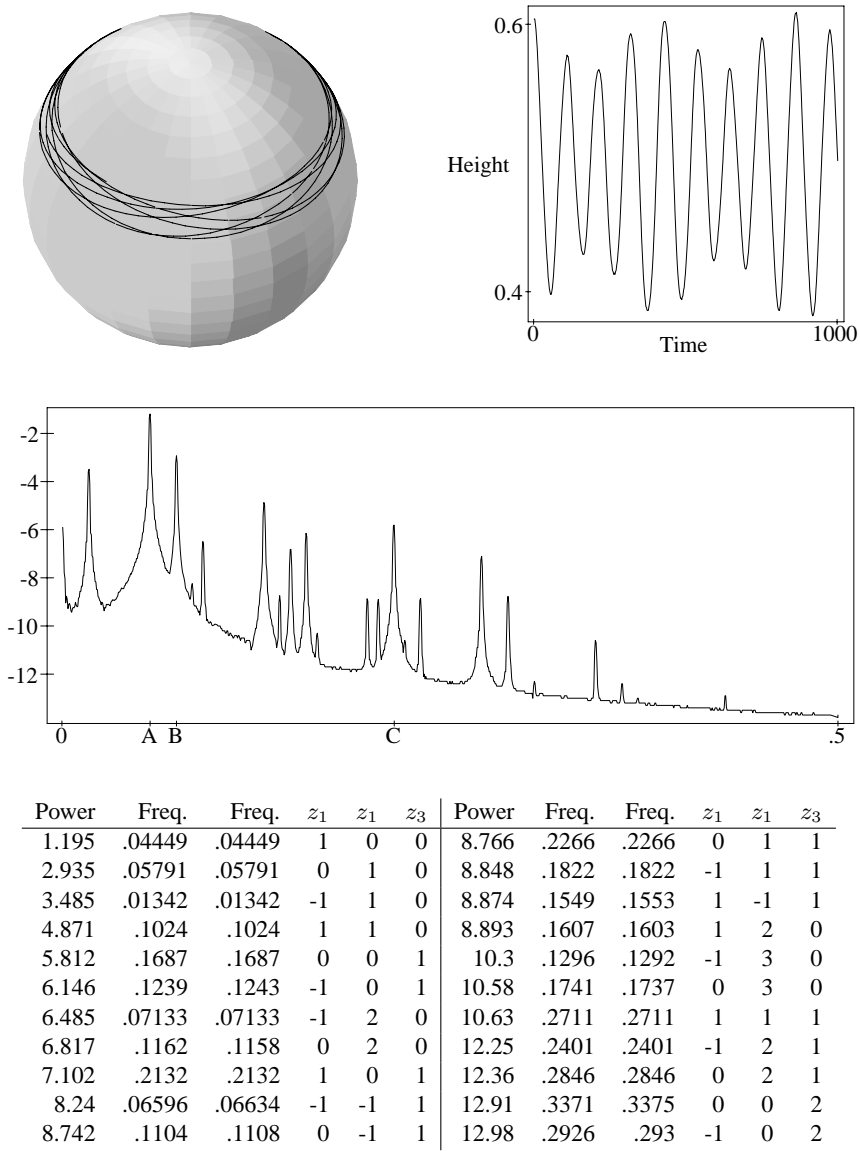


Fig. 3 Some results of the simulation of the coupled rod system for a perturbation corresponding to a elliptic relative equilibrium of the drift system. *Top left*: the evolution of the rotation vector projected onto the unit sphere. *Top right*: the time evolution of the height corresponding to the top left. *Middle*: the power spectrum of a signal derived from the drifting motion; note the vertical logarithmic scale. *Bottom table*: the peaks in the power spectrum, sorted by power. The second column tabulates the frequencies of the peaks and the third column tabulates the harmonics $z_1A + Z_2B + Z_3C$ of the frequencies A, B and C indicated on the frequency axis of the power spectrum.

the top, so that η_e is pointing down and away from you as you look upon the Figure. The height of the rotation vector has been graphed in the top right of Fig. (3), and is visibly quasiperiodic with about two frequencies. The measured average height of the motion was .4970, compared with the prediction of .5300. In the middle of Fig. (3) is the power spectrum of one of the horizontal components of τ . As is shown in the table immediately below the power spectrum, the power spectrum is consistent with the 3 fundamental angular frequencies

$$(66) \quad .04449, \quad .05791, \quad .1687,$$

all other peaks being harmonics of these. The first two of these are by far the largest peaks (note the vertical logarithmic scale in the power spectrum) and they agree with the the predicted frequencies (65). The third observed frequency in (66) corresponds to a far less prominent peak and is likely a frequency associated to the higher order terms that have been truncated in the drift approximation.

To give an idea of the speed of the drift, on the short simulation corresponding to the top left and right of Fig. (3), the system rotated about 300 times while $\bar{\tau}$ rotated about 7 times. Thus the drift was about 50 times slower than the original relative equilibrium. Due to this, these kinds of simulations can be long time: over the duration of the simulation that generated the power spectrum in Fig. (2) the rods rotated about 10,000 times.

5.2.3 Third comparison: a singularity-induced phase jump As shown in Sect. (4.2), the singularities of the $-j_1 = j_2$ reduced phase spaces of the drift system are occupied by an unstable equilibria corresponding to $SO(3)$ relative equilibria of the drift system. A single homoclinic orbit, tracing out the intersection of each reduced space with the plane $w_2 = 0$, emanates from every such equilibrium. These homoclinic orbits are interesting features of the drift system, in part because they provide an opportunity to investigate dynamics near singularities of reduced spaces. I will show here that, due to the singularity of the reduced space, the drift system suffers a jump in its reconstruction phase as initial conditions traverse the homoclinic orbits, and (numerically) that this feature of the drift system persists to the coupled rod system. The reconstruction phase jump shows the presence of Hamiltonian monodromy in the completely integrable drift system; for more information on Hamiltonian monodromy see Cushman and Bates [1997], page 175, the summary on page 403 of the same reference, as well as Bates [1991].

So choose j_1 small and consider perturbations of the relative equilibrium

$$\pi = j_1 \mathbf{k}, \quad x = 0, \quad A = \text{Id},$$

where by abuse of notation j_1 serves as the constant value of the Casimir (36) of the same name. This relative equilibrium corresponds to the equilibrium

$$(67) \quad w_1 = w_2 = 0, \quad \pi_3 = j_1$$

on the $SO(3) \times S^1$ reduced space. Near (67) the motion on the reduced space of the drift system is periodic along the intersection of the $w_2 = h$ plane where h is small: for a long time the system remains near the singularity, and then moves off, passes near $\pi_3 = j_1$ and then returns to the singularity, similar to, for example, the motion of an inverted pendulum. One S^1 reconstruction gives the motion on the phase space $\{(x, \pi)\}$ and, since the motion is periodic on the $SO(3) \times S^1$ reduced space, there is a well defined S^1 reconstruction phase. On the phase space $\{(\pi, x)\}$, the only points on the level sets of j_1 and j_2 that map to (67) are

$$(68) \quad \pi = j_1 \mathbf{k}, \quad x = 0,$$

since $\pi_3 = j_1 = -j_2$ implies by (36) and (37) that $\pi_1 = \pi_2 = x_1 = x_2 = 0$. Thus, when the $SO(3) \times S^1$ reduced system is near (67), the reduced system with phase space $\{(x, \pi)\}$ is near (68). An $SO(3)$ reconstruction gives the motion on the phase space $\mathbb{R} \times T^*SO(3) = \{(x, A, \pi)\}$, starting, say, at $A = \text{Id}$. By conservation of the momentum $J = A\pi$, the variable A is nearly a rotation about \mathbf{k} whenever the reduced system is near (68). Thus, the motion of the point $A\mathbf{k}$ on the unit sphere is this: *for a long time $A\mathbf{k}$ remains near \mathbf{k} , then moves off, passes near $-\mathbf{k}$, and then returns near to \mathbf{k} . This motion repeats but rotated (with respect to the previous excursion from \mathbf{k}) by a excursion-independent angle about \mathbf{k} . This rotation is what I mean by the $SO(3)$ reconstruction phase shift of the motion.*

By simulation of the coupled rod system I have verified that the gross details of the motion of $A\mathbf{k}$ in the drift system also occur in the coupled rod system (see the left of Fig. (4)). This situation is robust; no particular care is required, in choosing the coupled rod system's initial conditions to evoke these kinds of motions. The robust nature under addition of higher order terms of the homoclinic orbit is expected through averaging theorems such as the one in Guckenheimer and Holmes [1983], page 168.

It is the behavior of the reconstruction phase of the drift system near to $\pi = 0$ that is relevant for predictions of the motion of the coupled rod system near to its resonant relative equilibrium. One way to examine this behavior is to numerically integrate initial conditions of the drift system that are reconstructions of initial conditions starting at a small circle surrounding the singularity of the $SO(3) \times S^1$ reduced space. For the S^1 reconstruction phase shift the result is a constant phase shift of $\pm\pi/2$ with a jump of π as the homoclinic orbit is traversed. The corresponding phase shift of $A\mathbf{k}$ in fact becomes undefined as the perturbation vanishes. The reason for this

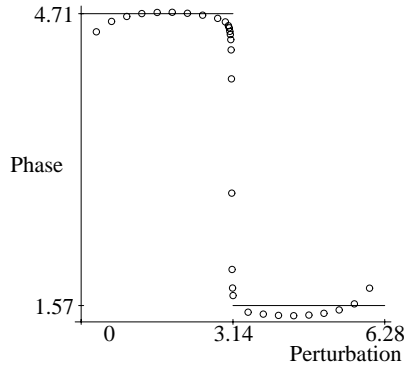
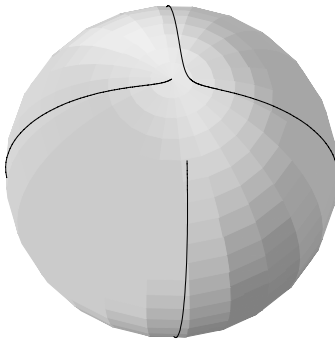


Fig. 4 *Left*: the motion of $\bar{\tau}$ for a perturbation corresponding to being near the homoclinic orbit of the reduced space of the drift system. The vector $\bar{\tau}$ begins near the top of the sphere and moves off to the left, comes around the sphere, makes an approximate $\pi/2$ turn, then moves away from you, and goes around the sphere again. *On the right*: the $SO(3)$ reconstruction phase of the coupled rod system as initial conditions are rotated around the singularity of the reduced phase space of the drift system.

is that the equation of motion for A , namely Equation (35), becomes, near to (68), the equation

$$\frac{dA}{dt} = (I_2 j_1 + a) \mathbf{k}^\wedge,$$

while $A(t)\mathbf{k}$ is not exactly \mathbf{k} due to the presence of the perturbation itself. Consequently the $A\mathbf{k}$ picks up a rotation of angular frequency $I_2 j_1 + a$ during its long visit of (68). As the perturbation vanishes this long visit becomes an eternity and this undefines the $SO(3)$ reconstruction phase shift as an asymptotic effect. However, viewed from a frame that corotates with the same angular frequency, namely $I_2 j_1 + a$, one can expect a well defined phase shift. I have numerically verified this: from the corotating frame the $SO(3)$ reconstruction phase shift of the drift system is nearly $\pm\pi/2$ with a jump of π as the homoclinic orbit is traversed. This compares favorably with the $SO(3)$ reconstruction phase shift of the coupled rod system shown in right of Fig. (4).

Finally, so that the comparison of these phases of the drift system and the coupled rod system are not entirely numerical, I give a calculation showing that, asymptotically as the homoclinic orbit is approached, these phases arise mostly from the singularity of the reduced space. The S^1 reconstruction phase of the drift system may be calculated by a slight modification (to allow reparameterization) of the usual reconstruction method found in Abraham and Marsden [1978]: Generally, suppose H is a Hamiltonian on a symplectic phase space P and that a curve $c_\mu(s)$ is a reparameterization an evolution $c_\mu(t)$ on some Marsden-Weinstien reduced phase space $P_\mu = J^{-1}(\mu)/G_\mu$.

Choose a curve $d(s) \in J^{-1}(\mu)$ such that $c = \pi_\mu \circ d$, where $\pi_\mu : J^{-1}(\mu) \rightarrow P_\mu$ is the quotient projection. Then there is a unique smooth curve $\xi(s) \in \mathfrak{g}_\mu$ and a unique smooth function $a(s)$ such that

$$(69) \quad X_H(d(s)) = \xi(s)p(s) + a(s)d'(s).$$

By differentiation, the curve $g(s(t))d(s(t))$ satisfies Hamilton’s equations on P if

$$g^{-1} \frac{dg}{ds} = \frac{1}{a(s)} \xi(s), \quad \frac{dt}{ds} = \frac{1}{a(s)}.$$

Perfectly obvious generalizations hold if d is just a 1-manifold in $J^{-1}(\mu)$ covering the image of c_μ . Particularly, by setting $\pi_2 = 0$, d can be the subset of P_{drift} defined by

$$(70) \quad \pi_1^2 + \pi_2^2 = j_1^2, \quad \pi_2 = 0, \quad x_1 = \frac{h}{2\pi_1}, \quad x_2 = \frac{-w_1}{2\pi_1},$$

$$(71) \quad w_1^2 + h^2 = 8(\pi_3^2 - j_1^2)(\pi_3 - j_1),$$

in which case one calculates using (34), (69), (70) and (71) that

$$(72) \quad \frac{d\theta}{d\pi_3} = \frac{-h\pi_3}{(j_1^2 - \pi_3^2)w_1}, \quad \frac{d\pi_3}{dt} = -\frac{\kappa w_1}{2}$$

where θ is the S^1 reconstruction phase as the counterclockwise angle of the vector $(\pi_1(t), \pi_2(t))$ from the $\pi_2 = 0$ axis.

Let $r_1(h) < r_2(h) < r_3(h)$ be the roots of the cubic $8(\pi_3 - j_1)^2(\pi_3 + j_1) - h^2$. By standard perturbation arguments

$$\begin{aligned} r_1 &= -j_1 + \frac{1}{32j_1^2}h^2 + O(h^3), \\ r_2 &= j_1 - \frac{1}{4\sqrt{j_1}}h + O(h^2), \\ r_3 &= j_1 + \frac{1}{4\sqrt{j_1}}h + O(h^2). \end{aligned}$$

The S^1 reconstruction phase over the curve $w_1^2 + h^2 = 8(\pi_3^2 - j_1^2)(\pi_3 - j_1)$ is calculated as follows. Let the reduced system start at $w_1 = 0, \pi_3 = r_2$ at time $t = t_0$, move to $w_1 = 0, \pi_3 = r_1$ at time $t = t_1$ and then complete its periodic orbit by moving back to $w_1 = 0, \pi_3 = r_2$ at time t_2 . Then the contribution of (72) to phase shift over the interval $[t_0, t_1]$ is

$$\phi(t_1) - \phi(t_0) = \int_{t_0}^{t_1} \frac{d\phi}{dt} dt = \int_{\pi_3(t_0)}^{\pi_3(t_1)} \frac{d\phi}{dt} \frac{dt}{d\pi_3} d\pi_3 = \int_{r_2}^{r_1} \frac{d\phi}{d\pi_3} d\pi_3,$$

where the positive square root must be used when solving for w_1 in (71), since π_3 must immediately decrease after time $t = t_0$, and by the second of (72), w_1 is positive over the interval $[t_0, t_1]$. The contribution of (72) over the interval $[t_1, t_2]$ is identical (use the negative root here). Thus, the total phase shift $\phi \equiv \phi(t_2) - \phi(t_0)$ over the loop as the loop approaches the homoclinic orbit through $h > 0$ is

$$(73) \quad \phi = -2 \lim_{h \rightarrow 0^+} \int_{r_1}^{r_2} \frac{-h\pi_3}{(j_1^2 - \pi_3^2)\sqrt{8(\pi_3 - r_1)(r_2 - \pi_3)(r_3 - \pi_3)}} d\pi_3.$$

For small h the integrand of (73) is small away from its two singularities at $\pi_3 = r_1$ and $\pi_3 = r_2$. I begin with the left singularity at $\pi_3 = r_1$, so I calculate

$$\phi_1 \equiv \lim_{h \rightarrow 0^+} \int_{r_1}^0 \frac{-h\pi_3}{(j_1^2 - \pi_3^2)\sqrt{(\pi_3 - r_1)(r_2 - \pi_3)(r_3 - \pi_3)}} d\pi_3.$$

Elementary estimates show that zero error is made as $h \rightarrow 0$ by the replacement of

$$\frac{\pi_3}{(j_1 - \pi_3)\sqrt{(\pi_3 - r_2)(\pi_3 - r_3)}}$$

with its limit as $h \rightarrow 0$ of its evaluation at $\pi_3 = r_1$, which is

$$\lim_{h \rightarrow 0^+} \frac{r_1}{(j_1 - r_1)\sqrt{(r_2 - r_1)(r_3 - r_1)}} = \frac{-j_1}{2j_1(\sqrt{2j_1})^2} = \frac{-1}{4j_1}.$$

Thus

$$\begin{aligned} \phi_1 &= \lim_{h \rightarrow 0^+} \frac{-h}{4j_1\sqrt{8}} \int_{r_1}^0 \frac{-1}{(j_1 + \pi_3)\sqrt{\pi_3 - r_1}} d\pi_3 \\ &= \lim_{h \rightarrow 0^+} \frac{h}{4j_1\sqrt{8}} \frac{2}{\sqrt{j_1 + r_1}} \arctan \left(\frac{\sqrt{\pi_3 - r_1}}{\sqrt{j_1 + r_1}} \right) \Bigg|_{\pi_3=r_1}^{\pi_3=0} \\ &= \frac{h}{4j_1\sqrt{8}} \frac{2}{\sqrt{\frac{h^2}{32j_1^2}}} \frac{\pi}{2} \\ &= \frac{\pi}{2}. \end{aligned}$$

The right singularity of (73) arises corresponds to the singularity of the reduced spaces of the drift system, and it gives a phase shift of

$$\begin{aligned}
 \phi_2 &\equiv \lim_{h \rightarrow 0^+} \int_0^{r_2} \frac{-h\pi_3}{(j_1^2 - \pi_3^2)\sqrt{8(\pi_3 - r_1)(r_2 - \pi_3)(r_3 - \pi_3)}} d\pi_3 \\
 &= \frac{-h}{\sqrt{8}} \lim_{h \rightarrow 0^+} \frac{\pi_3}{(j_1 + r_2)\sqrt{r_2 - r_1}} \\
 &\quad \times \lim_{h \rightarrow 0^+} \int_0^{r_2} \frac{1}{(j_1 - \pi_3)\sqrt{(r_2 - \pi_3)(r_3 - \pi_3)}} d\pi_3 \\
 &= \frac{-h}{\sqrt{8}} \frac{j_1}{2j_1\sqrt{2j_1}} \lim_{h \rightarrow 0^+} \left(\frac{-1}{\sqrt{(r_3 - j_1)(j_1 - r_2)}} \right. \\
 &\quad \left. \times \arctan \left(\frac{1}{2} \frac{\pi_3(2j_1 - r_2 - r_3) + 2r_2r_3 - j_1(r_2 + r_3)}{\sqrt{(r_3 - j_1)(j_1 - r_2)(r_2 - \pi_3)(r_3 - \pi_3)}} \right) \right) \Bigg|_{\pi_3=0}^{\pi_3=r_2} \\
 &= \frac{-h}{\sqrt{8}} \frac{j_1}{2j_1\sqrt{2j_1}} \frac{-1}{4\sqrt{j_1}} \frac{-\pi}{2} \\
 &= -\frac{\pi}{4},
 \end{aligned}$$

so that the phase shift as the homoclinic orbit is approached through positive h is

$$\phi = -2(\phi_1 + \phi_2) = -2 \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = -\frac{\pi}{2}.$$

The same calculation for h negative gives $\phi_1 = -\pi/2$ and $\phi_2 = \pi/4$ for a total phase shift of $\phi = \pi/2$.

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