Two Axially Symmetric Coupled Rigid Bodies: Relative Equilibria, Stability, Bifurcations, and a Momentum Preserving Symplectic Integrator

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Abstract

The dynamics of two rigid bodies coupled by an ideal spherically symmetric joint is studied. Except for preliminary material, attention is restricted to the case of two identical bodies with two equal moments of inertia and joined along their axes of symmetry. The system admits the symmetry group $SO(3) \times (S^1)^2$. All relative equilibria are explicitly computed. The nonlinear stability of all these relative equilibria is determined after a study of nonlinear stability of relative equilibria in general. The bifurcations (as momentum is varied) of the relative equilibria are determined.

The construction of symplectic integration algorithms using generating functions of type 1 is discussed. Without the assumption of constant kinetic energy metric, expansions of the generating function of type 1 for the flow of a simple mechanical system are found. The order of the integration algorithm constructed from a given expansion is determined in terms of the order of the expansion itself. Simple methods of constructing invariant generating functions are presented—they result in momentum preserving algorithms. The use of these ideas is illustrated by the construction of a momentum preserving symplectic integrator for two identical axially symmetric rigid bodies coupled by an ideal spherically symmetric joint.

Jerrold E. Marsden

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For Princess

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Introduction

Just as deduction should be supplemented by intuition, so the impulse to progressive generalization must be tempered and balanced by respect and love for colorful detail. The individual problem should not be degraded to the rank of special illustration of lofty general theories. In fact, general theories emerge from consideration of the specific, and they are meaningless if they do not serve to clarify and order the more particular substance below. The interplay between generality and individuality, deduction and construction, logic and imagination—this is the profound essence of live mathematics. Any one or the other of these aspects of mathematics can be at the center of a given achievement. In a far reaching development all of them will be involved. Generally speaking, such a development will start from the "concrete" ground, then discard ballast by abstraction and rise to the lofty layers of thin air where navigation and observation are easy; after this flight comes the crucial test of landing and reaching specific goals in the newly surveyed low plains of individual "reality". In brief, the flight into abstract generality must start from and return to the concrete and specific.

—*Richard Courant*

The system of a single rigid body in absence of external forces has often been a paradigm of a finite dimensional Hamiltonian system with symmetry [1][3][10][13][29]. In modern language, the configuration space for this system is the smooth manifold $SO(3)$, each element of which gives the configuration of the body by reference to some fixed configuration. The directional isotropy of space implies that the system admits the symmetry group $SO(3)$, which acts on the configuration space of this example by left multiplication [1]. In the generic case, where all three principle moments of inertia are different, the results of applying to this example methods in the theory of Hamiltonian systems with symmetry are an indication of what can be achieved by such an exercise:

1. The symmetry implies the existence of other descendant conservative systems having smaller dimension and less symmetry, one for each value of the angular momentum, and each being completely integrable. The phase spaces of these reduced systems are spheres, excepting the case arising from zero total angular momentum, where the phase space is a point [1].

2. If the body's motion is just that of simple rotation about an axis fixed in space, then that axis must coincide with one of the principle axes of inertia. Conversely, arbitrary rates of rotation about the principle axes of inertia are dynamic evolutions of the system. These simple motions correspond to fixed points on the reduced spaces, and there are six such on each reduced space associated to a nonzero value of angular momentum [1].

3. The motions of rotating about the longest and shortest axes are stable, while those of rotating about the intermediate axis are not stable, in the sense that small perturbations of former result in motions of the body that are close to rotations about those axes, whereas perturbing the latter will result in some tumbling behavior [10].

So much for the single rigid body—reduced systems of lower dimension can be found explicitly, special solutions (relative equilibria) associated to the symmetry group exist and can be explicitly computed, the nonlinear stability of those solutions can be determined, and these solutions do not bifurcate as angular momentum is varied. So as an example, this system has shortcomings, traceable to two sources: the group of symmetries is rather small, as is the dimension of the configuration space. And this problem is only made more acute by moving to the more symmetric cases in which two of the principle moments of inertia are equal, since although the symmetry becomes greater, all the reduced phase spaces in that case become points. The subject of Hamiltonian systems with symmetry is a powerful blend of group theory and symplectic geometry, a sophisticated machine which easily overruns the example that is a single rigid body moving in the absence of external forces.

Of course, there are many examples of Hamiltonian systems with symmetry: the Lagrange top with its $(S^1)^2$ symmetry, systems of planar coupled rigid bodies, multibody gravitational systems, as well as several infinite dimensional examples, just to name a few. But still there is a need for a moderately complex mechanical system with a large symmetry group. So, in 1988, Grossman, Krishnaprasad and Marsden, after being encouraged by success the planar situation, considered the problem of two rigid bodies moving in three dimensional space and coupled by an ideal spherical joint [8]. They generated the basic physical system by first showing that the configuration space for this example is $SO(3)^2$, and second deriving the Hamiltonian and Lagrangian functions. This system, like that of the single rigid body, has no potential, and so is just a geodesic flow, and spatial isotropy implies an $SO(3)$ symmetry on its phase space $SO(3)^2$ by diagonal left multiplication.

If a single rigid body is too simple to stretch the theory of Hamiltonian systems with symmetry, the generic case of two coupled rigid bodies, with unequal principle moments of inertia and joined at odd places, is probably too hard an example for that purpose. This thesis is entirely concerned with the following special case: the case of identical bodies, with two equal moments of inertia, and joined along their axes of symmetry. At a single stroke this assumption achieves two things: a reduction of algebraic complexity of the problem, and an enlargement of the symmetry group from $SO(3)$ to $SO(3) \times (S^1)^2$. The system that results is complex enough to provide and arena for the exploration of the theory of Hamiltonian systems with symmetry, yet not so complex as to unduly task current technology of computerized symbolic manipulation.

We begin then, in chapter (1), by recalling the Grossman, Krishnaprasad and Marsden results in a way adapted to our pervasive assumption of axial symmetry. At this point, a fruit of acquiring a larger symmetry group is apparent: In thinking about a particular configuration where the symmetry axes of the bodies are collinear, you will realize that same configuration can be achieved by rotating the entire system by some amount (corresponding to the action of an element of $SO(3)$), and then separately turning each body back by the same amount (corresponding to the action of some element in the diagonal of $S^1 \times S^1$). This is emblematic of a fact about the symmetry of this system: some points of phase space have nontrivial isotropy. Points of isotropy are interesting because they represent singularities in the reduced phase spaces [2], and thus places where standard methods of analysis fail. Thus, our system forms a test case for the situation where isotropy exists.

After these preliminaries, we next find all of the relative equilibria. Following a typical pattern, the analysis of course locates the obvious relative equilibria, where the bodies are rotating on their fixed axes of symmetry, and locates as well as a plethora of other more or less complicated motions. The relative equilibria are parameterized by $\mathbb{R}^3$—for a given element of $\mathbb{R}^3$ there is one relative equilibrium (these parameters have no independent significance.) The finding of these relative equilibria provides a lesson in the uses of the abstract: conceivably, relative equilibria can be found by employing the ansatz $t \mapsto \exp(\xi t)p$, where ξ is in the Lie algebra of the group of symmetries and p is a point in phase space, but it is much easier to use the general principle of symmetric criticality [13] to find them. Throughout this thesis, abstract results, although usually very simple when viewed on their own, enter powerfully into the analysis as guides through fairly complicated calculations, by organizing those calculations, and by suggesting simpler routes that could not be easily seen amongst the morass of detail and computer output.

In the literature, verifying the nonlinear stability of relative equilibria is equated with establishing the Liapunov stability of the corresponding fixed point of the flow on the reduced phase space [10][13][29]. A defect of this approach is the absence of a fundamental interpretation of nonlinear stability in terms of the dynamics on the original phases spaces. Thinking about the motion of a single rigid body rotating about its longest or shortest principle axis of inertia, then perturbing this motion in such a way that the body only rotates more quickly, you can see there results two orbits in phase space that gradually separate from one another. But notice that, after arbitrary time, the endpoints of these two orbits can be brought together by multiplying by an element in the group of rotations about that axis, and that this group is the isotropy group of the angular momentum vector. That this is a common situation is the content of the first theorem of the thesis, which may be viewed as a dynamical interpretation of the assertion of Liapunov stability on the reduced space. This theorem is then used to establish stability results for the relative equilibria when isotropy is not involved. When isotropy is involved, it comes in two forms: isotropy of configurations and the more fundamental isotropy of points in phase space. In both cases, we make progress by establishing results that justify restriction to a subgroup that acts locally freely.

Next we consider the problem of counting the number of relative equilibria on each reduced space. Since all of the relative equilibria are known, in the sense that they have been parameterized, this problem can be reduced to counting the number of parameters whose associated relative equilibria have a given value of momentum. This is a classical bifurcation problem, but the algebraic complexity of the more complicated relative equilibria gives some difficulty. Nevertheless, we we conclude the analysis of the relative equilibria by solving this problem, and there results a rather complete picture of the bifurcation of relative equilibria as momentum is varied.

When investigating the phase portrait of a complicated Hamiltonian system with symmetry, like the system of two coupled rigid bodies, a numerical integrator is of obvious utility. Recently, attention has been attracted towards the class of symplectic integrators [6]. These algorithms are discrete flows obtained by iterating a symplectic mapping of phase space that approximates the exact time flow for small time, and appear to better preserve behaviors intimately associated with the conservative nature of these systems[6]. One method to construct the symplectic time step map is by generating it using an approximation to the generating function of type 1 for the actual flow. Showing that a certain order algorithm results from a given order approximation to that function is the first subject of chapter (3), an issue which is complicated by the fact that generating functions of type 1 must have singularities at zero time if they are to generate maps close to the identity when time is close to zero. The current literature on symplectic integration algorithms does not include expansions of the generating function of type 1 when the kinetic energy is nontrivial, an obvious defect when dealing with geodesic flows. We compute expansions sufficient to construct symplectic integration algorithms for simple mechanical systems with nontrivial kinetic energy metric up to order three, thus extending some results of Channell and Scovel [6].

Now, the Noether theorem, by combining symmetry and symplectic geometry, associates conserved quantities to symmetries [1]. So if a symplectic integration method is chosen that also respects the symmetry, then there is hope that it will conserve momentum. By an observation of Ge and Marsden [7], this is in fact the case: if a generating function of type 1 is used to construct a symplectic integration algorithm, and if the generating function is invariant under the group of symmetries, then the resulting algorithm will conserve momentum. Unfortunately, while the exact generating function of the time evolution of a system will be invariant, approximations to that function need not be. We consider two methods for constructing invariant approximations: the use of a coordinate system adapted to the given group action and the modifying of an approximation that is not invariant using a section to the group action. At points with isotropy, sections are unavailable, and if appropriate coordinates are also unavailable due to complications with the symmetry group, then we have to accept that some of the momenta will not be conserved by the algorithm nearby to some points of phase space. After these generalities, this thesis is closed by briefly describing how these ideas have been used to construct a numerical integrator for the system of two axially symmetric coupled rigid bodies that conserves all the momenta to machine precision in one open subset of phase space, and similarly conserves all momenta but one in another open subset, these two open subsets covering the entire phase space.

In as far a possible, the thesis adheres to the notation of [1]. All momentum maps are Ad^* equivariant, and all vectors are represented as column vectors. A Hamiltonian system with symmetry is denoted by (P, ω, H, G, J), where P is a smooth manifold, ω is a symplectic form, G is a Lie group acting on P, H is a G invariant Hamiltonian function on P, and J is an Ad^* equivariant momentum mapping; if the momentum map J is not needed, it is simply omitted from this quintuple. A simple mechanical system with symmetry is denoted by (Q, G, V), where Q is a Riemannian manifold, G is a Lie group acting on Q, and V is a G invariant potential function. The co-adjoint action of a Lie group on its Lie algebra is denoted by CoAd. The relation $\overset{!}{=}$ denotes equality up to a well known and obvious identification. The reference system is straightforward: a reference number of the form (n) refers to an equation in the chapter in which the reference occurs, while references across chapters are prefixed by the chapter number, like (c.n). An external reference appears as [n], and may be found in the bibliography. If an external reference to a specific page p is required, it appears as [n : p].

Chapter 1

Two Coupled Rigid Bodies

For conservative physical systems, the Lagrangian description of the system is fundamental. We begin by recalling this description in the case of the free motion of two rigid bodies coupled by an ideal spherical joint, as derived in [8].

The first element of a Lagrangian description is the configuration space. To derive the configuration space of the system of two coupled rigid bodies, fix an inertial frame, and let ρ_1 and ρ_2 be two distributions of order zero and compact support on $\mathbb{R}^3$, representing the densities of the two bodies in some reference configuration. Without loss of generality, assume that the spherical joint in this configuration is located at the origin. Then take as configuration space the manifold $SO(3)^2 \times \mathbb{R}^3$: a point $q \in \mathbb{R}^3$ on body $i \in \{1,2\}$ in the reference configuration has position $A_i q + w$ in the configuration (A_1, A_2, w).

The second element of a Lagrangian description is the Lagrangian function. For this, consider the state described by the tangent vector $(A_i, w, \dot{A}_i, \dot{w})$. The kinetic energy of this state is easily computed:

$$
\begin{aligned}
KE &\stackrel{\text{def}}{=} \frac{1}{2}\int |\dot{A}_1 q + \dot{w}|^2 \rho_1(q)\, dq + \frac{1}{2}\int |\dot{A}_2 q + \dot{w}|^2 \rho_2(q)\, dq \\
&= \frac{1}{2}\int \left(|\dot{A}_1 q|^2 + 2\dot{A}_1 q \cdot \dot{w} + |\dot{w}|^2\right) \rho_1(q)\, dq + (1 \leftrightarrow 2) \\
&= \frac{1}{2}\int \left(\mathrm{trace}((\dot{A}_1 q)(\dot{A}_1 q)^t) + 2\dot{A}_1 q \cdot \dot{w} + |\dot{w}|^2\right) \rho_1(q)\, dq + (1 \leftrightarrow 2) \\
&= \frac{1}{2}\mathrm{trace}\left(\dot{A}_1 \left[\int q q^t \rho_1(q)\, dq\right] \dot{A}_1{}^t\right) + \dot{A}_1 \left[\int q \rho_1(q)\, dq\right] \cdot \dot{w} \\
&\qquad + \frac{1}{2}\left[\int \rho_1(q)\, dq\right] |\dot{w}|^2 + (1 \leftrightarrow 2).
\end{aligned}
$$

Let m_i be the mass and d_i be the center of mass of body i in the reference configuration:

$$
\begin{aligned}
m_i &\stackrel{\text{def}}{=} \int \rho_i(q)\, dq, \\
d_i &\stackrel{\text{def}}{=} \frac{1}{m_i}\int q \rho_i(q)\, dq,
\end{aligned}
$$

and let $m = m_1 + m_2$ be the total mass of the system. Also, let I_i be the coefficient of inertia matrix of body i in the reference configuration, but with respect to the center of mass, instead of the origin:

$$\begin{aligned} I_i &\stackrel{\text{def}}{=} \int (q-d_i)(q-d_i)^t \rho_i(q)\,dq \\ &= \int \left[qq^t - d_i q^t - q d_i{}^t + d_i d_i{}^t \right] \rho_i(q)\,dq \\ &= \int qq^t \rho_i(q)\,dq - m_i d_i d_i{}^t. \end{aligned}$$

By direct substitution, then,

$$\begin{aligned} KE &= \frac{1}{2}\mathrm{trace}(\dot{A}_1 I_1 \dot{A}_1{}^t) + \frac{1}{2}\mathrm{trace}(\dot{A}_2 I_2 \dot{A}_2{}^t) + \frac{m_1}{2}|\dot{A}_1 d_1|^2 + \frac{m_2}{2}|\dot{A}_2 d_2|^2 \\ &\qquad + (m_1 \dot{A}_1 d_1 + m_2 \dot{A}_2 d_2)\cdot \dot{w} + \frac{m}{2}|\dot{w}|^2. \end{aligned}$$

The kinetic energy is manifestly quadratic in the velocities and the potential energy is zero, so this function is also the total energy and the Lagrangian.

Of course, just as in the case of a single rigid body, the translational parts of this physical system can be separated from its other parts. Indeed, the group $\mathbb{R}^3$ acts on configuration space by addition to the variable w, and this action lifts to an action on velocity phase space (also addition to w) under which the total energy E is invariant. Standard symplectic reduction applies, and in this case offers no difficulty. Equivalently, one may assume that the total linear momentum is zero, and then use the fact that the center of mass of the system is stationary to eliminate w and $\dot{w}$. Proceeding with the latter, note that the center of mass in the configuration (A_i, w) has position

$$\begin{aligned} &\frac{m_1(A_1 d_1 + w) + m_2(A_2 d_2 + w)}{m_1 + m_2} \\ &\qquad = \frac{m_1 A_1 d_1 + m_2 A_2 d_2}{m} + w. \end{aligned}$$

This is constant, so that

$$\dot{w} = -\frac{m_1 \dot{A}_1 d_1 + m_2 \dot{A}_2 d_2}{m},$$

and the Lagrangian becomes

$$\begin{aligned} L &= \frac{1}{2}\mathrm{trace}(\dot{A}_1 I_1 \dot{A}_1{}^t) + \frac{1}{2}\mathrm{trace}(\dot{A}_2 I_2 \dot{A}_2{}^t) + \frac{m_1}{2}|\dot{A}_1 d_1|^2 + \frac{m_2}{2}|\dot{A}_2 d_2|^2 \\ &\qquad - \frac{1}{2m}|m_1 \dot{A}_1 d_1 + m_2 \dot{A}_2 d_2|^2 \\ &= \frac{1}{2}\mathrm{trace}(\dot{A}_1 I_1 \dot{A}_1{}^t) + \frac{1}{2}\mathrm{trace}(\dot{A}_2 I_2 \dot{A}_2{}^t) + \frac{\epsilon}{2}|\dot{A}_1 d_1 - \dot{A}_2 d_2|^2, \end{aligned}$$

where $\epsilon = m_1 m_2 / m$ is the reduced mass. Thus, the basic system consists of this Lagrangian on the configuration space $SO(3)^2$.

Since the configuration space is $SO(3)^2$, and since much of the symmetry group will be a subgroup of $SO(3)^4$, the Lie group $SO(3)$ will be ubiquitous. Let us recall the basic facts which identify the Lie algebra, tangent bundle, and cotangent bundle of this Lie group. By viewing $SO(3)$ as a submanifold of $\mathbb{R}^9$ and linearizing the equation $BB^t = \mathrm{Id}$, the Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ is the set of pairs $(\mathrm{Id}, \dot{B})$ where $\dot{B}$ is antisymmetric, with Lie bracket $[(\mathrm{Id}, \dot{A}_1), (\mathrm{Id}, \dot{A}_2)] = (\mathrm{Id}, \dot{A}_1\dot{A}_2 - \dot{A}_2\dot{A}_1)$. If $a \in \mathbb{R}^3$, define the maps

$$a^\wedge \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{bmatrix}^\vee \stackrel{\text{def}}{=} a,$$

and note the following useful identities: if $b \in \mathbb{R}^3$ and $B \in SO(3)$ then

$$(a^\wedge b^\wedge - b^\wedge a^\wedge) = (a^\wedge b)^\wedge = -(b^\wedge a)^\wedge = (a \times b)^\wedge,$$
$$Ba^\wedge B^t = (Ba)^\wedge.$$

The first of these identities justifies the identification of $\mathfrak{so}(3)$ with the Lie algebra of $\mathbb{R}^3$ having Lie bracket the vector cross product, so that $(\mathrm{Id}, \dot{A}) \stackrel{!}{=} \dot{A}^\vee$, while the second shows that, through this identification, the adjoint action becomes

$$\mathrm{Ad}_B a = Ba,$$

where $a \in \mathfrak{so}(3) \stackrel{!}{=} \mathbb{R}^3$. Then $TSO(3)$ is identified with $SO(3) \times \mathbb{R}^3$ by left translation, so that

$$(A, \Omega) \stackrel{!}{=} \left.\frac{d}{dt}\right|_{t=0} A\exp(t\Omega^\wedge).$$

The cotangent bundle $T^*SO(3)$ is identified with $SO(3) \times \mathbb{R}^3$ as well by the decree that

$$\langle (A, \pi), (A, \Omega) \rangle = \pi \cdot \Omega \quad \forall \Omega \in \mathbb{R}^3,$$

an equation that defines how $(A, \pi) \in T_A^*SO(3)$ acts on $(A, \Omega) \in T_ASO(3)$. In fact, this is the identification of $T^*SO(3)$ by left translation, provided that $\mathfrak{so}(3)^*$ is identified with $\mathbb{R}^3$ by the same decree without the base point A. Since the standard inner product of $\mathbb{R}^3$ is $SO(3)$ invariant, the co-adjoint action becomes

$$\mathrm{CoAd}_B \pi = B\pi,$$

where $\pi \in \mathfrak{so}(3)^* \stackrel{!}{=} \mathbb{R}^3$.

This said, the Lagrangian L becomes a function on $SO(3)^2 \times (\mathbb{R}^3)^2$ through this space's identification with $TSO(3)^2$. For the explicit expression, label the elements of $SO(3)^2 \times (\mathbb{R}^3)^2$ by (A_i, Ω_i), so $\Omega_i = ({A_i}^t \dot{A}_i)^\vee$, and compute as follows:

$$\begin{aligned} L &= \frac{1}{2}\mathrm{trace}(\dot{A}_1 I_1 {\dot{A}_1}^t) + \frac{1}{2}\mathrm{trace}(\dot{A}_2 I_2 {\dot{A}_2}^t) + \frac{\epsilon}{2}|\dot{A}_1 d_1 - \dot{A}_2 d_2|^2 \\ &= \frac{1}{2}\mathrm{trace}({A_1}^t \dot{A}_1 I_1 {\dot{A}_1}^t A_1) + \frac{1}{2}\mathrm{trace}({A_2}^t \dot{A}_2 I_2 {\dot{A}_2}^t A_2) \\ &\qquad + \frac{\epsilon}{2}|A_1 {A_1}^t \dot{A}_1 d_1 - A_2 {A_2}^t \dot{A}_2 d_2|^2 \\ &= \frac{1}{2}\mathrm{trace}({\Omega_1}^\wedge I_1 ({\Omega_1}^\wedge)^t) + \frac{1}{2}\mathrm{trace}({\Omega_2}^\wedge I_2 ({\Omega_2}^\wedge)^t) + \frac{\epsilon}{2}|A_1 {\Omega_1}^\wedge d_1 - A_2 {\Omega_2}^\wedge d_2|^2 \\ &= \frac{1}{2}\mathrm{trace}(-({\Omega_1}^\wedge)^2 I_1) + \frac{1}{2}\mathrm{trace}(-({\Omega_2}^\wedge)^2 I_2) + \frac{\epsilon}{2}|A_1 {d_1}^\wedge \Omega_1 - A_2 {d_2}^\wedge \Omega_2|^2. \end{aligned}$$

Concerning the first two terms, temporarily suppress the index 1 or 2 and verify directly the identity $(\Omega^\wedge)^2 = \Omega\Omega^t - |\Omega|^2\mathrm{Id}$, so that

$$\begin{aligned}\mathrm{trace}(-(\Omega^\wedge)^2 I) &= \mathrm{trace}(|\Omega|^2 I - \Omega\Omega^t I)\\ &= \Omega^t\,\mathrm{trace}(I)\Omega - \Omega^t I\Omega\\ &= \Omega^t(\mathrm{trace}(I)\mathrm{Id} - I)\Omega\\ &= \Omega^t J\Omega,\end{aligned}$$

where J is the moment of inertia matrix,

$$J \stackrel{\mathrm{def}}{=} \mathrm{trace}(I)\mathrm{Id} - I = \begin{bmatrix} I^{22}+I^{33} & -I^{12} & -I^{13}\\ -I^{21} & I^{11}+I^{33} & -I^{23}\\ -I^{31} & -I^{32} & I^{11}+I^{22}\end{bmatrix}.$$

Thus,

$$\begin{aligned} L &= \frac{1}{2}\Omega_1{}^t J_1\Omega_1 + \frac{1}{2}\Omega_2{}^t J_2\Omega_2 + \frac{\epsilon}{2}|A_1(d_1\times\Omega_1) - A_2(d_2\times\Omega_2)|^2\\ &= \frac{1}{2}\Omega_1{}^t J_1\Omega_1 + \frac{1}{2}\Omega_2{}^t J_2\Omega_2 + \frac{\epsilon}{2}\Big(\Omega_1{}^t(d_1{}^\wedge)^t d_1{}^\wedge\Omega_1\\ &\quad - 2A_1(d_1\times\Omega_1)\cdot A_2(d_2\times\Omega_2) + \Omega_2{}^t(d_2{}^\wedge)^t d_2{}^\wedge\Omega_2\Big)\\ &= \frac{1}{2}\Omega_1{}^t\left[J_1 - \epsilon(d_1{}^\wedge)^2\right]\Omega_1 + \frac{1}{2}\Omega_2{}^t\left[J_2 - \epsilon(d_2{}^\wedge)^2\right]\Omega_2\\ &\quad - \epsilon A_1(d_1\times\Omega_1)\cdot A_2(d_2\times\Omega_2).\end{aligned}$$

Letting $A = A_1{}^t A_2$, $\tilde{J}_i = J_i - \epsilon(d_i{}^\wedge)^2$, and

$$J(A) \stackrel{\mathrm{def}}{=} \begin{bmatrix} \tilde{J}_1 & \epsilon d_1{}^\wedge A d_2{}^\wedge\\ \epsilon d_2{}^\wedge A^t d_1{}^\wedge & \tilde{J}_2\end{bmatrix},$$

one obtains

$$\boxed{\begin{gathered} L = \frac{1}{2}\Omega_1{}^t\tilde{J}_1\Omega_1 + \frac{1}{2}\Omega_2{}^t\tilde{J}_2\Omega_2 - \epsilon A_1(d_1\times\Omega_1)\cdot A_2(d_2\times\Omega_2)\\ L = \frac{1}{2}\begin{bmatrix}\Omega_1{}^t & \Omega_2{}^t\end{bmatrix} J(A)\begin{bmatrix}\Omega_1\\ \Omega_2\end{bmatrix}\end{gathered}}$$

The following completely trivial lemma is useful in proving proposition (1) below and for further reference:

Lemma 1. *Let $SO(3)^4$ act on $SO(3)^2$ by*

$$(B_1, B_2, \tilde{B}_1, \tilde{B}_2)\cdot(A_1, A_2) \stackrel{\mathrm{def}}{=} (B_1 A_1\tilde{B}_1{}^t, B_2 A_2\tilde{B}_2{}^t).$$

Then the lifts of this action to $TSO(3)^2$ and $T^*SO(3)^2$ are

$$\begin{aligned}(B_i,\tilde{B}_i)\cdot(A_i,\Omega_i) &= (B_iA_i\tilde{B}_i{}^t,\tilde{B}_i\Omega_i),\\ (B_i,\tilde{B}_i)\cdot(A_i,\pi_i) &= (B_iA_i\tilde{B}_i{}^t,\tilde{B}_i\pi_i),\end{aligned}$$

respectively.

Proof. For the first statement,

$$\begin{aligned}(B_i,\tilde{B}_i)\cdot(A_i,\Omega_i) &= \left.\frac{d}{dt}\right|_{t=0}(B_i,\tilde{B}_i)\cdot(A_i\exp(t\Omega_i{}^\wedge))\\ &= (B_iA_i\tilde{B}_i{}^t, B_iA_i\Omega_i{}^\wedge\tilde{B}_i{}^t)\\ &\stackrel{!}{=} (B_iA_i\tilde{B}_i{}^t,(\tilde{B}_i\Omega_i{}^\wedge\tilde{B}_i{}^t)^\vee)\\ &= (B_iA_i\tilde{B}_i{}^t, B_i\Omega_i),\end{aligned}$$

and for the second statement, if the Ω_i are arbitrary,

$$\begin{aligned}\langle(B_i,\tilde{B}_i)\cdot(A_i,\pi_i),(B_iA_i\tilde{B}_i{}^t,\Omega_i)\rangle &= \langle(A_i,\pi_i),(B_i,\tilde{B}_i)^{-1}\cdot(B_iA_i\tilde{B}_i{}^t,\Omega_i)\rangle\\ &= \langle(A_i,\pi_i),(A_i,\tilde{B}_i{}^t\Omega_i)\rangle\\ &= \pi_1\cdot\tilde{B}_1{}^t\Omega_1+\pi_2\cdot\tilde{B}_2{}^t\Omega_2\\ &= \langle(B_iA_i\tilde{B}_i{}^t,\tilde{B}_i\pi_i),(B_iA_i\tilde{B}_i{}^t,\tilde{B}_i\Omega_i)\rangle,\end{aligned}$$

and the result follows. □

Of course, the symmetries of the Lagrangian for two coupled rigid bodies are of intense interest. The following proposition, the proof of which is routine, summarizes the situation.

Proposition 1. *L is invariant under the lift of the action of $SO(3)$ on configuration space given by*

$$B\cdot(A_1,A_2)\stackrel{\text{def}}{=}(BA_1,BA_2).$$

Furthermore, there are additional symmetries for special parameter values, as follows:

1. *If $e_1\in\mathbb{R}^3$ is a unit vector, $d_1=l_1e_1$, and $\exp(te_1{}^\wedge)I_1=I_1\exp(te_1^\wedge)$ for all $t\in\mathbb{R}$, then L is invariant under the lift of the action of $SO(3)\times S^1$ on configuration space given by*

$$(B,\theta_1)\cdot(A_1,A_2)\stackrel{\text{def}}{=}(BA_1\exp(-\theta_1e_1{}^\wedge),BA_2).$$

   The analogous statement with the index 1 instead of 2 is also true.

2. *If for both $i\in\{1,2\}$, $e_i\in\mathbb{R}^3$ are unit vectors, $d_i=l_ie_i$, and $\exp(te_i^\wedge)I_i=I_i\exp(te_i{}^\wedge)$ for all $t\in\mathbb{R}$, then L is invariant under the lift of the action of $SO(3)\times(S^1)^2$ on configuration space given by*

$$(B,\theta_1,\theta_2)\cdot(A_1,A_2)\stackrel{\text{def}}{=}(BA_1\exp(-\theta_1e_1{}^\wedge),BA_2\exp(-\theta_2e_2{}^\wedge)).$$

3. *If $I_1 = I_2$ and $d_1 = d_2$, then L is invariant under the lift of the action of the involution of configuration space given by*

$$(A_1, A_2) \mapsto (A_2, A_1).$$

4. *If $d_1 = 0$ or $d_2 = 0$, then L is invariant under the lift of the action of $SO(3)^2$ on configuration space given by*

$$(B_1, B_2) \cdot (A_1, A_2) \stackrel{\text{def}}{=} (B_1 A_1, B_2 A_2).$$

5. *If $d_1 = 0$ and I_1 is a constant multiple of the identity, then L is invariant under the lift of the action of $SO(3)$ on $SO(3)^2$ given by*

$$B \cdot (A_1, A_2) \stackrel{\text{def}}{=} (A_1 B^t, A_2).$$

The analogous statement with the index 1 instead of 2 is also true.

Remark. Further to item (5), if $d_1 = 0$, then the Lagrangian may be written

$$L = \frac{1}{2}\text{trace}(\dot{A}_1 I_1 \dot{A}_1^t) + \frac{1}{2}\text{trace}(\dot{A}_2(I_2 + \epsilon d_2 {d_2}^t)\dot{A}_2^t),$$

so the system evolves as two uncoupled rigid bodies, even if $d_2 \neq 0$. If one thinks of two rods with the joint at the end of one rod and the center of mass of the other, this is somewhat counter intuitive, but it is a direct consequence of the Lagrangian formulation of the system. Since the single rigid body is so well understood, the case of $d_1 = 0$ or $d_2 = 0$ will not be seriously considered in this thesis.

Remark. In item (1), considering the transformation rule for coefficient of inertia matrices, e_1 is obviously an axis of symmetry of body 1. Thus, this is the case of a rigid body coupled along an axis of symmetry. As a matter of terminology, items (1) and (2) will be referred to as the S^1-symmetric and $(S^1)^2$-symmetric cases, respectively, while the case where no material symmetry is to be assumed will be referred to as the $SO(3)$-symmetric case. A body giving rise to an S^1 symmetry as above will be called *axially symmetric*.

Remark. Without thought, one might expect additional symmetry when one of the bodies is spherically symmetric, but the hoped for $SO(3)$ symmetry is broken to S^1 by the axis connecting the body to the joint, whenever the joint is not at that body's center of mass.

Recall that, in the context of a Hamiltonian system with symmetry, a relative equilibrium is a point of phase space which evolves along a one parameter orbit of the attendant Lie group. In the case of a Lagrangian system of kinetic plus potential type, the following theorem, called the *principle of symmetric criticality* is useful in determining the relative equilibria. For an elegant proof, see [13].

Theorem 1. *Let (Q, G, V) be a simple mechanical system with symmetry. For $\xi_e \in \mathfrak{g}$, define the function V_{ξ_e} by*

$$V_{\xi_e}(q) = V(q) - \frac{1}{2}|\xi_e(q)|^2.$$

Then v_{q_e} is a relative equilibrium with evolution $t \mapsto \exp(t\xi_e) \cdot v_{q_e}$ if and only if $v_{q_e} = \xi_e(q_e)$ and $dV_{\xi_e}(q_e) = 0$

The Lagrangian system for two coupled rigid bodies, in its variously symmetric cases, is ready for an application of the principle of symmetric criticality. Without further assumption, it appears difficult to obtain a full and explicit solution to the resulting equations, and indeed the finding of such solutions is the object of a numerical study in [30]. There is one interesting fact, though, that is generally true: *in a state of relative equilibrium, the rotation axis and the axes from the joints to the centers of mass are coplanar.* This was first noticed in the $S^1 \times S^1$-symmetric case in [15] and proved in the $SO(3)$-symmetric case in [30].

Proposition 2. *Regardless of the symmetry, a relative equilibrium satisfies the equations*

$$\Omega \times \Big(A_1 \tilde{J}_1 \Omega_1 + \epsilon((A_1 d_1) \cdot \Omega) A_2 d_2\Big) = 0, \tag{1a}$$

$$\Omega \times \Big(A_2 \tilde{J}_2 \Omega_2 + \epsilon((A_2 d_2) \cdot \Omega) A_1 d_1\Big) = 0, \tag{1b}$$

$$\Omega \cdot (A_1 d_1 \times A_2 d_2) = 0, \tag{1c}$$

for some $\Omega \in \mathbb{R}^3$. Thus Ω, $A_1 d_1$, and $A_2 d_2$ are coplanar. Furthermore, the relative equilibria are exactly the (A_i, Ω_i) such that there exist Ω and σ_i satisfying equations (1a)–(1c) *together with one of the pairs of equations*

$$\Omega_1 = A_1{}^t\Omega, \quad \Omega_2 = A_2{}^t\Omega, \tag{2a}$$

$$\Omega_1 = A_1{}^t\Omega - \sigma_1 e_1, \quad \Omega_2 = A_2{}^t\Omega, \tag{2b}$$

$$\Omega_1 = A_1{}^t\Omega - \sigma_1 e_1, \quad \Omega_2 = A_2{}^t\Omega - \sigma_2 e_2, \tag{2c}$$

in the $SO(3)$, S^1, and $(S^1)^2$-symmetric cases respectively.

Proof. In the $(S^1)^2$-symmetric case, the infinitesimal generator corresponding to $(\Omega, \sigma_1, \sigma_2)$ in the Lie algebra of $SO(3) \times (S^1)^2$ is

$$\begin{aligned}
(\Omega, \sigma_i)(A_i) &= \frac{d}{dt}\Big|_{t=0} (\exp(t\Omega^\wedge) A_i \exp(-t\sigma_i e_i{}^\wedge)) \\
&= (\Omega^\wedge A_1 - \sigma_1 A_1 e_1{}^\wedge, \Omega^\wedge A_2 - \sigma_2 A_2 e_2{}^\wedge) \\
&\stackrel{!}{=} ((A_1{}^t \Omega^\wedge A_1 - \sigma_1 e_1{}^\wedge)^\vee, (A_2{}^t \Omega^\wedge A_2 - \sigma_2 e_2{}^\wedge)^\vee) \\
&= (A_1{}^t\Omega - \sigma_1 e_1, A_2{}^t\Omega - \sigma_2 e_2),
\end{aligned} \tag{3}$$

and in the less symmetric cases the infinitesimal generator is this with one or both of the σ_i set to zero. Thus, the last part of the proposition (i.e. concerning equations (2a)–(2c)) is exactly the condition $V_{q_e} = \xi_e(q_e)$ in theorem (1).

It is left to verify that the equations (1a)–(1c) are equivalent to the vanishing of $\boldsymbol{d}V_\Omega$, $\boldsymbol{d}V_{(\Omega,\sigma_1)}$ or $\boldsymbol{d}V_{(\Omega,\sigma_1,\sigma_2)}$, as the case may be. Taking up the latter case, and since the potential is zero, $V_{(\Omega,\sigma_1,\sigma_2)}$ is the negative of half of the result of substituting (2c) into L. Then one can compute the derivative of this function using

$$\boldsymbol{d}V_{(\Omega,\sigma_1,\sigma_2)}(A_1, A_2)(w_1, w_2) = \frac{d}{dt}\Big|_{t=0} V_{(\Omega,\sigma_1,\sigma_2)}(A_1 \exp(tw_1{}^\wedge), A_2 \exp(tw_2{}^\wedge))$$

This differentiation is most easily accomplished using the chain rule: set

$$\Omega_i = (A_i \exp(tw_i{}^\wedge))^t\Omega - \sigma_i e_i,$$

note that

$$\dot{\Omega}_i \stackrel{\text{def}}{=} \left.\frac{d}{dt}\right|_{t=0} \Omega_i(t) = -w_i{}^\wedge A_i\Omega = -w_i \times A_i\Omega,$$

and note also that

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} A_i \exp(tw_i{}^\wedge)(d_i \times \Omega_i) &= \left.\frac{d}{dt}\right|_{t=0} (A_i \exp(tw_i{}^\wedge)d_i) \times (\Omega - \sigma_i A_i \exp(tw_i{}^\wedge)e_i) \\
&= \left.\frac{d}{dt}\right|_{t=0} (A_i \exp(tw_i{}^\wedge)d_i) \times \Omega \\
&= (A_i w_i{}^\wedge d_i) \times \Omega \\
&= (A_i w_i \times A_i d_i) \times \Omega.
\end{aligned}$$

Then, remembering that the Ω_i now depend on t,

$$\begin{aligned}
&-2dV_{(\Omega,\sigma_1,\sigma_2)}(A_1, A_2)(w_1, w_2) \\
&= \left.\frac{d}{dt}\right|_{t=0} V_{(\Omega,\sigma_1,\sigma_2)}(A_1 \exp(tw_1{}^\wedge), A_2 \exp(tw_2{}^\wedge)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\frac{1}{2}\Omega_1{}^t\tilde{J}_1\Omega_1 + \frac{1}{2}\Omega_2{}^t\tilde{J}_2\Omega_2 - \epsilon A_1(d_1 \times \Omega_1) \cdot A_2(d_2 \times \Omega_2)\right) \\
&= \dot{\Omega}_1{}^t\tilde{J}_1\Omega_1 - \left(\epsilon \left.\frac{d}{dt}\right|_{t=0} A_1 \exp(tw_1{}^\wedge)(d_1 \times \Omega_1)\right) \cdot A_2(d_2 \times \Omega_2) + (1 \leftrightarrow 2) \\
&= -w_1 \times A_1{}^t\Omega \cdot \tilde{J}_1\Omega_1 - \epsilon(A_1 w_1 \times A_1 d_1) \times \Omega \cdot (A_2 d_2) \times \Omega + (1 \leftrightarrow 2) \\
&= -w_1 \cdot A_1{}^t\Omega \times \tilde{J}_1\Omega_1 - \epsilon A_1 w_1 \times A_1 d_1 \cdot \Omega \times ((A_2 d_2) \times \Omega) + (1 \leftrightarrow 2) \\
&= -A_1 w_1 \cdot \Omega \times A_1\tilde{J}_1\Omega_1 - \epsilon A_1 w_1 \cdot A_1 d_1 \times (\Omega \times ((A_2 d_2) \times \Omega)) + (1 \leftrightarrow 2) \\
&= -A_1 w_1 \cdot \Big[\Omega \times A_1\tilde{J}_1\Omega_1 + \epsilon A_1 d_1 \times (\Omega \times ((A_2 d_2) \times \Omega))\Big] + (1 \leftrightarrow 2) \\
&= -A_1 w_1 \cdot \Big[\Omega \times A_1\tilde{J}_1\Omega_1 + \epsilon(A_1 d_1 \cdot (A_2 d_2) \times \Omega)\Omega - \epsilon(A_1 d_1 \cdot \Omega)A_2 d_2 \times \Omega\Big] + (1 \leftrightarrow 2) \\
&= -A_1 w_1 \cdot \Big[\Omega \times (A_1\tilde{J}_1\Omega_1 + \epsilon(A_1 d_1 \cdot \Omega)A_2 d_2) + \epsilon(\Omega \cdot A_1 d_1 \times A_2 d_2)\Omega\Big] + (1 \leftrightarrow 2).
\end{aligned}$$

Setting $w_2 = 0$ shows that the term enclosed in square brackets vanishes, and then since the two vectors therein are linearly independent, they vanish separately, giving (1a) and (1c). Setting $w_1 = 0$ gives (1b), so the proof in the $(S^1)^2$-symmetric case is complete.

The proof in the less symmetric cases is similar, except that (2a) and (2b) are substituted into L instead of (2c). This is equivalent to setting one or both of the σ_i to zero in the proof of the $(S^1)^2$-symmetric case, so one again obtains (1a)–(1c), since those equations do not depend on the σ_i. □

We consider next some generalities concerning Poisson reduction of cotangent bundles, with an eye towards reducing the system of two coupled rigid bodies by the action of $SO(3)$. Let Φ be a free left action of a Lie group G on a configuration manifold Q, so G

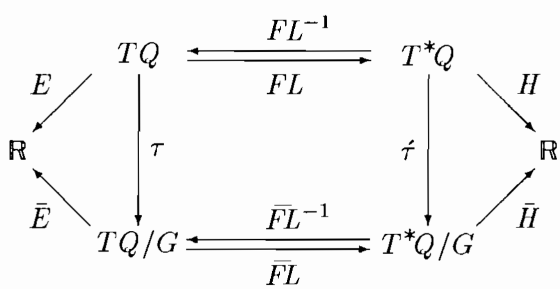
also acts freely on TQ and T^*Q by $gv_q = T\Phi_g v_q$ and $g\alpha_q = T^*\Phi_{g^{-1}}\alpha_q$. Let $L : TQ \to \mathbb{R}$ be some G invariant Lagrangian and suppose that the quotient manifolds $\tau : TQ \to TQ/G$ and $\acute{\tau} : T^*Q \to T^*Q/G$ exist. Then the Legendre transformation is equivariant:

$$\begin{aligned}
\langle FL(gv_q), w_q\rangle &= \left.\frac{d}{dt}\right|_{t=0} L(T\Phi_g v_q + tw_q) \\
&= \left.\frac{d}{dt}\right|_{t=0} L(T\Phi_g(v_q + tT\Phi_{g^{-1}} w_q)) \\
&= \left.\frac{d}{dt}\right|_{t=0} L(v_q + tT\Phi_{g^{-1}} w_q) \\
&= \langle FL(v_q), T\Phi_{g^{-1}} w_q\rangle \\
&= \langle gFL(v_q), w_q\rangle,
\end{aligned}$$

and hence induces a smooth map $\bar{FL} : TQ/G \to T^*Q/G$. Suppose this map is a diffeomorphism. Then $FL : TQ \to T^*Q$ is a diffeomorphism (this is a general property when passing to the quotient by a free action), so L is hyperregular. Under these conditions, then, TQ is a symplectic manifold, TQ/G and T^*Q/G are Poisson manifolds and $\bar{FL} : TQ/G \to T^*Q/G$ is a Poisson diffeomorphism. In addition, the energy function is G invariant:

$$E(gv_q) = \langle FL(gv_q), gv_q\rangle - L(gv_q) = \langle FL(v_q), v_q\rangle - L(v_q) = E(v_q),$$

as is the Hamiltonian $H = E \circ FL^{-1}$, so these functions induce functions $\bar{E} : TQ/G \to \mathbb{R}$ and $\bar{H} : T^*Q/G \to \mathbb{R}$. To summarize: if $\bar{FL}$ is a diffeomorphism, then FL is hyperregular, and one has the following commutative diagram of Poisson maps and functions:



For the system of two coupled rigid bodies, the following result of Krishnaprasad and Marsden [11] may be used to compute the Poisson bracket on the reduced phase space $T^*SO(3)^2/SO(3)$:

Proposition 3. *Let G be a Lie group and consider the lift to T^*G^2 of the left diagonal action of G on G^2. Then this action is symplectic, and the map*

$$\lambda : T^*G^2 \mapsto \mathfrak{g}^2 \times G, \qquad \lambda(\mu_g, \nu_h) = (T^*L_g\mu_g, T^*L_h\nu_h, g^{-1}h)$$

is a quotient map. Suppose f is a function on $\mathfrak{g}^2 \times G$ and let $p = (\mu, \nu, g) \in \mathfrak{g}^2 \times G$, use the identification of $\mathfrak{g}^{**}$ with $\mathfrak{g}$ to recognize that $\mathbf{d}_1 f(p) \in \mathfrak{g}$ and $\mathbf{d}_2 f(p) \in \mathfrak{g}$, and note *that* $\boldsymbol{d}_3 f(p) \in T^*G$*. Then the Poisson bracket on* $\mathfrak{g}^2 \times G$ *induced by* λ *is*

$$\begin{aligned}\{f_1, f_2\}(p) &= -\langle \mu, [\boldsymbol{d}_1 f_1(p), \boldsymbol{d}_1 f_2(p)]\rangle - \langle \nu, [\boldsymbol{d}_2 f_1(p), \boldsymbol{d}_2 f_2(p)]\rangle \\ &\quad -\langle \boldsymbol{d}_3 f_1(p), TR_g \boldsymbol{d}_1 f_2(p) - TL_g \boldsymbol{d}_2 f_2(p)\rangle \\ &\quad +\langle \boldsymbol{d}_3 f_2(p), TR_g \boldsymbol{d}_1 f_1(p) - TL_g \boldsymbol{d}_2 f_1(p)\rangle.\end{aligned}$$

Applying this reduction theorem to the momentum phase space of two coupled rigid bodies yields the reduced phase space $(\mathfrak{so}(3)^*)^2 \times SO(3)$, which is identified with $(\mathbb{R}^3)^2 \times SO(3)$. The quotient map is then

$$\acute{r} : T^*SO(3)^2 \to (\mathbb{R}^3)^2 \times SO(3), \quad (\mu_{A_1}, \nu_{A_2}) \mapsto (TL_{A_1}\mu_{A_1}, TL_{A_2}\nu_{A_2}, {A_1}^t A_2).$$

Now the trivialization $T^*SO(3) \cong SO(3) \times \mathfrak{so}(3)^*$ by left translation may be used to write the Poisson bracket on $(\mathbb{R}^3)^2 \times SO(3)$ in matrix form. Let f and h be two functions on $(\mathbb{R}^3)^2 \times SO(3)$, and temporarily write $p = (\pi_i, A)$. Then one computes as follows:

$$\begin{aligned}\{f, h\}(p) &= -\langle \pi_1, [\boldsymbol{d}_1 f(p), \boldsymbol{d}_1 h(p)]\rangle - \langle \pi_2, [\boldsymbol{d}_2 f(p), \boldsymbol{d}_2 h(p)]\rangle \\ &\quad - \langle \boldsymbol{d}_3 f(p), TR_A \boldsymbol{d}_1 h(p) - TL_A \boldsymbol{d}_2 h(p)\rangle \\ &\quad + \langle \boldsymbol{d}_3 h(p), TR_A \boldsymbol{d}_1 f(p) - TL_A \boldsymbol{d}_2 f(p)\rangle \\ &= -\langle \pi_1, [\boldsymbol{d}_1 f(p), \boldsymbol{d}_1 h(p)]\rangle - \langle \pi_2, [\boldsymbol{d}_2 f(p), \boldsymbol{d}_2 h(p)]\rangle \\ &\quad - \langle \boldsymbol{d}_3 f(p), TL_A \mathrm{Ad}_{A^{-1}} \boldsymbol{d}_1 h(p) - TL_A \boldsymbol{d}_2 h(p)\rangle \\ &\quad + \langle \boldsymbol{d}_3 h(A), TL_A \mathrm{Ad}_{A^{-1}} \boldsymbol{d}_1 f(p) - TL_A \boldsymbol{d}_2 f(p)\rangle \\ &= -\pi_1 \cdot (\boldsymbol{d}_1 f(p) \times \boldsymbol{d}_1 h(p)) - \pi_2 \cdot (\boldsymbol{d}_2 f(p) \times \boldsymbol{d}_2 h(p)) \\ &\quad - \boldsymbol{d}_3 f(p) \cdot (A^{-1} \boldsymbol{d}_1 h(p) - \boldsymbol{d}_2 h(p)) \\ &\quad + \boldsymbol{d}_3 h(p) \cdot (A^{-1} \boldsymbol{d}_1 f(p) - \boldsymbol{d}_2 f(p)) \\ &= \boldsymbol{d}_1 f(p) \cdot (\pi_1 \times \boldsymbol{d}_1 h(p)) + \boldsymbol{d}_2 f(p) \cdot (\pi_2 \times \boldsymbol{d}_2 h(p)) \\ &\quad - \boldsymbol{d}_3 f(p) \cdot A^t \boldsymbol{d}_1 h(p) + \boldsymbol{d}_3 f(p) \cdot \boldsymbol{d}_2 h(p) \\ &\quad + \boldsymbol{d}_3 h(p) \cdot A^t \boldsymbol{d}_1 f(p) - \boldsymbol{d}_3 h(p) \cdot \boldsymbol{d}_2 f(p) \\ &= \boldsymbol{d}_1 f(p)^t {\pi_1}^\wedge \boldsymbol{d}_1 h(p) + \boldsymbol{d}_2 f(p)^t {\pi_2}^\wedge \boldsymbol{d}_2 h(p) \\ &\quad - \boldsymbol{d}_3 f(p)^t A^t \boldsymbol{d}_1 h(p) + \boldsymbol{d}_3 f(p)^t \boldsymbol{d}_2 h(p) \\ &\quad + \boldsymbol{d}_1 f(p)^t A \boldsymbol{d}_3 h(p) - \boldsymbol{d}_2 f(p)^t \boldsymbol{d}_3 h(p) \\ &= \boldsymbol{d} f(p)^t\, \Pi(\pi_1, \pi_2, A)\, \boldsymbol{d} h(p),\end{aligned}$$

where Π is the 9×9 matrix

$$\Pi(\pi_1, \pi_2, A) = \begin{bmatrix} {\pi_1}^\wedge & 0 & A \\ 0 & {\pi_2}^\wedge & -\mathrm{Id} \\ -A^t & \mathrm{Id} & 0 \end{bmatrix}.$$

This completes the computation of the Poisson bracket on the Poisson reduced phase space $(\mathbb{R}^3)^2 \times SO(3)$.

As an application of this form, the Hamiltonian vector field of a function f may be computed as follows: if h is an arbitrary function, then

$$\boldsymbol{d} h(p)^t \boldsymbol{X}_f(p) = \{h, f\}(p) = \boldsymbol{d} h(p)^t \Pi(p) \boldsymbol{d} f(p),$$

so that

$$\boldsymbol{X}_f(p) = \Pi(p)\boldsymbol{d}f(p).$$

For yet another application, the symplectic leaves of the reduced phase space are easily determined: By simple row and column operations, the rank of $\Pi(\pi_1, \pi_2, A)$ is found to be 8 if $\pi_1 + A\pi_2 \neq 0$ and 6 otherwise:

$$\begin{bmatrix} \pi_1{}^\wedge & 0 & A \\ 0 & \pi_2{}^\wedge & -\mathrm{Id} \\ -A^t & \mathrm{Id} & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \pi_1{}^\wedge & 0 & A \\ \pi_2{}^\wedge A^t & 0 & -\mathrm{Id} \\ 0 & \mathrm{Id} & 0 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} \pi_1{}^\wedge + A\pi_2{}^\wedge A^t & 0 & A \\ 0 & 0 & -\mathrm{Id} \\ 0 & \mathrm{Id} & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} (\pi_1 + A\pi_2)^\wedge & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{bmatrix}.$$

The momentum map for the $SO(3)$ action on momentum phase space is $A_1\pi_1 + A_2\pi_2$, so a Casimir for the reduced phase space is $c_1 = |\pi_1 + A\pi_2|^2$. Indeed, this can be verified directly by computing that

$$\boldsymbol{d}c_1(\pi_1, \pi_2, A) = \begin{bmatrix} \pi_1 + A\pi_2 \\ A^t\pi_1 + \pi_2 \\ -(A^t\pi_1) \times \pi_2 \end{bmatrix},$$

and then $\boldsymbol{X}_{c_1} = 0$ by matrix multiplication with Π. As the level sets of c_1 are connected imbedded submanifolds which are everywhere the same dimension as the rank of Π, the level sets of c_1 are exactly the symplectic leaves of the reduced phase space.

In a similar way, the map

$$\tau : TSO(3)^2 \to (\mathbb{R}^3)^2 \times SO(3), \quad (A_i, \dot{A}_i) \mapsto ((A_i{}^t\dot{A}_1)^\vee, A_1{}^tA_2)$$

may be used to identify the quotient $TSO(3)^2/SO(3)$ with $(\mathbb{R}^3)^2 \times SO(3)$, and so induces the maps

$$\bar{E} : (\mathbb{R}^3)^2 \times SO(3) \to \mathbb{R}, \quad \bar{FL} : (\mathbb{R}^3)^2 \times SO(3) \to (\mathbb{R}^3)^2 \times SO(3),$$

which are easily calculated: note that $\tau(\mathrm{Id}, A, \Omega_1{}^\wedge A\Omega_2{}^\wedge) = (\Omega_1, \Omega_2, A)$, so

$$\bar{E}(\Omega_i, A) = E(\mathrm{Id}, A, \Omega_1{}^\wedge, A\Omega_2{}^\wedge), \quad \bar{FL}(\Omega_i, A) = \hat{\tau} FL(\mathrm{Id}, A, \Omega_1, A\Omega_2).$$

For the computation of $\bar{E}$, use any of the forms for the total energy in terms of the angular velocities, and for $\bar{FL}$, since FL is fiber preserving, there exist $\pi_1, \pi_2 \in \mathbb{R}^3$ such that $FL(\mathrm{Id}, A, \Omega_1, A\Omega_2) = (\pi_1, T^*L_{A^{-1}}\pi_2)$. Then for arbitrary $w_1, w_2 \in \mathbb{R}^3$

$$\begin{aligned} \begin{bmatrix} w_1{}^t & w_2{}^t \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} &= \pi_1 \cdot w_1 + \pi_2 \cdot w_2 \\ &= \langle FL(\mathrm{Id}, A, \Omega_1, A\Omega_2), (\mathrm{Id}, A, w_1, Aw_2) \rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} L(\mathrm{Id}, A, \Omega_1 + tw_1, A\Omega_2 + tAw_2) \\ &= \left.\frac{d}{dt}\right|_{t=0} \frac{1}{2} \begin{bmatrix} \Omega_1{}^t + tw_1{}^t & \Omega_2{}^t + tw_2{}^t \end{bmatrix} J(A) \begin{bmatrix} \Omega_1 + tw_1 \\ \Omega_2 + tw_2 \end{bmatrix} \\ &= \begin{bmatrix} w_1{}^t & w_2{}^t \end{bmatrix} J(A) \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix}, \end{aligned}$$

so that

$$\bar{FL}(\Omega_1, \Omega_2, A) = \left(J(A) \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix}, A \right).$$

Suppose $J(A)$ is invertible for all $A \in SO(3)$; for example, J_1 and J_2 positive definite will imply this. Then $\bar{FL}$ is a diffeomorphism,

$$\bar{FL}^{-1}(\pi_1, \pi_2, A) = \left(J(A)^{-1} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}, A \right),$$

and

$$\begin{aligned} \bar{H}(\pi_1, \pi_2, A) &= \bar{E} \circ \bar{FL}^{-1}(\pi_1, \pi_2, A) \\ &= \frac{1}{2} \left(J(A)^{-1} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \right)^t J(A) \left(J(A)^{-1} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \right) \\ &= \frac{1}{2} \begin{bmatrix} {\pi_1}^t & {\pi_2}^t \end{bmatrix} J(A)^{-1} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}. \end{aligned}$$

Chapter 2

Axially Symmetric Identical Bodies

The compact expressions obtained for the Lagrangian of two coupled rigid bodies hide some complexity. Indeed, even after we observe that the matrices $\tilde{J}_i$ may be assumed diagonal by choice of the reference configuration, there remain 12 parameters. Worse yet, computations involving the evolution vector fields necessitate the computation of $J(A)^{-1}$ and its derivatives. The subject of this chapter is the case of maximal symmetry which is still nontrivial: the $(S^1)^2$-symmetric case where the bodies are identical. Specifically, assume the following:

- $I_1 = I_2 \stackrel{\text{def}}{=} I$, I is diagonal, and $I^{11} = I^{22}$.
- $d_1 = d_2 \stackrel{\text{def}}{=} l\boldsymbol{k}$.

This is the general identical body $(S^1)^2$-symmetric case, since one may always arrange the initial configuration so that the axes of the bodies are along $\boldsymbol{k}$.

We begin by consolidating the parameters in the Lagrangian. Define

$$\alpha \stackrel{\text{def}}{=} \frac{2I^{11}}{I^{11} + I^{33} + \epsilon l^2}, \quad \beta \stackrel{\text{def}}{=} \frac{\epsilon l^2}{I^{11} + I^{33} + \epsilon l^2}, \quad \tilde{J} \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}. \tag{1}$$

Then,

$$L = (I^{11} + I^{33} + \epsilon l^2)\left(\frac{1}{2}{\Omega_1}^t \tilde{J}\Omega_1 + \frac{1}{2}{\Omega_2}^t \tilde{J}\Omega_2 - \beta A_1(\boldsymbol{k} \times \Omega_1) \cdot A_2(\boldsymbol{k} \times \Omega_2)\right),$$

and

$$J(A) = (I^{11} + I^{33} + \epsilon l^2)\begin{pmatrix} \tilde{J} & \beta \boldsymbol{k}^\wedge A \boldsymbol{k}^\wedge \\ \beta \boldsymbol{k}^\wedge A^t \boldsymbol{k}^\wedge & \tilde{J} \end{pmatrix}.$$

The nonzero multiplier of L and $J(A)$ merely reparameterizes evolution curves, and so may be discarded for the purpose of analyzing the dynamics. Thus, the number of parameters in the Lagrangian is reduced to two, just α and β.

Not every choice of α and β corresponds to a physically reasonable Lagrangian, which we take to be a Lagrangian obtained from parameters in the set

$$PS \stackrel{\text{def}}{=} \{ (I^{11}, I^{33}, \epsilon l^2) \mid I^{11} \geq 0, I^{33} \geq 0, \epsilon l^2 \geq 0 \}.$$

The equations (1) defining α and β map PS onto the triangular set

$$PS' \stackrel{\text{def}}{=} \{ (\alpha, \beta) \mid \alpha \geq 0, \beta \geq 0, \alpha + 2\beta \leq 2 \}.$$

Indeed, $0 \leq \alpha \leq 2$ and $0 \leq \beta \leq 1$ are obvious, and a small amount of algebra shows

$$I^{11}(2 - \alpha - 2\beta) = I^{33}\alpha,$$

so $2 - \alpha - 2\beta \geq 0$ or $I^{11} = 0$. In the latter case, $\alpha = 0$, so

$$2 - \alpha - 2\beta = 2(1 - \beta) \geq 0$$

as well, which shows that equations (1) map PS into PS'. They also map onto PS': a right inverse is

$$\begin{aligned} 2 - \alpha - 2\beta \neq 0 : \quad & I^{11} = \frac{\alpha}{2 - \alpha - 2\beta}, \quad \epsilon l^2 = \frac{2\beta}{2 - \alpha - 2\beta}, \quad I^{33} = 1, \\ 2 - \alpha - 2\beta = 0 : \quad & I^{11} = \alpha, \quad \epsilon l^2 = 2 - \alpha, \quad I^{33} = 0. \end{aligned}$$

In addition to these constraints on α and β, the determinant of the matrix $J(A)$ is

$$\det J(A) = \alpha^2(1 - \beta^2)(1 - \beta^2(A^{33})^2),$$

so FL is a diffeomorphism only for $\alpha \neq 0$ and $\beta \neq 1$. In this chapter, then,

$$\boxed{\begin{gathered} L = \frac{1}{2}\Omega_1{}^t \tilde{J}\Omega_1 + \frac{1}{2}\Omega_2{}^t \tilde{J}\Omega_2 - \beta A_1(\boldsymbol{k} \times \Omega_1) \cdot A_2(\boldsymbol{k} \times \Omega_2) \\ J(A) = \begin{pmatrix} \tilde{J} & \beta \boldsymbol{k}^\wedge A \boldsymbol{k}^\wedge \\ \beta \boldsymbol{k}^\wedge A^t \boldsymbol{k}^\wedge & \tilde{J} \end{pmatrix} \\ \alpha > 0, \quad \beta \geq 0, \quad \alpha + 2\beta \leq 2 \end{gathered}}$$

Some intuition on the meaning of α and β may be obtained by examining the boundary of the set PS':

- $\alpha = 0, \beta = 1$. Then $I^{11} = I^{33} = 0$, so this is the case of two point masses coupled by massless rods.
- $\alpha = 0, \beta \neq 1$. Then $I^{11} = 0$ while $I^{33} \neq 0$. Both bodies are rods with zero diameter.
- $\alpha + 2\beta = 2$, $\alpha \neq 0$. Then $I^{33} = 0$ while $I^{11} \neq 0$. Both bodies are two disks of zero thickness coupled by massless rods through the centers of the disks.
- $\beta = 0$. Then $\epsilon l^2 = 0$; the spherical joint lies at the center of mass, so no torque is exerted by the joint, and the bodies are uncoupled.

Owing to these observations, β will be called the *coupling constant*.

Relative Equilibria

Our first task will be to find the relative equilibria in the $(S^1)^2$-symmetric, identical body case by solving equations of proposition (1.2). Let us say that two relative equilibria are *equivalent* if they lie in the same group orbit. Searching for the relative equilibria on a set which intersects each group orbit only once is an efficient way to prevent consideration of different relative equilibria that are equivalent. Thus, the following proposition is relevant; the idea is that the group action can be used to represent a point in phase space by just a joint angle plus velocities, as long as the bodies are not overlapped or opposed, in which case some velocities belong to the same orbit. To avoid indices upon indices in large expressions, write

$$\Omega_1 \stackrel{\text{def}}{=} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \qquad \Omega_2 \stackrel{\text{def}}{=} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Proposition 1.

1. *Let M be the union of the disjoint subsets of phase space $TSO(3)^2$*

$$\begin{aligned}
M_0 &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \exp(\theta \boldsymbol{j}^\wedge), \Omega_1, \Omega_2) \mid 0 < \theta < \pi \,\}, \\
M_1 &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \mathrm{Id}, \Omega_1, \Omega_2) \mid v_1 > 0, v_2 = 0 \,\}, \\
M_{11} &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \mathrm{Id}, \Omega_1, \Omega_2) \mid v_1 = v_2 = 0, w_1 > 0, w_2 = 0 \,\}, \\
M_{111} &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \mathrm{Id}, \Omega_1, \Omega_2) \mid v_1 = v_2 = w_1 = w_2 = 0 \,\}, \\
M_2 &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \exp(\pi \boldsymbol{j}^\wedge), \Omega_1, \Omega_2) \mid v_1 > 0, v_2 = 0 \,\}, \\
M_{22} &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \exp(\pi \boldsymbol{j}^\wedge), \Omega_1, \Omega_2) \mid v_1 = v_2 = 0, w_1 > 0, w_2 = 0 \,\}, \\
M_{222} &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \exp(\pi \boldsymbol{j}^\wedge), \Omega_1, \Omega_2) \mid v_1 = v_2 = w_1 = w_2 = 0 \,\}.
\end{aligned}$$

Then M intersects each $SO(3) \times (S^1)^2$ orbit exactly once.

2. *Let M' be the union of the disjoint subsets of configuration space $SO(3)^2$*

$$\begin{aligned}
M'_0 &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \exp(\theta \boldsymbol{j}^\wedge)) \mid 0 < \theta < \pi \,\}, \\
M'_1 &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \mathrm{Id}) \,\}, \\
M'_2 &\stackrel{\text{def}}{=} \{\, (\mathrm{Id}, \exp(\pi \boldsymbol{j}^\wedge)) \,\}.
\end{aligned}$$

Then M' intersects each $SO(3) \times (S^1)^2$ orbit exactly once.

Proof. First, each orbit intersects M at least once: from lemma (1.1), the $SO(3) \times (S^1)^2$ action is

$$(B, \theta_1, \theta_2) \cdot (A_i, \Omega_i) = (BA_i \exp(-\theta_i \boldsymbol{k}^\wedge), \exp(\theta_i \boldsymbol{k}^\wedge)\Omega_i).$$

Parameterize ${A_1}^t A_2$ by Euler angles ϕ_1, ϕ_2, ϕ_3:

$${A_1}^t A_2 = \exp(\phi_1 \boldsymbol{k}^\wedge) \exp(\phi_2 \boldsymbol{j}^\wedge) \exp(\phi_3 \boldsymbol{k}^\wedge).$$

Then

$$(\exp(-\phi_1 \boldsymbol{k}^\wedge)A_1{}^t, -\phi_1, \phi_3) \cdot (A_i, \Omega_i) = (\mathrm{Id}, \exp(\phi_2 \boldsymbol{j}^\wedge), \exp(-\phi_1 \boldsymbol{k}^\wedge)\Omega_1, \exp(\phi_3 \boldsymbol{k}^\wedge)\Omega_2),$$

so if $\phi_2 \neq 0$ and $\phi_2 \neq \pi$ then the orbit of (A_i, Ω_i) meets M_0, while otherwise this shows $A_1 = A_2 = \mathrm{Id}$ or $A_1 = \mathrm{Id}$ and $A_2 = \exp(\pi \boldsymbol{j}^\wedge)$ without loss of generality. These two cases are similar, so assume the former. Note that, for arbitrary θ,

$$(\exp(\theta \boldsymbol{j}^\wedge), \theta, \theta) \cdot (\mathrm{Id}, \mathrm{Id}, \Omega_i) = (\mathrm{Id}, \mathrm{Id}, \exp(\theta \boldsymbol{k}^\wedge)\Omega_i),$$

and that $\exp(\theta \boldsymbol{k}^\wedge)$ is the rotation of angle θ about $\boldsymbol{k}$. Thus, if Ω_1 is not parallel to $\boldsymbol{k}$, then there is a θ so that the projection of $\exp(\theta \boldsymbol{k}^\wedge)\Omega_1$ on the $\boldsymbol{i}, \boldsymbol{j}$ plane is aligned with $\boldsymbol{i}$, and then the orbit of $(\mathrm{Id}, \mathrm{Id}, \Omega_i)$ meets M_1. Similarly, if Ω_1 is parallel to $\boldsymbol{k}$ but Ω_2 is not, then the orbit meets M_{11}. Finally, if Ω_1 and Ω_2 are both parallel to $\boldsymbol{k}$, then (A_i, Ω_i) is already in M_{111}.

To show that each orbit meets M only once, note that the functions

$$A_1\boldsymbol{k} \cdot A_2\boldsymbol{k}, \quad v_1{}^2 + v_2{}^2, \quad w_1{}^2 + w_2{}^2$$

are $SO(3) \times (S^1)^2$ invariant and have distinct values on the M_n. Therefore, the sets $SO(3) \times (S^1)^2 \cdot M_n$ are disjoint, and one is reduced to showing that the orbit of a point in one of the M_n meets that same set only once. For the set M_0, suppose $0 < \theta < \pi$, and that for some $0 < \bar{\theta} < \pi$ and $\bar{\Omega}_i \in \mathbb{R}^3$,

$$\begin{aligned} &(B, \theta_1, \theta_2) \cdot (\mathrm{Id}, \exp(\theta \boldsymbol{j}^\wedge), \Omega_i) \\ &= (B\exp(-\theta_1 \boldsymbol{k}^\wedge), B\exp(\theta \boldsymbol{j}^\wedge)\exp(-\theta_2 \boldsymbol{k}^\wedge), \exp(\theta_i \boldsymbol{k}^\wedge)\Omega_i) \\ &= (\mathrm{Id}, \exp(\bar{\theta} \boldsymbol{j}^\wedge), \bar{\Omega}_i). \end{aligned}$$

Then, by comparing the first components, $B = \exp(\theta_1 \boldsymbol{k}^\wedge)$, so the second components give

$$\exp(\theta_1 \boldsymbol{k}^\wedge)\exp(\theta \boldsymbol{j}^\wedge)\exp(-\theta_2 \boldsymbol{k}^\wedge) = \exp(\bar{\theta} \boldsymbol{j}^\wedge),$$

which are two Euler angle representations of an element of $SO(3)$ that is not a rotation about $\boldsymbol{k}$. Thus, $\exp(\theta_i \boldsymbol{k}^\wedge) = \mathrm{Id}$ and $\theta = \bar{\theta}$, and the proof for the set M_0 is complete. The remaining M_n are even easier to deal with, the proof amounting just to the observation that if one of the Ω_i is not parallel to $\boldsymbol{k}$, then the rotation about $\boldsymbol{k}$ that aligns the projection of that Ω_i in the $\boldsymbol{i}, \boldsymbol{j}$ plane with $\boldsymbol{i}$ is unique.

The second statement of the proposition is proved similarly. □

Concerning the proof of lemma (1), an attentive reader may have noticed that some points of phase space and some configurations admit nontrivial isotropy by the action of $SO(3) \times (S^1)^2$. It is vital to recognize the symmetry of points in phase space and configuration space: Symmetric points of phase space are exactly the places that the constant levels of momentum mappings cease to be locally Euclidean, and reduction fails to be well defined [2]. The cotangent bundle reduction theorem [1] fails precisely at symmetric points of configuration space, a fact which complicates the determination of the nonlinear stability of relative equilibria overtop of symmetric configurations. Near symmetric configurations, the absence of a section to the group action prevents simple-minded approaches to obtaining group invariant generating functions, and this in turn bears on the construction of momentum preserving symplectic integrators. The following proposition identifies these interesting points; the proof is an easy exercise in the use of Euler angles.

Proposition 2.

1. *The set of points of phase space* $TSO(3)^2$ *with nontrivial isotropy is the* $SO(3)\times(S^1)^2$ *orbit of the set* $M_{111}\cup M_{222}$. *The isotropy group of an element in* M_{111} *is*

$$G_1 \stackrel{\text{def}}{=} \{\, (\exp(\theta \boldsymbol{k}^\wedge), -\theta, -\theta) \mid \theta\in\mathbb{R}\,\}\cong S^1,$$

while the isotropy group of an element in M_{222} *is*

$$G_2 \stackrel{\text{def}}{=} \{\, (\exp(\theta \boldsymbol{k}^\wedge), \theta, -\theta) \mid \theta\in\mathbb{R}\,\}\cong S^1.$$

2. *The set of points of configuration space* $SO(3)^2$ *with nontrivial isotropy is the* $SO(3)\times(S^1)^2$ *orbit of the set* $M'_1\cup M'_2$. *The isotropy group of an element in* M'_i *is* G_i.

The next proposition explicitly gives all the relative equilibria in M, and hence, by $SO(3)\times(S^1)^2$ translation, all relative equilibria on the phase space $TSO(3)^2$.

Proposition 3. *For typographical convenience, define the coefficient*

$$\kappa_\gamma^{t_1t_2t_3} \stackrel{\text{def}}{=} \frac{(t_1\cos t_3 - t_2)(\beta t_2-\gamma t_1)}{\alpha t_1 \sin t_3}.$$

Then for every triple $t_1,t_2,t_3\in\mathbb{R}$, *the point* $(\mathrm{Id},\exp(\theta\boldsymbol{j}^\wedge),\Omega_1,\Omega_2)$ *calculated in the following list is a relative equilibrium:*

$$\theta=0,\Omega_1=t_1\boldsymbol{k},\Omega_2=t_2\boldsymbol{k} \tag{2a}$$
$$\theta=\pi,\Omega_1=t_1\boldsymbol{k},\Omega_2=t_2\boldsymbol{k} \tag{2b}$$
$$\theta=0,\Omega_1=t_1\boldsymbol{i}+t_2\boldsymbol{k},\Omega_2=\Omega_1;\ 0<t_1 \tag{2c}$$
$$\theta=\pi,\Omega_1=t_1\boldsymbol{i}+t_2\boldsymbol{k},\Omega_2=-\Omega_1;\ 0<t_1 \tag{2d}$$
$$\theta=t_3,\Omega_1=t_1\boldsymbol{k},\Omega_2=t_2\boldsymbol{k};\ 0<t_3<\pi \tag{2e}$$
$$\theta=t_3,\Omega_1=t_1\boldsymbol{i}-\kappa_1^{t_1t_2t_3}\boldsymbol{k},\Omega_2=t_2\boldsymbol{i}+\kappa_1^{t_2t_1t_3}\boldsymbol{k};\ t_1\neq 0,t_2\neq 0,0<t_3<\pi \tag{2f}$$
$$\beta=0,\theta=t_3,\Omega_1=t_1\boldsymbol{k},\Omega_2=t_2(\alpha\sin t_3\,\boldsymbol{i}-\cos t_3\,\boldsymbol{k});\ t_2\neq 0,0<t_3<\pi \tag{2g}$$
$$\beta=0,\theta=t_3,\Omega_1=t_1(\alpha\sin t_3\,\boldsymbol{i}+\cos t_3\,\boldsymbol{k}),\Omega_2=t_2\boldsymbol{k};\ t_1\neq 0,0<t_3<\pi \tag{2h}$$

No two distinct relative equilibria in this list are equivalent.

Proof. When restricted to M, and in the symmetric case being considered, the equations (1.1a)–(1.1c) and (1.2c) of proposition (1.2) become

$$\Omega\times\left(\tilde J\Omega_1+\beta(\boldsymbol{k}\cdot\Omega)A_2\boldsymbol{k}\right)=0, \tag{3a}$$
$$\Omega\times\left(A_2\tilde J\Omega_2+\beta((A_2\boldsymbol{k})\cdot\Omega)\boldsymbol{k}\right)=0, \tag{3b}$$
$$\Omega\cdot(\boldsymbol{k}\times A_2\boldsymbol{k})=0, \tag{3c}$$
$$\Omega_1=\Omega-\sigma_1\boldsymbol{k}, \tag{3d}$$
$$\Omega_2={A_2}^t\Omega-\sigma_2\boldsymbol{k}. \tag{3e}$$

The task is to find all $(A_i, \Omega_i) \in M$ such that there exist Ω and σ_i with (3a)–(3e) satisfied. There are three distinct cases:

Case 1. $(A_i, \Omega_i) \in M_0$. Then equation (3c) is

$$\Omega \cdot \boldsymbol{k} \times \exp(\theta \boldsymbol{j}^\wedge)\boldsymbol{k} = \Omega \cdot \boldsymbol{j} \sin\theta = 0,$$

so (3c) is equivalent to $\Omega \cdot \boldsymbol{j} = 0$, since $\sin\theta \neq 0$ on M_0. By (3d) and (3e), it follows that $v_2 = w_2 = 0$:

$$v_2 = \boldsymbol{j} \cdot \Omega_1 = \boldsymbol{j} \cdot (\Omega - \sigma_1 \boldsymbol{k}) = 0,$$
$$w_2 = \boldsymbol{j} \cdot \Omega_2 = \boldsymbol{j} \cdot (\exp(-\theta \boldsymbol{j}^\wedge)\Omega - \sigma_2 \boldsymbol{k}) = \Omega \cdot \boldsymbol{j} = 0.$$

Eliminating Ω from (3d) and (3e) yields $A_2\Omega_2 - \Omega_1 = \sigma_1 \boldsymbol{k} - \sigma_2 A_2 \boldsymbol{k}$; that is

$$\begin{bmatrix} w_1\cos\theta + w_3\sin\theta - v_1 \\ 0 \\ -w_1\sin\theta + w_3\cos\theta - v_3 \end{bmatrix} = \begin{bmatrix} -\sigma_2\sin\theta \\ 0 \\ \sigma_1 - \sigma_2\cos\theta \end{bmatrix},$$

which may be solved for σ_1 and σ_2:

$$\sigma_1 = -v_3 + \frac{v_1\cos\theta - w_1}{\sin\theta},$$
$$\sigma_2 = -w_3 + \frac{v_1 - w_1\cos\theta}{\sin\theta}.$$

By (3d), then,

$$\Omega = \Omega_1 + \sigma_1 \boldsymbol{k} = v_1 \boldsymbol{i} + \frac{v_1\cos\theta - w_1}{\sin\theta}\boldsymbol{k}.$$

There remain equations (3a) and (3b). Notice that both vectors in the cross product in (3a) are orthogonal to $\boldsymbol{j}$, so only the $\boldsymbol{j}$ component of that product can be nonzero. This component is easily computed:

$$\begin{aligned} \boldsymbol{j} \cdot \Omega \times (\tilde{J}\Omega_1 + \beta(\boldsymbol{k}\cdot\Omega)A_2\boldsymbol{k}) &= \begin{vmatrix} 0 & 1 & 0 \\ v_1 & 0 & \boldsymbol{k}\cdot\Omega \\ v_1 + \beta(\boldsymbol{k}\cdot\Omega)\sin\theta & 0 & \alpha v_3 + \beta(\boldsymbol{k}\cdot\Omega)\cos\theta \end{vmatrix} \\ &= -\alpha v_1 v_3 - (\boldsymbol{k}\cdot\Omega)(\beta v_1\cos\theta - v_1 - \beta(\boldsymbol{k}\cdot\Omega)\sin\theta) \\ &= -\alpha v_1 v_3 - \frac{(v_1\cos\theta - w_1)(\beta w_1 - v_1)}{\sin\theta}, \end{aligned}$$

which shows equation (3a) may be replaced by

$$v_1 v_3 = -\frac{(v_1\cos\theta - w_1)(\beta w_1 - v_1)}{\alpha\sin\theta}. \tag{4}$$

Similarly, one uses (3e) and ${A_2}^t$ times (3b) to replace (3b) by

$$w_1 w_3 = \frac{(w_1\cos\theta - v_1)(\beta v_1 - w_1)}{\alpha\sin\theta}. \tag{5}$$

If $\beta \neq 0$, then (4) and (5) imply that $v_1 = 0$ if and only if $w_1 = 0$. Thus, if $\beta \neq 0$, there are two subcases: $v_1 = w_1 = 0$, so (4) and (5) are satisfied for arbitrary v_3 and w_3, and there results (2e), or $v_1 \neq 0$ and $w_1 \neq 0$, so (4) and (5) may be divided by v_1 and w_1 respectively, and there results (2f). If $\beta = 0$ then one has these solutions, as well as two other subcases: $v_1 = 0$ and $w_1 \neq 0$ or $v_1 \neq 0$ and $w_1 = 0$. For the former, (4) is satisfied and (5) yields

$$w_3 = -\frac{w_1 \cos\theta}{\alpha \sin\theta},$$

and so there results (2g). Similarly, the latter subcase results in (2h).

Case 2. $(A_i, \Omega_i) \in M_1 \cup M_{11} \cup M_{111}$. Then equation (3c) is satisfied, since $A_2 = \mathrm{Id}$. By subtracting (3d) and (3e), and by observing that $v_2 = 0$ by definition of the M_i, (3d) and (3e) may be replaced by

$$v_1 = w_1 = \Omega^1, \quad v_3 = \Omega^3 - \sigma_1, \quad w_3 = \Omega^3 - \sigma_2. \tag{6}$$

Thus, as in case (1), there remain (3a) and (3b), and only the j component of (3a) is nonzero. Its value is, using (3d) to write Ω in terms of Ω_1 and σ_1,

$$\begin{vmatrix} 0 & 1 & 0 \\ v_1 & 0 & v_3 + \sigma_1 \\ v_1 & 0 & \alpha v_3 + \beta(v_3 + \sigma_1) \end{vmatrix} = v_1(v_3(1 - \alpha - \beta) + \sigma_1(1 - \beta)),$$

and similarly with (3b). Thus, (3a) and (3b) may be replaced by

$$v_1(v_3(1 - \alpha - \beta) + \sigma_1(1 - \beta)) = 0, \tag{7}$$

$$w_1(w_3(1 - \alpha - \beta) + \sigma_2(1 - \beta)) = 0. \tag{8}$$

Now from (6), either $v_1 = w_1 = 0$ or $v_1 \neq 0$ and $w_1 \neq 0$. The most general solution to the former results in (2a), since (7) and (8) are satisfied, and (6) may be satisfied with arbitrary v_3 and w_3 by setting $\Omega^3 = 0$, $\sigma_1 = -v_3$ and $\sigma_2 = -w_3$. The latter results in (2c): divide (7) and (8) by v_1 and w_1 respectively and subtract the results to obtain

$$(1 - \alpha - \beta)(v_3 - w_3) + (1 - \beta)(\sigma_1 - \sigma_2) = 0. \tag{9}$$

Then by (6), $v_3 - w_3 = -(\sigma_1 - \sigma_2)$, so (9) gives $\alpha(v_3 - w_3) = 0$, which implies $v_3 = w_3$ and $\sigma_1 = \sigma_2$. Thus, the most general solution is obtained by choosing $v_3 = w_3$ arbitrarily, setting

$$\sigma_1 = \sigma_2 = \frac{\alpha + \beta - 1}{1 - \beta} v_3,$$

and then from (6)

$$\Omega^3 = \sigma_1 + v_3 = \frac{\alpha v_3}{1 - \beta}.$$

Case 3. $(A_i, \Omega_i) \in M_2 \cup M_{22} \cup M_{222}$. In a way similar to case (2), equations (3a)–(3e) may be replaced with the following analogues of (6), (7) and (8):

$$v_1 = -w_1 = \Omega^1, \quad v_3 = \Omega^3 - \sigma_1, \quad w_3 = -\Omega^3 - \sigma_2,$$

$$v_1(v_3(1 - \alpha + \beta) + \sigma_1(1 + \beta)) = 0,$$

$$w_1(w_3(1 - \alpha + \beta) + \sigma_2(1 + \beta)) = 0.$$

Then $v_1 = w_1 = 0$ gives the solution $\Omega^3 = 0, \sigma_1 = -v_3, \sigma_2 = -w_3$, resulting in (2b), while $v_1 \neq 0$ and $w_1 \neq 0$ gives

$$\sigma_1 = -\sigma_2 = \frac{\alpha - \beta - 1}{1+\beta} v_3, \quad \Omega^3 = \frac{\alpha}{1+\beta} v_3,$$

resulting in (2d). □

Remark. In [15] one finds an identical list with (2g) and (2h) omitted. That error is not serious, for the $\beta = 0$ case is $SO(3)^2 \times (S^1)^2$-symmetric, so it would be unwise to deal with (2g) and (2h) with the less symmetric $SO(3) \times (S^1)^2$ techniques.

Remark. Let us call a relative equilibrium *regular* if its isotropy is discrete, otherwise it will be *degenerate*, and call it *symmetric* if it has nontrivial isotropy. From proposition (2) it is evident that the set of points of phase space which admit nontrivial continuous isotropy is exactly the set of degenerate relative equilibria (2a) and (2b).

Interpreting these relative equilibria is not difficult. In the course of proving proposition (3), at least one triple Ω, σ_i has been constructed for each relative equilibrium (A_i, Ω_i). By theorem (1.1), the Lie algebra elements Ω, σ_i give the motion, which may be visualized by thinking of Ω as the axis of overall rotation with angular velocity $|\Omega|$, and σ_i as the spins of the bodies. These important parameters are listed in the next proposition:

Proposition 4. *In correspondence with the list of relative equilibria* (2a)-(2h), *the Lie algebra elements* Ω, σ_i *are*

$$\sigma_1 = -t_1, \sigma_2 = -t_2, \Omega = 0 \tag{10a}$$

$$\sigma_1 = -t_1, \sigma_2 = -t_2, \Omega = 0 \tag{10b}$$

$$\sigma_1 = \sigma_2 = \frac{(\alpha + \beta - 1)t_2}{1-\beta}, \Omega = t_1 \boldsymbol{i} + \frac{\alpha t_2}{1-\beta} \boldsymbol{k} \tag{10c}$$

$$\sigma_1 = -\sigma_2 = \frac{(\alpha - \beta - 1)t_2}{1+\beta}, \Omega = t_1 \boldsymbol{i} + \frac{\alpha t_2}{1+\beta} \boldsymbol{k} \tag{10d}$$

$$\sigma_1 = -t_1, \sigma_2 = -t_2, \Omega = 0 \tag{10e}$$

$$\sigma_1 = \kappa_{1-\alpha}^{t_1 t_2 t_3}, \sigma_2 = -\kappa_{1-\alpha}^{t_2 t_1 t_3}, \Omega = t_1 \boldsymbol{i} + \frac{t_1 \cos t_3 - t_2}{\sin t_3} \boldsymbol{k} \tag{10f}$$

$$\sigma_1 = -t_1 - \alpha t_2, \sigma_2 = (1-\alpha) t_2 \cos t_3, \Omega = \alpha t_2 \boldsymbol{k} \tag{10g}$$

$$\sigma_1 = -(1-\alpha) t_1 \cos t_3, \sigma_2 = -t_2 + \alpha t_1, \Omega = \alpha t_1 \exp(t_3 \boldsymbol{j}^\wedge) \boldsymbol{k} \tag{10h}$$

For rough intuition, the relative equilibria are as follows:

- *Classes* (10a),(10b) *and* (10e). *The angle that the bodies make with the joint is arbitrary, they are both spinning, and there is no overall rotation.*
- *Class* (10c). *The bodies are aligned on top of one another, with identical spin, so they rotate together as a unit, and the entire ensemble is rotating about some axis which is tilted with respect to the body axes.*

- *Class* (10d). *The bodies are aligned opposed to one another, with opposite spin, so that they rotate together as a unit, and the entire ensemble is rotating about some axis which is tilted with respect to the body axes.*

- *Class* (10f). *The joint angle is arbitrary, the bodies are spinning, and there is some overall rotation depending on the spin and the joint angle.*

- *Classes* (10g) *and* (10h). *Recall that a single axially symmetric rigid bodies' general motion consists of spinning while the symmetry axis precesses about some line in space. The relative equilibria in the classes* (10g) *and* (10h) *consist of setting one of the bodies in this state and the other body with arbitrary spin and axis of symmetry along the precession axis of the first body.*

Stability

Before considering the stability of the equilibria, we embark on a theoretical excursion. First, the meaning of stability itself: orbital stability of relative equilibria in Hamiltonian systems with symmetry cannot be expected—loosely speaking, the motion close to a relative equilibrium tends to drift in the direction of the isotropy group of the momentum. The following is a weaker notion:

Definition 1. *Let (P, ω, H, G) be a Hamiltonian system with symmetry and let G' be a subgroup of G. Then a relative equilibrium z_e is called G'-stable, or stable modulo G',if for all G' invariant open neighborhoods V of $G' \cdot z_e$, there is an open neighborhood $U \subseteq V$ of z_e which is invariant under the Hamiltonian evolution.*

To avoid redundantly specifying the subgroup G', we adopt the convention that a relative equilibrium with momentum μ is *stable* if it is G_μ stable. Also, if G' is compact, then any open neighborhood of $G' \cdot z_e$ contains a G' invariant open neighborhood of $G' \cdot z_e$ (use the tube lemma of elementary topology [14]), so that in definition (1) the phrase "G' invariant open neighborhoods V" may be replaced with "open neighborhoods V" in that case.

In the process of determining the stability of relative equilibria, the following easy lemma is often useful:

Lemma 1. *Let A and B be bilinear forms on a finite dimensional vector space. Suppose that A is positive semidefinite and that B is positive definite on $\ker A$. Then there exists an $r > 0$ such that $A + \epsilon B$ is positive definite for all $\epsilon \in (0, r)$.*

Proof. Let the vector space be E, let $|\cdot|$ be a norm on E, and write $E = E' \oplus \ker A$. Then A is positive definite on E', so there is a constant $c_1 > 0$ such that

$$A(x_1, x_1) \geq c_1 |x_1|^2 \qquad \forall x_1 \in E'.$$

Also choose $M > 0$ and $c_2 > 0$ so that

$$\begin{aligned} B(x_2, x_2) &\geq c_2 |x_2|^2 \qquad &\forall x_2 \in \ker A, \\ |B(x, y)| &\leq M |x||y| \qquad &\forall x, y \in E. \end{aligned}$$

Then if $x_1 \in E'$ and $x_2 \in \ker A$,

$$\begin{aligned}(A+\epsilon B)(x_1+x_2, x_1+x_2) &= A(x_1,x_1)+\epsilon B(x_1,x_1)+2\epsilon B(x_1,x_2)+\epsilon B(x_2,x_2)\\ &\geq c_1|x_1|^2-\epsilon M|x_1|^2-2\epsilon M|x_1||x_2|+\epsilon c_2|x_2|^2\\ &= (c_1-\epsilon M)|x_1|^2-2\epsilon M|x_1||x_2|+\epsilon c_2|x_2|^2.\end{aligned}$$

Viewed as a quadratic polynomial in $|x_1|$ and $|x_2|$, the discriminant of the last expression is

$$\begin{aligned}&4\epsilon^2M^2-4(c_1-\epsilon M)\epsilon c_2\\ &\quad= -4\epsilon(c_1c_2-\epsilon M(M+c_2)),\end{aligned}$$

which is negative as long as $\epsilon < c_1c_2/M(M+c_2)$. □

Remark. The following alternative proof was suggested by Alan Weinstein: use bilinearity to reduce the domains to a unit sphere in E. Then use this fact: on a compact space S, if $f_1 : S \to \mathbb{R}$ is continuous and nonnegative and $f_2 : S \to \mathbb{R}$ is continuous and positive on $f_1^{-1}(0)$, then $f_1+\epsilon f_2$ is positive for all sufficiently small positive ϵ.

The next result, inspired by the energy momentum method [13], gives sufficient conditions for a relative equilibrium to be stable.

Theorem 1. *Let (P,ω,H,G,J) be a Hamiltonian system with symmetry. Suppose z_e is a regular relative equilibrium with evolution $t \mapsto \exp(\xi_e t)\cdot z_e$, $J(z_e)=\mu_e$, the action of G_{μ_e} on P is proper, and $\mathfrak{g}$ admits an inner product invariant under the* Ad *action of G_{μ_e}. Then $\mathbf{d}(H-J_{\xi_e})(z_e)=0$, and z_e is stable if it is formally stable; that is, if $\mathbf{d}^2(H-J_{\xi_e})(z_e) \mid T_{z_e}J^{-1}(\mu_e)$ is positive or negative definite on some (and hence any) complement to $\mathfrak{g}_{\mu_e}\cdot z_e$ in $T_{z_e}J^{-1}(\mu_e)$.*

Proof. That $\mathbf{d}(H-J_{\xi_e})(z_e)=0$ is a trivial computation using $\mathbf{X}_H(z_e)=\xi_e(z_e)$. It is also easy to see that the kernel of $\mathbf{d}^2(H-J_{\xi_e})(z_e) \mid T_{z_e}J^{-1}(\mu_e)$ contains $\mathfrak{g}_{\mu_e}\cdot z_e$: if $\eta \in \mathfrak{g}_{\mu_e}$ and $v \in T_{z_e}J^{-1}(\mu_e)$, then

$$\begin{aligned}\mathbf{d}^2(H-J_{\xi_e}(z_e))(\eta(z_e),v) &= \mathbf{d}(\mathbf{i}_\eta\mathbf{d}(H-J_{\xi_e}))(z_e)v\\ &= \mathbf{d}\{J_{\xi_e},J_\eta\}\\ &= \mathbf{d}J_{[\xi_e,\eta]}(z_e)v\\ &= 0.\end{aligned}$$

Thus if $\mathbf{d}^2(H-J_{\xi_e})(z_e)$ is definite on one complement to $\mathfrak{g}_{\mu_e}\cdot z_e$ in $T_{z_e}J^{-1}(\mu_e)$, then it is definite on any such complement.

Now for the proof that z_e is stable. Obviously, the positive definite case may be assumed without loss of generality. The proof is obtained by modifying $H-J_{\xi_e}$, thereby constructing a G_{μ_e} invariant function f in an open neighborhood of $G_{\mu_e}\cdot z_e$ which has $G_{\mu_e}\cdot z_e$ as a manifold of critical points and positive definite Hessian in directions complementary to $G_{\mu_e}\cdot z_e$. The Morse lemma is then used on a submanifold tangent to these complementary directions, and the proof is completed by establishing control on the time evolution of the function f.

Since the action of G_{μ_e} on P is proper, it admits a relatively compact slice at z_e; that is, there is a submanifold S containing z_e with compact closure and a map χ from an open neighborhood U_{z_e} of z_e in $G_{\mu_e}\cdot z_e$ to G_{μ_e} such that:

- If $gz_e = z_e$ then $gS = S$.
- If $gS \cap S \neq \emptyset$ then $gz_e = z_e$.
- The map χ satisfies the following: $\chi(z_e) = \mathrm{Id}$, $\chi(u)z_e = u$ for all $u \in U_{z_e}$ and the map $U_{z_e} \times S \to P$ by $(u, z) \mapsto \chi(u) \cdot z$ is a diffeomorphism from $U_{z_e} \times S$ to some open neighborhood of z_e.

Indeed, S may be constructed as follows: the isotropy group I_{z_e} of z_e in $G_{\mu_e} \cdot z_e$ is compact since the G_{μ_e} action on P is proper. Thus, there is a Riemannian metric on P such that the I_{z_e} action is isometric. Then S can be set the the image under the metric exponential map of a sufficiently small ball in the orthogonal complement of $\mathfrak{g}_{\mu_e} \cdot z_e$ (it is the second property requires the assumption that the action is proper.)

Note that $G_{\mu_e} \cdot S$ is an open neighborhood of $G_{\mu_e} \cdot z_e$. Construct $\pi : G_{\mu_e} \cdot S \to G_{\mu_e} \cdot z_e$ by the requirement

$$\pi(gz) = gz_e \qquad \forall z \in S, g \in G_{\mu_e}.$$

The map π is well defined since if $gz = g'z'$ then $(g^{-1}g'S) \cap S \neq \emptyset$, so $g^{-1}g'z_e = z_e$ and hence $gz_e = g'z_e$. Also, π is smooth since, by the definition of a slice, it is locally just a projection. Now every point in $G_{\mu_e} \cdot z_e$ is a regular relative equilibrium, so there is a smooth function $\tilde{\Psi} : G_{\mu_e} \to \mathfrak{g}$ such that $\mathbf{X}_H(u) = \tilde{\Psi}(u)(u)$; that is, the evolution of $u \in G_{\mu_e} \cdot z_e$ is $t \mapsto \exp(\tilde{\Psi}(u)t)u$. It is immediate from this definition that $\tilde{\Psi}(gz) = \mathrm{Ad}_g \tilde{\Psi}(z)$ for all $g \in G_{\mu_e}$; thus the map $\Psi \stackrel{\mathrm{def}}{=} \tilde{\Psi} \circ \pi$ has this property too, since π intertwines the action of G_{μ_e}. To summarize, we have constructed a map $\Psi : G_{\mu_e} \cdot S \to \mathfrak{g}$ such that

$$\Psi(gx) = \mathrm{Ad}_g \Psi(x) \qquad \forall g \in G_{\mu_e}, \tag{11}$$

and

$$\Psi(z_e) = \xi_e, \quad \mathrm{Image}\Psi = G_{\mu_e} \cdot \xi_e, \quad \mu_e \circ \Psi = \langle \mu_e, \xi_e \rangle. \tag{12}$$

Consider the function $f_1 = H - J_\Psi + \langle \mu_e, \xi_e \rangle - H(z_e)$. First, f_1 is G_{μ_e} invariant: if $g \in G_{\mu_e}$ then

$$\begin{aligned} f_1(gx) - f_1(x) &= \langle J(gx), \Psi(gx) \rangle - \langle J(x), \Psi(x) \rangle \\ &= \langle \mathrm{CoAd}_g J(x), \mathrm{Ad}_g \Psi(x) \rangle - \langle J(x), \Psi(x) \rangle \\ &= 0. \end{aligned}$$

Also, $\boldsymbol{d}f_1(z_e) = 0$: let $c(t)$ be a curve at z_e tangent to $v \in T_{z_e}P$. Then

$$\begin{aligned} \boldsymbol{d}f_1(z_e)v &= \boldsymbol{d}H(z_e)v - \left.\frac{d}{dt}\right|_{t=0} \langle J(c(t)), \Psi(c(t)) \rangle \\ &= \boldsymbol{d}H(z_e)v - \langle \boldsymbol{d}J(z_e)v, \Psi(z_e) \rangle - \left.\frac{d}{dt}\right|_{t=0} \langle \mu_e, \Psi(c(t)) \rangle \\ &= [\boldsymbol{d}H(z_e)v - \boldsymbol{d}J_{\xi_e}(z_e)v] - \left.\frac{d}{dt}\right|_{t=0} \langle \mu_e, \xi_e \rangle \\ &= 0. \end{aligned}$$

Additionally, define the function $f_2 = |J - \mu_e|^2$, where the norm is obtained from the CoAd invariant inner product induced from the hypothesized Ad invariant inner product on $\mathfrak{g}$. Obviously, f_2 shares with f_1 the properties that it is G_{μ_e} invariant and has zero derivative at z_e. Now let Y be a complement to $T_{z_e}J^{-1}(\mu_e)$ in $T_{z_e}S$; that is, suppose

$$T_{z_e}S = (T_{z_e}S \cap T_{z_e}J^{-1}(\mu_e)) \oplus Y \stackrel{\text{def}}{=} Z \oplus Y.$$

Then Z is a complement to $\mathfrak{g}_{\mu_e} \cdot z_e$ in $T_{z_e}J^{-1}(\mu_e)$ and one computes that f_1 and $H - J_{\xi_e}$ differ be a constant on S, so by hypothesis $\boldsymbol{d}^2(f_1|S)(z_e)$ is positive definite on Z. Moreover, $\boldsymbol{d}^2(f_2|S)(z_e)$ is positive semidefinite and has kernel Z. Thus, by lemma (1), there is an $a \in \mathbb{R}$ such that $f = af_1 + f_2$ has $\boldsymbol{d}^2(f|S)(z_e)$ positive definite.

Then, given a G_{μ_e} invariant neighborhood V of $G_{\mu_e} \cdot z_e$, one can use the Morse lemma, and perhaps shrink S, to find an $\epsilon > 0$ such that $f \geq 0$ on S and

$$f^{-1}[0,\epsilon) \cap S \subseteq V, \tag{13}$$

$$\mathrm{Cl}_P(f^{-1}[0,\epsilon) \cap S) \subseteq S. \tag{14}$$

Concerning the time evolution of f, there is the following estimate: if F_t is the Hamiltonian flow, if $z \in S$, and if $F_t(z) \in G_{\mu_e} \cdot S$, then

$$\begin{aligned} f(F_t(z)) - f(z) &= J_\Psi(F_t(z)) - J_\Psi(z) \\ &= \langle J(F_t(z)), \Psi(F_t(z))\rangle - \langle J(z), \Psi(z)\rangle \\ &= \langle J(z) - \mu_e, \Psi(F_t(z)) - \Psi(z)\rangle + \langle \mu_e, \Psi(F_t(z))\rangle - \langle \mu_e, \Psi(z)\rangle \\ &= \langle J(z) - \mu_e, \Psi(F_t(z)) - \Psi(z)\rangle, \end{aligned}$$

since the evaluation of μ on the image of Ψ is $\langle \mu_e, \xi_e\rangle$. Thus,

$$\begin{aligned} 0 \leq f(F_t(z)) &\leq f(z) + |\langle J(z) - \mu_e, \Psi(F_t(z)) - \Psi(z)\rangle| \\ &\leq f(z) + |J(z) - \mu_e|\,(|\Psi(F_t(z))| + |\Psi(z)|) \\ &= f(z) + 2|\xi_e||J(z) - \mu_e| \end{aligned} \tag{15}$$

By continuity of f and J, there is some neighborhood $S' \subseteq S$ of z_e such that $|f(z)| \leq \epsilon/2$ and $|J(z) - \mu_e| \leq \epsilon/4|\xi_e|$ on S'. The proof will be complete if it is shown that

$$F_t(S') \subseteq f^{-1}[0,\epsilon] \cap G_{\mu_e} \cdot S \stackrel{\text{def}}{=} A, \tag{16}$$

for then $U \stackrel{\text{def}}{=} \bigcup_t F_t(G_{\mu_e} \cdot S') \subseteq A \subseteq V$, by (13) and G_{μ_e} invariance of everything in sight, and U is invariant under the Hamiltonian flow. To show (16), *suppose it is false for some positive* t. Then for some $z \in S'$,

$$t_f \stackrel{\text{def}}{=} \sup\{\, t \mid F_s(z) \in A \quad \forall s < t \,\} < \infty.$$

Obviously, $u_f \stackrel{\text{def}}{=} F_{t_f}(z) \notin A$; otherwise, since A is open, $F_t(z)$ would be contained in A for a time longer than t_f. On the other hand, $u_f \in \mathrm{Cl}_P A$, since t_f is the smallest time of escape from A. Thus, there are sequences $z_i \in S$ and $g_i \in G_{\mu_e}$ such that $g_i z_i \to u_f$. Since S is relatively compact, one may assume $z_i \to z \in \mathrm{Cl}_P S$, and then since G_{μ_e} acts properly, some subsequence of g_i converges, so one may assume $g_i \to g \in G_{\mu_e}$. Using (15), $f(z) < \epsilon$, and then using (14) gives $z \in S$. Thus, $u_f = gz \in A$, *a contradiction*. The proof that (16) is true for t negative is similar. □

Remark. The conclusion that $\boldsymbol{d}(H - J_{\xi_e})(z_e)$, and the definition of formal stability are, of course, not predicated on the assumption that G_{μ_e} acts properly or on the existence of an Ad invariant inner product.

Remark. At a regular relative equilibrium, the Marsden-Weinstein reduction is well defined, at least locally, and z_e passes to an equilibrium there. Formal stability is equivalent to the Hessian of the reduced Hamiltonian being positive or negative definite at that equilibrium: to see this simply use a small section to the G_{μ_e} action through z_e and within $J^{-1}(\mu_e)/G_{\mu_e}$ as an open neighborhood of the equilibrium in the reduced space.

Remark. The same conclusion follows if the hypotheses are verified with the Hamiltonian H replaced by any G_{μ_e} invariant conserved quantity with the same derivative as H at the relative equilibrium.

When applied to simple mechanical systems with symmetry, theorem (1) is quite inefficient, since it fails to recognize a fact known *a priori*: that the kinetic energy is positive. Accounting for this reduces the size of the Hessian to be considered by a factor of two when the symmetry group is Abelian, though somewhat less when it is not. In a further refinement known as *block diagonalization*, part of the Hessian may be related to a new Lagrangian system obtained from the original one by "locking" the "internal" degrees of freedom. This last nuance is not very helpful for the system of two coupled rigid bodies, but is provided here for its intrinsic interest.

Consider, then, the simple mechanical system with symmetry (Q, G, V). Recall the following basic notations [1]:

- Let $\Lambda = \{\, q \in Q \mid \xi \mapsto \xi(q) \text{ is not injective} \,\}$; Λ will be called the set of *symmetric configurations*. Then Λ is closed and G invariant, and so $Q \setminus \Lambda$ is a G invariant open submanifold of Q.

- Given $\mu \in \mathfrak{g}^*$, define the one form β_μ on $Q \setminus \Lambda$ by

$$\langle \beta_\mu, \eta(q) \rangle \stackrel{\text{def}}{=} \langle \mu, \eta \rangle \qquad \forall \eta \in \mathfrak{g},$$
$$\langle \beta_\mu, v_q \rangle \stackrel{\text{def}}{=} 0 \qquad \forall v_q \perp \mathfrak{g} \cdot q.$$

  Although defined differently, this one form corresponds the the one form α_μ of [1:343]. The one form β_μ is G_μ invariant.

- The *locked inertia tensor* I is the Q dependent bilinear form on $\mathfrak{g}$ given by

$$I(q)(\eta_1, \eta_2) \stackrel{\text{def}}{=} \langle \eta_1(q), \eta_2(q) \rangle.$$

  For $q \notin \Lambda$, $I(q)$ is an inner product.

- Given μ, the *effective*, or *amended potential* is the smooth function $V_\mu : Q \setminus \Lambda \to \mathbb{R}$

$$V_\mu \stackrel{\text{def}}{=} V + \frac{1}{2}|\beta_\mu|^2 = V + \frac{1}{2}|\mu|^2_{I(q)}.$$

  From G_μ invariance of β_μ and V, it follows that V_μ is G_μ invariant.

Having dealt with this elementary jargon, the cotangent bundle reduction theorem may now be used to replace the function $H - J_{\xi_e}$ with V_{μ_e}, when considering the stability of a relative equilibrium whose base point does not lie in the set Λ.

Theorem 2. *Let $z_e \in T^*Q$ with base point $q_e \notin \Lambda$ and momentum μ_e. Then z_e is a relative equilibrium if and only if $z_e = \beta_{\mu_e}(q_e)$ and $\mathbf{d}V_{\mu_e}(q_e) = 0$. If z_e is a relative equilibrium [and $\dim Q = \dim G$] then z_e is formally stable if and only if $\mathbf{d}^2 V_{\mu_e}(z_e)$ is positive definite [definite], on one (and hence any) complement to $\mathfrak{g}_{\mu_e} \cdot q_e$ in $T_{q_e}Q$.*

Proof. For the sake of clarity, the proof will be presented under the additional hypotheses that the quotient Q/G_{μ_e} exists and that μ_e is a regular value of J; if not, then $q_e \notin \Lambda$ validates the use of local analogues to both Q/G_{μ_e} and to the constructions below.

Recall the cotangent bundle reduction theorem: Denoting the annihilator of a vector subspace by ^{0}, there is a vector bundle map $\Psi : \mathfrak{g}_{\mu_e} \cdot Q^0 \to T^*(Q/G_{\mu_e})$ defined by the equation

$$\langle \Psi(\alpha_q), T\pi\, v_q \rangle = \langle \alpha_q, v_q \rangle,$$

where $\pi : Q \to Q/G_{\mu_e}$ is the projection. This Ψ is a linear isometry on each fiber, by definition of the metric on the fibers of the bundle $T^*(Q/G_{\mu_e})$. The fiber translation $\alpha_q \mapsto \alpha_q - \beta_{\mu_e}(q)$ is a bijection between $J^{-1}(\mu_e)$ and $\mathfrak{g} \cdot Q^0$, which is a subbundle of $\mathfrak{g}_{\mu_e} \cdot Q^0$. Composing this fiber translation with Ψ establishes a quotient map for the G_{μ_e} action on $J^{-1}(\mu_e)$, and hence identifies the reduced space $J^{-1}(\mu_e)/G_{\mu_e}$ with a subbundle P of $T^*(Q/G_{\mu_e})$. Note that the fibers of P have dimension zero if and only if $\dim Q = \dim G$.

If $\alpha_q \in J^{-1}(\mu_e)$, then $\alpha_q - \beta_{\mu_e}(q)$ is the orthogonal projection of α_q on $\mathfrak{g} \cdot Q^0$, and $\beta_{\mu_e}(q) \in (\mathfrak{g} \cdot Q^{\perp})^0 = (\mathfrak{g} \cdot Q^0)^{\perp}$, so

$$\begin{aligned} H(\alpha_q) &= \frac{1}{2}|\alpha_q|^2 + V(q) \\ &= \frac{1}{2}|\alpha_q - \beta_{\mu_e}(q)|^2 + \frac{1}{2}|\beta_{\mu_e}(q)|^2 + V(q) \\ &= \frac{1}{2}|\Psi(\alpha_q - \beta_{\mu_e}(q))|^2 + V_{\mu_e}(q). \end{aligned}$$

From this, the Hamiltonian on P becomes $\left(\bar\alpha_{\bar q} \mapsto \frac{1}{2}|\bar\alpha_{\bar q}|^2 + \bar V_{\mu_e}(\bar q)\right)|P$, where $\bar V_{\mu_e}$ is the function on $T^*(Q/G_{\mu_e})$ induced by V_{μ_e}. Thus, z_e is a relative equilibrium if and only if $\Psi(z_e - \beta_{\mu_e}(q)) = 0$ is an equilibrium of the reduced dynamics, that is, if and only if $\Psi(z_e - \beta_{\mu_e}(q)) = 0$ and $\bar q_e \stackrel{\text{def}}{=} \pi(q_e)$ is a critical point of $\bar V_{\mu_e}$, and then z_e is formally stable if and only if $\mathbf{d}^2 \tilde V_{\mu_e}(q_e)$ is positive definite [definite if the fibers of P have dimension zero]. The statements in the theorem now follow from the fact that Ψ has trivial kernel and that V_{μ_e} is G_{μ_e} invariant. □

Given $q \in \Lambda$, define the locked system at q as follows: immerse G in Q by $\iota(g) = gq$ and consider the system constrained to the submanifold $\iota(G)$. The Lagrangian becomes

$$\begin{aligned} \bar L(TL_g \eta) &= L(T\iota\, TL_g \eta) \\ &= \frac{1}{2}|T\Phi_g T\iota\, \eta|^2 + V(gq_e) \end{aligned}$$

$$= \frac{1}{2}|\eta(q_e)|^2 + V(q_e)$$
$$= \frac{1}{2}I(q_e)(\eta, \eta) + V(q_e).$$

Apply theorem (2) to this new mechanical system; call the corresponding one form and amended potential $\tilde{\beta}_{\mu_e}$ and $\tilde{V}_{\mu_e}$ respectively. Since ι is an isometric immersion which intertwines left multiplication and Φ, and since $\mathfrak{g}\cdot Q$ is G invariant, it follows that $\tilde{\beta}_{\mu_e} = \iota^*\beta_{\mu_e}$ and hence $\tilde{V}_{\mu_e} = \iota^* V_{\mu_e}$. Thus, if q_e is a critical point for V_{μ_e}, then Id is a critical point for $\tilde{V}_{\mu_e}$, and if $\boldsymbol{d}^2 V_{\mu_e}(q_e)$ is definite, then so is $\boldsymbol{d}^2\tilde{V}_{\mu_e}(\mathrm{Id})$. This proves the following proposition:

Proposition 5. *A relative equilibrium for the original system is a relative equilibrium for the system locked at the equilibrium configuration. If the locked system is not formally stable at the relative equilibrium, then neither is the relative equilibrium of the original system.*

Remark. This innocuous result gives an interesting insight: consider any system of two or more rigid bodies coupled by any joints whatever and in a state of relative equilibrium corresponding to the spatial action of $SO(3)$. Then the locked system is exactly the mechanical system of a single rigid body obtained by fixing the joint degrees of freedom, and the motion must be a rotation about a principle axis of inertia thereby obtained; moreover, this rotation must be about the *shortest principle axis of inertia for the relative equilibrium to be formally stable.*

Now suppose the relative equilibrium for the locked system is formally stable. If $\dim Q = \dim G$ then this is equivalent to formal stability of the original system, and the subspace V_{INT} below is trivial. Otherwise, choose a subspace Z_1 complementary to $\mathfrak{g}_{\mu_e}$ in $\mathfrak{g}$, and set $V_{\mathrm{RIG}} = Z_1 \cdot q_e$. Choose a complement Z_2 to $T_{q_e}(G_{\mu_e}\cdot q_e)$ in $T_{q_e}Q$ which contains V_{RIG}. Then $\boldsymbol{d}^2 V_{\mu_e}(q_e)|V_{\mathrm{RIG}}$ is positive definite by theorem (2) and $\tilde{V}_{\mu_e} = \iota^* V_{\mu_e}$, so

$$V_{\mathrm{INT}} = \{\, v \in Z_2 \mid \boldsymbol{d}^2 V_{\mu_e}(q_e)(v, w) = 0 \quad \forall w \in V_{\mathrm{RIG}} \,\}$$

complements V_{RIG} in Z_2. Note that V_{INT} is a complement to $T_{q_e}(G\cdot q_e)$. This discussion thus proves the following proposition:

Proposition 6. *Let the locked system be formally stable at the relative equilibrium. Then there is a subspace V_{INT} complementary to $T_{q_e}(G\cdot q_e)$ such that the original system is stable at the relative equilibrium if and only if $\boldsymbol{d}^2 V_{\mu_e}(q_e)|V_{\mathrm{INT}}$ is positive definite.*

Remark. Since the derivative of the potential in the direction of the group action is zero, $\boldsymbol{d}^2 V_{\mu_e}(q_e)(v, \eta(q_e))$ has an expression solely in term of derivatives of the locked inertia tensor [29]. Thus, proposition (6) can acquire practical use, in that information about V_{INT} may be obtained without first computing the full Hessian $\boldsymbol{d}^2 V_{\mu_e}$.

Remark. The cotangent bundle reduction theorem also specifies the symplectic form as being the canonical symplectic form on $T^*(Q/G_{\mu_e})$ plus a "magnetic" term [1]. Moreover, the decomposition (after fiber translation by α_{μ_e}) into vertical and horizontal components, refined by the split into V_{INT} and V_{RIG}, may be lifted to give a basis of $T_{z_e}T^*Q$. These ideas pursued result in a basis at z_e in which *both* the Hessian $\boldsymbol{d}^2 H_{\xi_e}$ *and* the canonical symplectic form on T^*Q have a remarkably simple structure [29].

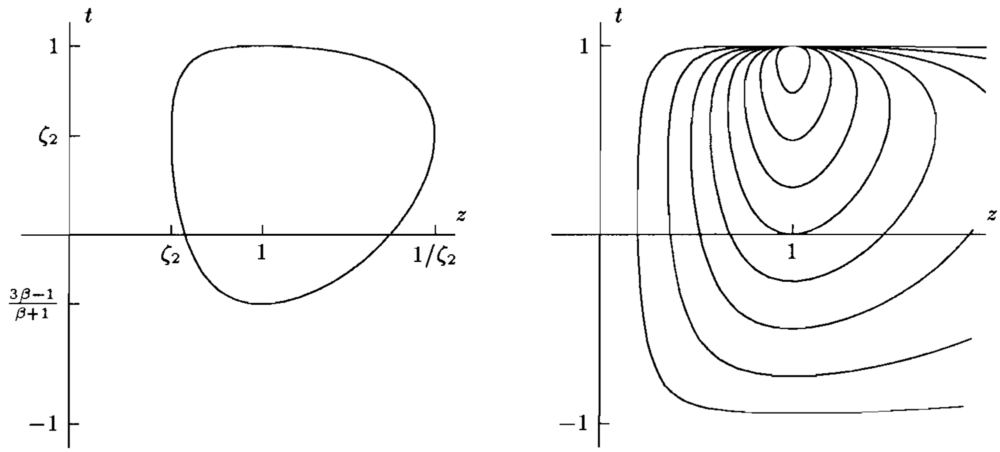


Figure 1: The zero sets of p^{Δ}

This much theory is sufficient to determine the stability of the relative equilibria (2e) and (2f). The proof in the case of class (2f) relies on information about a certain polynomial, which we collect in a separate lemma:

Lemma 2.

1. *For* $0 < \beta < 1$*, the quadratic* $\beta z^2 - 2z + \beta$ *has exactly two roots—namely* ζ_1 *and* $1/\zeta_1$*, where*

$$\zeta_1 \stackrel{\text{def}}{=} \frac{\beta}{1+\sqrt{1-\beta^2}}.$$

2. *For* $0 < \beta < 1$*, the cubic*

$$p_1 \stackrel{\text{def}}{=} (2-\beta^2)z^3 - 3\beta z^2 + 3\beta z - \beta$$

has exactly one root, say ζ_2*, and furthermore,*

$$0 < \zeta_1 < \beta < \zeta_2 < 1. \tag{17}$$

3. *If* $\beta = 0$*, the polynomial*

$$\begin{aligned} p^{\Delta}(z,t) \stackrel{\text{def}}{=} & -z^2(1-\beta^2)(\beta z^2 - 2z + \beta)t^2 - 2\beta z(z^2 - 2\beta z + 1)^2 t \\ & + \beta z^6 - 4\beta^2 z^5 + (4+3\beta^2)\beta z^4 - 2(1+3\beta^2)z^3 \\ & + (4+3\beta^2)\beta z^2 - 4\beta^2 z + \beta. \end{aligned}$$

is zero exactly on the lines $t = \pm 1$ *and* $z = 0$*. When* $\beta \neq 0$ *and* $t \in (-1,1)$*,* $p^{\Delta}(z,t)$ *is strictly positive for all* $z \in (-\infty, \zeta_2) \cup (1/\zeta_2, \infty)$*. Moreover, the set of zeros* (z,t) *of* p^{Δ} *with* $z \in [\zeta_2, 1/\zeta_2]$ *is a continuous one dimensional submanifold* $\mathcal{Z}_\beta$ *that is accurately represented in figure* (1)*, in that:*

- $\mathcal{Z}_\beta$ *is entirely contained in the rectangle* $[\zeta_2, 1/\zeta_2] \times (-1,1]$ *and meets the boundary of this rectangle exactly at the points* $(1,1)$*,* (ζ_2, ζ_2) *and* $(1/\zeta_2, \zeta_2)$*.*

- *$\mathcal{Z}_\beta$ meets each vertical line above the interval $(\zeta_2, 1/\zeta_2)$ exactly twice.*
- *$\mathcal{Z}_\beta$ contains the point $(1, (3\beta-1)/(\beta+1))$.*

Proof. This proof is supported by the **MAPLE** symbolic manipulator through [26]. The first statement and the fact that $\zeta_1 < \beta$ are grade school verifications. For the second statement, the derivative of the polynomial p_1 is a quadratic in z with discriminant $-36\beta^2(1-\beta^2)$, so p_1 has exactly one root, say ζ_2, and since the leading coefficient of p_1 is positive, p_1 is negative to the left and positive to the right of ζ_2, so the evaluation $p_1(1) = 2(1-\beta^2) > 0$ shows $\zeta_2 < 1$. Also, the evaluation $p_1(\beta) = -\beta(1-\beta^2)^2$ shows $\beta < \zeta_2$.

Now the proof that $p^\Delta(z,t) > 0$ for $t \in (-1,1)$ and $z \in (-\infty, \zeta_2) \cup (1/\zeta_2, \infty)$ is immediate from the following two observations:

- $p^\Delta(z,t) > 0$ for $z \in (-\infty, \zeta_1] \cup [1/\zeta_1, \infty)$ and $t \in (-1,1)$: Indeed, for $\beta z^2 - 2z + \beta > 0$ (that is for $z \in (-\infty, \zeta_1) \cup (1/\zeta_1, \infty)$), p^Δ is a concave down quadratic polynomial in t and

$$p^\Delta(z,1) = \beta(z-1)^2(z^2 - 2\beta z + 1)^2 > 0,$$
$$p^\Delta(z,-1) = \beta(z+1)^2(z^2 - 2\beta z + 1)^2 > 0,$$

so $p^\Delta(z,t) > 0$ for z in the union of these two open intervals. Moreover, by substituting $\beta = 2\zeta_1/(1+\zeta_1^2)$ into p^Δ,

$$p^\Delta(\zeta_1, t) = {\zeta_1}^6 p^\Delta(1/\zeta_1, t) = \frac{2\zeta_1(1-\zeta_1^2)^4(\zeta_1^2 - 2\zeta_1 t + 1)}{(1+\zeta_1^2)^3} > 0,$$

as required.

- $p^\Delta(z,t) \neq 0$ for $z \in (\zeta_1, \zeta_2) \cup (1/\zeta_2, 1/\zeta_1)$ and $t \in (-1,1)$: The discriminant of the quadratic p^Δ is $4z^5(z-\beta)(\beta z - 1)p_1(z)p_1(1/z)$, and this is nonnegative exactly on the intervals $(-\infty, \beta]$, $[\zeta_2, 1/\zeta_2]$ and $[1/\beta, \infty)$, so p^Δ has no zeros over the intervals (β, ζ_2) and $(1/\zeta_2, 1/\beta)$. Moreover, by solving the quadratic, the zero set of p^Δ over $(\zeta_1, \beta]$ is the union of the graphs of two continuous functions, both graphs containing the point $z = \beta$, $t = 1/\beta$, since

$$p^\Delta(\beta, t) = \beta(1-\beta^2)(1-\beta t),$$

and furthermore, since

$$p^\Delta(z,1) = \beta(z-1)^2(z^2 - 2\beta z + 1)^2,$$

the zero set of p^Δ never crosses the line $t = 1$ on the interval $(\zeta_1, \beta]$. Therefore, $p^\Delta(z,t)$ does not vanish for $z \in (\zeta_1, \beta]$ and $t \in (-1,1)$, and similarly with the interval $[1/\beta, 1/\zeta_1)$. Therefore, $p^\Delta(z,t)$ does not vanish for $z \in (\zeta_1, \zeta_2)$ and $t \in (-1,1)$, since the interval (ζ_1, ζ_2) is the union of the two intervals $(\zeta_1, \beta]$ and (β, ζ_2), and similarly with the interval $(1/\zeta_2, 1/\zeta_1)$, as required.

As the discriminant of the quadratic p^Δ is nonnegative on the interval $[\zeta_2, 1/\zeta_2]$ and zero only at $z = \zeta_2$ or $z = 1/\zeta_2$, the set $\mathcal{Z}_\beta$ is the union of the graphs of two continuous functions which are smooth on $(\zeta_2, 1/\zeta_2)$ and meet exactly once at $z = \zeta_2$ and $z = 1/\zeta_2$. Therefore, $\mathcal{Z}_\beta$ is a continuous one dimensional submanifold, and moreover $\mathcal{Z}_\beta$ meets each vertical line over $(\zeta_2, 1/\zeta_2)$ exactly twice and meets each of the vertical lines $z = \zeta_2$ and $z = 1/\zeta_2$ exactly once; these lines meet $\mathcal{Z}_\beta$ at their respective z values and $t = \zeta_2$ since

$$p^\Delta(z,z) = -(1-z^2)(1-\beta z)p_1(z) \quad \text{and} \quad p^\Delta(z,1/z) = z^3(1-z^2)(z-\beta)p_1(1/z).$$

That $\mathcal{Z}_\beta$ contains the points $(1,1)$ and $(1,(3\beta-1)/(\beta+1))$ is simply a matter of substituting these values in p^Δ and verifying that the result is zero.

To complete the proof, it remains to be shown that $\mathcal{Z}_\beta$ is below the line $t = 1$ and above the line $t = -1$. For the former, note that $\mathcal{Z}_\beta$ is topologically a circle, and so $\mathcal{Z}_\beta \setminus \{(1,1)\}$ is connected and does not meet the line $t = 1$. For the latter, note that the connected set $\mathcal{Z}_\beta$ never meets the line $t = -1$, since

$$p^\Delta(z,-1) = \beta(z+1)^2(z^2 - 2\beta z + 1)^2,$$

which is not zero on the interval $[\zeta_2, 1/\zeta_2]$. □

Proposition 7.

1. *The relative equilibria* (2e) *are formally stable if and only if* $t_1 \neq 0$ *and* $t_2 \neq 0$.
2. *The relative equilibria* (2f) *are formally stable if and only if* $\beta \neq 0$ *and* $\beta {t_1}^2 - 2t_1t_2 + \beta {t_2}^2 \geq 0$.

Proof. With the the standard basis of $\mathfrak{so}(3) \times \mathbb{R}^2$, and left translating vectors on the configuration space $SO(3)^2$ to the identity, equation (1.3) shows that the map taking elements of $\mathfrak{so}(3) \times \mathbb{R}^2$ to infinitesimal generators of the $SO(3) \times (S^1)^2$ action may be represented by the 6×5, $SO(3)^2$ dependent matrix

$$IGN(A_1, A_2) \stackrel{\text{def}}{=} \begin{bmatrix} {A_1}^t & -\boldsymbol{k} & 0 \\ {A_2}^t & 0 & -\boldsymbol{k} \end{bmatrix}. \tag{18}$$

Then the locked inertia tensor is the 5×5, $SO(3)^2$ dependent matrix

$$I(A_1, A_2) = IGN(A_1, A_2)^t J({A_1}^{-1}A_2) IGN(A_1, A_2),$$

and the amended potential is

$$V_{\mu_e} = \frac{1}{2}{\mu_e}^t I(A_1, A_2)^{-1} \mu_e.$$

Use the standard basis $\{\boldsymbol{e}_k\}$ of $\mathfrak{so}(3)^2 \stackrel{!}{=} \mathbb{R}^6$ to form the 6-vector of 5×5 matrices I' by

$$I'_k(A_1, A_2) = \left.\frac{d}{dt}\right|_{t=0} I((A_1, A_2) \cdot \exp(t\boldsymbol{e}_k{}^\wedge)),$$

and the 6×6 array of 5×5 matrices I''

$$I''_{kl}(A_1, A_2) = \left.\frac{d}{dt}\right|_{t=0} I'_k((A_1, A_2) \cdot \exp(t\mathbf{e}_l{}^\wedge)).$$

Then noting that $\mu_e = I(q_e)\xi_e$, the Hessian of the amended potential at the relative equilibrium is

$$\begin{aligned}\left[d^2 V_{\mu_e}(q_e)\right]_{kl} &= {\mu_e}^t I^{-1} I_k' I^{-1} I_l' I^{-1} \mu_e - \frac{1}{2}{\mu_e}^t I^{-1} I_{kl}'' I^{-1} \mu_e \\ &= {\xi_e}^t (I_k' I^{-1} I_l' - \frac{1}{2} I_{kl}'')\xi_e.\end{aligned}$$

Theorem (2) and proposition (6) then suggest that the Hessian of V_{μ_e} be computed on a basis of $\mathfrak{g} \cdot q_e$ extended to a basis of $T_{q_e}Q$ in a way such that the first $\dim G_{\mu_e}$ vectors span $\mathfrak{g}_{\mu_e} \cdot q_e$. The the Hessian will be zero in the first $\dim G_{\mu_e}$ rows and columns, and have a middle 2×2 submatrix that is the Hessian for the locked system. In the case at hand, this basis in constructed as follows: First,

$$IGN(\mathrm{Id}, \exp(t_3\mathbf{j}^\wedge)) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \cos t_3 & 0 & \sin t_3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sin t_3 & 0 & \cos t_3 & 0 & -1 \end{bmatrix},$$

so that a quick look at the columns of IGN shows that the vector

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^t \tag{19}$$

complements $\mathfrak{g} \cdot q_e$ regardless of anything else, hence this vector may always be used as the last vector of the basis. If all of the first three components of μ_e are zero (that is, if the total angular momentum is zero),then G_{μ_e} is the full symmetry group $SO(3) \times (S^1)^2$, so one need only look at the Hessian restricted to the subspace spanned by (19). Otherwise, a basis of $\mathfrak{g}_{\mu_e}$ is

$$\begin{gathered}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t, \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}^t, \\ \begin{bmatrix} {\mu_e}^1 & {\mu_e}^2 & {\mu_e}^3 & 0 & 0 \end{bmatrix}^t.\end{gathered} \tag{20}$$

In fact, the second component of μ_e is zero, because at a relative equilibrium the body axes and the total rotation vector are co-planar, so the two vectors

$$\begin{gathered}\begin{bmatrix} -{\mu_e}^3 & 0 & {\mu_e}^1 & 0 & 0 \end{bmatrix}^t, \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^t,\end{gathered}$$

complete the basis of the vectors (20) to a basis of $\mathfrak{g}$. Then multiplying this basis of $\mathfrak{g}$ by $I(q_e)$ gives a basis of $\mathfrak{g} \cdot q_e$ which forms the first 5 vectors of the basis of $T_{q_e}Q$. As it turns out, the Hessian achieves the form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h^{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & h^{55} & h^{56} \\ 0 & 0 & 0 & 0 & h^{56} & h^{66} \end{bmatrix}, \tag{21}$$

so necessary and sufficient conditions for formal stability are

$$h^{44} > 0, \quad h^{55} > 0, \quad \Delta \stackrel{\text{def}}{=} h^{55}h^{66} - (h^{56})^2 > 0. \tag{22}$$

Set in this way, the computation has been performed by the MAPLE symbolic manipulator [4]. For the class (2e) relative equilibria [20], total angular momentum at the relative equilibria is zero if and only if $t_1 = t_2 = 0$, in which case the relative equilibrium is not formally stable, since the Hessian is zero. Otherwise, these relative equilibria are stable, since

$$h^{44} = \frac{\alpha^4({t_1}^2 + 2t_1t_2\cos t_3 + {t_2}^2)^2}{1 - \beta\cos t_3},$$
$$h^{55} = 2\alpha^2\frac{{t_1}^2 + 2\beta t_1t_2 + {t_2}^2}{1 - \beta^2},$$
$$\Delta = 4\frac{\alpha^4{t_1}^2{t_2}^2}{(1 - \beta^2)}.$$

For the class (2f) relative equilibria [21], set $z = t_1/t_2$ and $t = \cos(t_3)$, so $z \neq 0$ and $-1 < t < 1$, and then it is required to show that these relative equilibria are stable if and only if

$$\beta \neq 0, \quad \beta z^2 - 2z + \beta \geq 0. \tag{23}$$

The magnitude of the total angular momentum is

$$\frac{{t_2}^2(\beta z^2 - 2z + \beta)^2(z^2 - 2zt + 1)}{(1 - t^2)z^2},$$

which is zero if and only if $\beta z^2 - 2z + \beta = 0$, and the Hessian on the span of the single vector (19) at ζ_1 and $1/\zeta_1$ is

$$\frac{4t_2(\zeta_1^2 - 1)^2}{(1 - t^2)(\zeta_1^2 + 1)},$$

and $1/\zeta_1^2$ respectively, so the relative equilibria (2f) are stable when the total angular momentum is zero. Note that $\beta z^2 - 2z + \beta \neq 0$, as $\beta = 0$ and $\beta z^2 - 2z + \beta = 0$ simultaneously is impossible, since $z \neq 0$.

When the total angular momentum is non-zero (that is, when $\beta z^2 - 2z + \beta \neq 0$), one has

$$h^{44} = \frac{\beta {t_2}^4 (\beta z^2 - 2z + \beta)^3 (z^2 - 2zt + 1)^3}{z^4 (1-t^2)^2 (1-\beta t)},$$
$$h^{55} = \frac{2{t_2}^2(\beta z^2 - 2z + \beta)}{z^2(1-t^2)(1-\beta^2)} p^{55},$$
$$\Delta = \frac{4{t_2}^4(\beta z^2 - 2z + \beta)}{z^2(1-t^2)^2(1-\beta^2)} p^{\Delta},$$

where

$$p^{55}(z,t) \stackrel{\text{def}}{=} -2z^2(1-\beta^2)t + \beta z^4 - (1+3\beta^2)z^3 + 6\beta z^2 - (1+3\beta^2)z + \beta, \qquad (24)$$

and p^{Δ} is as in lemma (2). The factor $(\beta z^2 - 2z + \beta)^3$ of h^{44} shows that (23) are necessary for formal stability. The conditions (23) are also sufficient for formal stability: assuming (23) obviously gives $h^{44} > 0$, and further, $h^{55} > 0$ and $\Delta > 0$ for $-1 < t < 1$ if and only if $p^{55} > 0$ and $p^{\Delta} > 0$ for $-1 < t < 1$. Now $p^{\Delta} > 0$ for $-1 < t < 1$ by lemma (2), and p^{55} is a linear polynomial in t with negative slope so it suffices to show that the quartic in z

$$p^{55}(z,1) = \beta z^4 - (1+3\beta^2)z^3 + 2(\beta^2 + 3\beta - 1)z^2 - (1+3\beta^2)z + \beta$$

is positive off the interval $[\zeta_1, 1/\zeta_1]$. Indeed, $p^{55}(z,1) = z^2 \tilde{p}^{55}(z + 1/z)$, where

$$\tilde{p}^{55}(x) \stackrel{\text{def}}{=} \beta x^2 - (1+3\beta^2)x + 2(\beta^2 + 2\beta - 1),$$

so it is enough to check that $p^{55}(0,1) > 0$ and $\tilde{p}^{55}$ is positive on the union of intervals $(-\infty, -2] \cup [2/\beta, \infty)$, since $\zeta_1 + 1/\zeta_1 = 2/\beta$. But this is immediate, since $p^{55}(0,1) = \beta$, and $\tilde{p}^{55}$ is a concave up quadratic polynomial, and

$$\tilde{p}^{55}(-2) = 8\beta(1+\beta) > 0, \quad \tilde{p}^{55}(0) = (\beta+1)^2 - 2 < 0, \quad \tilde{p}^{55}(2/\beta) = \frac{2}{\beta}(1-\beta)^2(1+\beta).$$

□

Remark. Conditions (23) admit the following interpretation: let θ_i be the angle from the axis of overall rotation Ω to the axis of body i, measured in the same sense (for example, as right handed rotations about the cross product of the axis of body 1 and Ω). Then, from (2f), and since the axis of body 1 is along $\boldsymbol{k}$, while that of body 2 is along $\exp(t_3 \boldsymbol{j}^\wedge)$,

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\boldsymbol{j} \cdot \Omega \times \boldsymbol{k}}{\boldsymbol{j} \cdot \Omega \times \exp(t_3 \boldsymbol{j}^\wedge)} = \frac{t_1}{t_2} = z.$$

Then (23) become $\beta \neq 0$ and

$$\frac{\sin\theta_1}{\sin\theta_2} < \frac{\beta}{1+\sqrt{1-\beta^2}} \quad \text{or} \quad \frac{\sin\theta_1}{\sin\theta_2} > \frac{1+\sqrt{1-\beta^2}}{\beta}, \qquad (25)$$

inequalities which bound the bodies away from the overlapped configuration characterized by $\theta_1 = \theta_2$.

While still regular, the classes (2c) and (2d) reside at the symmetric overlapped or opposed *configurations* respectively; that is these relative equilibria have base points in the set Λ. Thus, theorem (2) is not directly applicable, although theorem (1) could be used. Since they are regular, the total angular momentum is not along the bodies' mutual axes of symmetry, so G_{μ_e} will in fact act freely on configuration space in these cases. This raises the possibility of using (2) with the action of G_{μ_e} instead of the entire group. Since the level set of the momentum map for the smaller group action contains that of the larger one, the notion of formal stability is stronger in the former situation than in the latter; that is, the test for formal stability associated to the G_{μ_e} action is not necessarily sharp. In fact, this is not a problem for the system of two coupled rigid bodies, because of the following proposition:

Proposition 8.

1. *Let (P, ω, H, G, J) be a Hamiltonian system with symmetry and let z_e be a regular relative equilibrium with evolution $t \mapsto \exp(\xi_e t)$ and momentum μ_e. If z_e is formally stable for the G_{μ_e} action, then it is formally stable for the G action. If z_e if formally stable for the G action, with positive [negative] definite Hessian at z_e, and if the symmetric bilinear form*
$$(\eta_1, \eta_2) \mapsto \langle \mu_e, [\eta_1, [\xi_e, \eta_2]] \rangle$$
is positive [negative] definite on one (and hence any) complement to $\mathfrak{g}_{\mu_e}$, then z_e is formally stable for the G_{μ_e} action.

2. *More specifically, suppose (Q, H, G) is a simple mechanical system with symmetry, $G = SO(3)^n \times A$ where $n \geq 0$ and A is Abelian, and $\dim Q \neq \dim G$. Then z_e is formally stable for the G action if and only if it is formally stable for the G_{μ_e} action.*

Proof. Note that
$$\mathfrak{g}_{\mu_e} = \{\, \eta \in \mathfrak{g} \mid \mu_e \in \ker(\mathrm{ad}_\eta{}^*) \,\}.$$
A momentum map for the G_{μ_e} action is $\tilde{J} = \iota^* \circ J$ where $\iota : \mathfrak{g}_{\mu_e} \to \mathfrak{g}$ is the inclusion and $\tilde{J}(z_e) = \mu_e|\mathfrak{g}_{\mu_e} \stackrel{\text{def}}{=} \tilde{\mu}_e$. Also, the two augmented Hamiltonians $H - J_{\xi_e}$ and $H - \tilde{J}_{\xi_e}$ are the same, since $\xi_e \in \mathfrak{g}_{\mu_e}$ by CoAd equivariance and conservation of momentum. Clearly, there is no loss of generality in assuming the positive definite case.

Suppose z_e is formally stable for the action of G_{μ_e}. Then $\boldsymbol{d}^2 H_{\xi_e}$ is positive definite on a complement $S \subset T_{z_e}\tilde{J}^{-1}(\tilde{\mu}_e)$ to the G_{μ_e} action. Since $J^{-1}(\mu_e) \subset \tilde{J}^{-1}(\tilde{\mu}_e)$, $S \cap T_{z_e}J^{-1}(\mu_e)$ is a complement to the G_{μ_e} action in $T_{z_e}J^{-1}(\mu_e)$, so z_e is formally stable for the action of G.

Now suppose z_e is stable for the action of G. Again, there is a complement $S \subset T_{z_e}J^{-1}(\mu_e)$ to the G_{μ_e} action on which $\boldsymbol{d}^2 H_{\xi_e}(z_e)$ is positive definite. Take any complement Z to $\mathfrak{g}_{\mu_e}$ in $\mathfrak{g}$ and let $T = Z \cdot z_e$. The proof is completed as follows:

- $S \oplus T \oplus \mathfrak{g}_{\mu_e} \cdot z_e = T_{z_e}\tilde{J}^{-1}(\tilde{\mu}_e)$: First note $S \cap T = 0$: if $\eta \in Z$ and $\eta(z_e) \in S$ then
$$0 = \left.\frac{d}{dt}\right|_{t=0} J(\exp(\eta t) z_e)$$

$$
\begin{aligned}
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{CoAd}(\exp(\eta t), \mu_e) \\
&= \mathrm{ad}^*_\eta \mu_e,
\end{aligned}
$$

so $\eta = 0$ since $Z \cap \mathfrak{g}_{\mu_e} = 0$. Obviously, $S \cap \mathfrak{g}_{\mu_e} \cdot z_e = 0$ and $T \cap \mathfrak{g}_{\mu_e} \cdot z_e = 0$, so the following dimension count proves this item:

$$
\begin{aligned}
\dim(S \oplus T \oplus \mathfrak{g}_{\mu_e} \cdot z_e) &= [\dim P - \dim G - \dim G_{\mu_e}] \\
&\quad + [\dim G - \dim G_{\mu_e}] + \dim G_{\mu_e} \\
&= \dim P - \dim G_{\mu_e} \\
&= \dim T_{z_e} \tilde{J}^{-1}(\tilde{\mu}_e).
\end{aligned}
$$

- H_{ξ_e} block diagonalizes on $S \oplus T$: Let $\eta \in Z$ and $v \in S$. Then

$$
\begin{aligned}
\boldsymbol{d}^2 H_{\xi_e}(z_e)(\eta(e), v) &= \langle \boldsymbol{d}(\boldsymbol{i}_\eta \boldsymbol{d} H_{\xi_e}), v \rangle \\
&= \langle \boldsymbol{d}(\boldsymbol{i}_\eta (\boldsymbol{d} H - \boldsymbol{d} J_{\xi_e})), v \rangle \\
&= -\langle \boldsymbol{d}(\boldsymbol{i}_\eta \boldsymbol{d} J_{\xi_e}), v \rangle \\
&= -\langle \boldsymbol{d} J_{[\xi_e, \eta]}(e), v \rangle \\
&= 0,
\end{aligned}
$$

since $v \in \ker \boldsymbol{d} J(e)$.

- $\boldsymbol{d}^2 H_\xi(e)$ is positive definite on T: From the expression for $\boldsymbol{d}^2 H_{\xi_e}(z_e)$ just computed,

$$
\begin{aligned}
\boldsymbol{d}^2 H_{\xi_e}(z_e)(\eta_1(z_e), \eta_2(z_e)) &= -\boldsymbol{i}_{\eta_2} \boldsymbol{d} J_{[\xi_e, \eta]}(z_e) \\
&= \langle \mu_e, [\eta_2, [\xi_e, \eta_1]] \rangle,
\end{aligned}
$$

which is positive definite by hypothesis.

For the sake of clarity, we show the second statement in the case $n = 1$ and A trivial; the proof of the general case is not more difficult. If $\mu_e = 0$ then there is nothing to prove, since $G = G_{\mu_e}$. If $\mu_e \neq 0$, then $\xi_e \neq 0$ since $\mu_e = I(z_e)\xi_e$, and $\xi_e = t\mu_e$ since $\xi_e \in \mathfrak{g}_{\mu_e}$. Moreover, $t > 0$ since

$$
t|\xi_e|^2_{I_{qe}} = t\xi_e \cdot \xi_e = \langle \mu_e, \xi_e \rangle_{I_{qe}} = |\mu|^2_{I_{qe}}.
$$

Then the bilinear form above, evaluated on the orthogonal complement to ξ_e with respect the $\mathbb{R}^3$ inner product, *not* the $I(q_e)$ inner product, is

$$
\begin{aligned}
(\eta_1, \eta_2) &\mapsto t\xi_e \cdot (\eta_1 \times (\xi_e \times \eta_2)) \\
&= t(|\xi_e|^2 \eta_1 \cdot \eta_2 - (\eta_1 \cdot \xi_e)(\eta_2 \cdot \xi_e)) \\
&= t|\xi_e|^2 \eta_1 \cdot \eta_2,
\end{aligned}
$$

which is positive definite. □

With this addition to the theory, the stability of the class (2c) and (2d) relative equilibria may now be determined.

Proposition 9. *The class* (2c) *relative equilibria are not formally stable. The class* (2d) *relative equilibria are formally stable.*

Proof. The total angular momentum of a relative equilibrium in (2d) is easily computed as

$$2\begin{bmatrix} (1+\beta)t_1 & 0 & \alpha t_2 \end{bmatrix},$$

which is nonzero, and so the group G_{μ_e} is the product of rotations about this axis and the two copies of S^1 from the axial symmetry. This fact observed, the argument is analogous to the proof of proposition (7), except that the smaller group is used. To summarize [19], the matrix IGN is computed by replacing the first 3 columns of (18) with the infinitesimal generator of $(\Omega, 0, 0)$ (here Ω refers to (10d)):

$$IGN(A_1, A_2) = \begin{bmatrix} {A_1}^t\Omega & -\boldsymbol{k} & 0 \\ {A_2}^t\Omega & 0 & -\boldsymbol{k} \end{bmatrix}. \tag{26}$$

Then the Hessian must be computed on a basis of $T_{q_e}Q$ whose first 3 vectors are the 3 columns of IGN evaluated at q_e. Since this evaluation is

$$\begin{bmatrix} t_1 & 0 & 0 \\ 0 & 0 & 0 \\ \dfrac{\alpha t_2}{1+\beta} & -1 & 0 \\ -t_1 & 0 & 0 \\ 0 & 0 & 0 \\ -\dfrac{\alpha t_2}{1+\beta} & 0 & -1 \end{bmatrix},$$

such a basis may be formed from these 3 column vectors and the vectors

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t .$$

With this choice of basis, the Hessian again takes the form (21), and so (22) are again necessary and sufficient conditions for formal stability. In the case at hand,

$$h^{44} = 8\beta {t_1}^2 + \frac{8\alpha^2\beta {t_2}^2}{(1+\beta)^2},$$
$$h^{55} = 2(1+\beta){t_1}^2 + \frac{\alpha^2(1+3\beta){t_2}^2}{(1+\beta)^2},$$
$$\Delta = \frac{4((1+\beta)^2{t_1}^2 + \alpha^2{t_2}^2)((1+\beta)^3{t_1}^2 + 2\alpha^2\beta{t_2}^2)}{(1+\beta)^3}.$$

Since all of these are obviously positive, this completes the proof of the formal stability of the class (2d) relative equilibria.

The proof that the class (2c) relative equilibria *are not* formally stable proceeds analogously [18]. There the corresponding basis of infinitesimal vector fields is completed to a basis of $T_{q_e}Q$ by the inclusion of the three vectors

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t,$$

and the Hessian again takes the form (21), but

$$h^{44} = -8\beta {t_1}^2 - \frac{8\alpha^2\beta {t_2}^2}{(1-\beta)^2},$$

which is negative. □

Degenerate relative equilibria can sometimes be viewed as being regular with respect to a subgroup of the full symmetry group, but a naive application of the methods discussed thus far using such a subgroup is likely to fail, in that the Hessians involved will normally be semidefinite. However, the existence of additional symmetries beyond the chosen subgroup implies the existence of additional conserved quantities. These conserved quantities can then be used to augment the usual Liapunov function, and this can sometimes lead to a resolution of the nonlinear stability question. In fact, such an argument usually gives stronger stability than might otherwise be expected, since that stability will refer *a priori* to some particular subgroup of the full symmetry group.

Theorem 3. *In the context of theorem* (2), *let G' be another Lie group with isometric action on Q which commutes with the action of G and is such that V is G' invariant. Let F be the projection onto $\mathfrak{g}'$ of the isotropy algebra of z_e for the product action of $G \times G'$, so that*

$$F \stackrel{\text{def}}{=} \{\, \eta' \in \mathfrak{g}' \mid \exists \eta \in \mathfrak{g} \text{ such that } \eta(z_e) = \eta'(z_e) \,\}.$$

For $\eta' \in \mathfrak{g}'$, define the function

$$f_{\eta'}(q) \stackrel{\text{def}}{=} \langle \beta_{\mu_e}(q), \eta'(q) \rangle.$$

Extend a basis η'_l of F to a basis of $\mathfrak{g}'$ using vectors $\eta''_m \in \mathfrak{g}'$, and consider the function

$$f \stackrel{\text{def}}{=} \sum_l a_l f_{\eta'_l} + \sum_m b_m \left(f_{\eta''_m} - f_{\eta''_m}(q_e)\right)^2,$$

where $a_l \in \mathbb{R}$ and $b_m \in \mathbb{R}$. Then $\mathbf{d}f(q_e) = 0$. Moreover, suppose that $\mathbf{d}^2 V_{\mu_e}(q_e)$ is positive semidefinite on a complement S to $\mathfrak{g}_{\mu_e} \cdot q_e$. Then z_e is G_{μ_e} stable if there exist a_l and b_m such that $\mathbf{d}^2 f(q_e)|S$ is positive definite on $\ker \mathbf{d}^2 V_{\mu_e}(q_e)|S$.

Proof. By linearity and elementary differentiation, showing $\boldsymbol{d}f_{\eta'}(z_e) = 0$ for $\eta' \in F$ is sufficient to show $\boldsymbol{d}f(z_e) = 0$. Suppose, then, that $\eta' \in F$ and choose $\eta \in \mathfrak{g}$ such that

$$\eta(z_e) + \eta'(z_e) = 0. \tag{27}$$

Note that

$$\begin{aligned} \hat{f}(q) &\stackrel{\text{def}}{=} \langle \beta_{\mu_e}, \eta(q) + \eta'(q) \rangle \\ &= \langle \mu_e, \eta \rangle + \langle \beta_{\mu_e}, \eta'(q) \rangle \\ &= \langle \mu_e, \eta \rangle + f_{\eta'}(q), \end{aligned}$$

so showing $\boldsymbol{d}f_{\eta'}(q_e) = 0$ is equivalent to showing $\boldsymbol{d}\hat{f}(q_e) = 0$. Using (27) and the identity $\boldsymbol{L}_X\alpha = \boldsymbol{d}\boldsymbol{i}_X\alpha + \boldsymbol{i}_X\boldsymbol{d}\alpha$,

$$\begin{aligned} \boldsymbol{d}\hat{f}(q_e) &= (\boldsymbol{L}_\eta \beta_{\mu_e} + \boldsymbol{L}_{\eta'}\beta_{\mu_e})(q_e) \\ &= \boldsymbol{L}_\eta \beta_{\mu_e}(q_e), \end{aligned}$$

since β_{μ_e} is easily seen to be G' invariant. As β_{μ_e} is G_{μ_e} invariant, $\boldsymbol{d}\hat{f}(q_e) = 0$ will follow if $\eta \in \mathfrak{g}_{\mu_e}$. But this is also a consequence of (27):

$$\begin{aligned} \text{coad}_\eta \mu_e &= \left.\frac{d}{dt}\right|_{t=0} \text{CoAd}_{\exp(\eta t)}\mu_e \\ &= \left.\frac{d}{dt}\right|_{t=0} J(\exp(\eta t)z_e) \\ &= -\left.\frac{d}{dt}\right|_{t=0} J(\exp(\eta' t)z_e) \\ &= 0, \end{aligned}$$

since J is G' invariant.

For the proof of G_{μ_e} stability, by theorem (1), it suffices to show that the G_{μ_e} invariant conserved quantity

$$C = H_{\xi_e} + \epsilon\left(\sum_l a_l J_{\eta'_l} + \sum_m b_m (J_{\eta''_m} - J_{\eta''_m}(z_e))^2\right)$$

is formally stable at z_e for some ϵ (by abuse of notation, the momentum mapping for both the G and G' actions have been designated by J). As in the proof of theorem (2), the cotangent bundle reduction theorem may be used to realize the reduced space $J^{-1}(\mu_e)/G_{\mu_e}$ as a subbundle of $T^*(Q/G_{\mu_e})$, and then the function $\bar{C}$ induced on P by C has positive definite Hessian at $\bar{q}_e$: To compute $\bar{C}$, note that G' acts on Q/G_{μ_e} and hence gives a momentum mapping $\bar{J}$ for the canonical symplectic structure on $T^*(Q/G_{\mu_e})$ as usual. This is *not* the function induced on P by $J_{\eta'}$ for $\eta' \in \mathfrak{g}'$—trivial diagram chasing shows *that* function is $\bar{J}_{\eta'} + \bar{f}_{\eta'}$, where $\bar{f}_{\eta'}$ is the function on Q/G_{μ_e} induced by $f_{\eta'}$. Thus $\bar{C}$ is the restriction to P of the function

$$\begin{aligned} &\bar{H}_{\xi_e} + \epsilon\left\{\sum_l a_l(\bar{J}_{\eta'_l} + \bar{f}_{\eta'_l}) + \sum_m b_m (\bar{J}_{\eta''_m} + \bar{f}_{\eta''_m} - f_{\eta''_m}(q_e))^2\right\} \\ &= \bar{H}_{\xi_e} + \epsilon\left\{\bar{f} + \left[\sum_l a_l \bar{J}_{\eta'_l} + \sum_m b_m (\bar{J}^2_{\eta''_m} + 2\bar{J}_{\eta''_m}(\bar{f}_{\eta''_m} - f_{\eta''_m}(q_e)))\right]\right\} \end{aligned} \tag{28}$$

Lapsing briefly to canonical coordinates on $T^*(Q/G_{\mu_e})$, if X^k are functions representing a vector field on Q/G_{μ_e}, then two derivatives with respect to q^k of the function $p_k X^k$ at $p_k = 0$ gives zero. This idea pursued shows the the Hessian of the square bracketed part of (28), when restricted to the subspace tangent to the zero section of $T^*(Q/G_{\mu_e})$, is zero at the equilibrium $0_{\bar{q}_e}$. Thus the Hessian of the curly bracketed part of the right side of equation (28) evaluated on this horizontal subspace is exactly $\mathbf{d}^2 \bar{f}(0_{\bar{q}_e})$. By hypothesis, this is positive definite on $\ker \mathbf{d}^2 \bar{H}_{\xi_e}(0_{\bar{q}_e})$, so an application of lemma (1) completes the proof. □

With this last addition to the theory, we can finally deal with the stability of last of the relative equilibria, that is degenerate classes (2a) and (2b).

Proposition 10.

1. *The class* (2a) *relative equilibria are* $\{\mathrm{Id}\} \times (S^1)^2$ *stable if* $t_1 + t_2 \neq 0$, $t_1 \neq 0$, *and* $t_2 \neq 0$.

2. *The class* (2b) *relative equilibria are* $\{\mathrm{Id}\} \times (S^1)^2$ *stable if* $t_1 - t_2 \neq 0$, $t_1 \neq 0$, *and* $t_2 \neq 0$.

Proof. Obviously, both classes (2a) and (2b) are relative equilibria for the subgroup $\{\mathrm{Id}\} \times (S^1)^2$ of the full symmetry group $SO(3) \times (S^1)^2$. The proof is obtained by using theorem (3), with

$$G = \{\mathrm{Id}\} \times (S^1)^2,$$
$$G' = SO(3) \times \{\mathrm{Id}\} \times \{\mathrm{Id}\}.$$

An application of theorem (2) with the action of G is trivial. Indeed, the matrix IGN is the 6×2 matrix found by deleting the first 3 columns of (18). Since

$$J(A) = \begin{bmatrix} 1 & 0 & 0 & -\beta A^{22} & \beta A^{21} & 0 \\ 0 & 1 & 0 & \beta A^{12} & -\beta A^{11} & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ -\beta A^{22} & \beta A^{12} & 0 & 1 & 0 & 0 \\ \beta A^{21} & -\beta A^{11} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \end{bmatrix} \qquad (29)$$

is the metric tensor at $A = A_1{}^t A_2$, the locked inertia tensor is α times the 2×2 identity matrix, so the amended potential is constant and $\mathbf{d}^2 V_{\mu_e}(q_e) = 0$.

Now a basis for $\mathfrak{g} \cdot q_e$ is the span of the vectors

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^t,$$

and these vectors are orthogonal in the kinetic energy metric to the vectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t,$$

as simple multiplication with $J(A)$ shows. Since for both classes

$$\mu_e = \alpha \begin{bmatrix} t_1 \\ t_2 \end{bmatrix},$$

the above orthogonal decomposition shows that

$$\beta_{\mu_e} \cdot \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix} = \alpha(t_1\Omega_1 \cdot \boldsymbol{k} + t_2\Omega_2 \cdot \boldsymbol{k}).$$

By proposition (2), the infinitesimal algebras of the product action for the class (2a) and (2b) relative equilibria are the spaces spanned by the single vectors

$$\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}^t,$$
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}^t,$$

respectively, so that in either case the subspace F is just the space spanned by the single vector $\boldsymbol{k}$. Then using $\boldsymbol{i}$ and $\boldsymbol{j}$ for the η''_m, the function f (for both classes) is

$$f(A_1, A_2) = a_1\alpha(t_1{A_1}^{33} + t_2{A_2}^{33}) + \alpha^2\sum_{i=1}^{2} b_i(t_1{A_1}^{i3} + t_2{A_2}^{i3})^2,$$

and obviously there is no loss of generality here if α and a_1 are set to unity.

For the class (2a), the Hessian of the function f is the 4×4 matrix

$$\begin{bmatrix} t_1(2b_2t_1 - 1) & 2b_2t_1t_2 & 0 & 0 \\ 2b_2t_1t_2 & t_2(2b_2t_2 - 1) & 0 & 0 \\ 0 & 0 & t_1(2b_1t_1 - 1) & 2b_1t_1t_2 \\ 0 & 0 & 2b_1t_1t_2 & t_2(2b_1t_2 - 1) \end{bmatrix}.$$

Since the two 2×2 blocks in this matrix are identical up to interchange of b_1 with b_2, it suffices to show that the upper block is positive or negative definite for some b_2 if $t_1 + t_2 \neq 0$, $t_1 \neq 0$, and $t_2 \neq 0$. But this is trivial: the determinant of this upper block is $-2b_2t_1t_2(t_1 + t_2) + 1$ which is positive for arbitrarily large positive or negative b_2, depending on the sign of the nonzero coefficient of b_2, and the upper diagonal element is nonzero as long as b_2 is chosen not equal to $1/2t_1$.

For the class (2b) relative equilibria, the Hessian of f is

$$\begin{bmatrix} t_1(2b_2t_1 - 1) & 2b_2t_1t_2 & 0 & 0 \\ 2b_2t_1t_2 & t_2(2b_2t_2 + 1) & 0 & 0 \\ 0 & 0 & t_1(2b_1t_1 - 1) & -2b_1t_1t_2 \\ 0 & 0 & -2b_1t_1t_2 & t_2(2b_1t_2 + 1) \end{bmatrix},$$

and so the proof in this case is entirely similar to the case (2a). □

The following theorem summarizes the relative equilibria and their stability:

Theorem 4. *The base integral curves of all relative equilibria for two axially symmetric, identical bodies are the* $(S^1)^2 \times SO(3)$ *translates of the curves in configuration space* $SO(3)^2$

$$t \to (\exp(t\Omega^\wedge)\exp(t\sigma_1 \boldsymbol{k}^\wedge), \exp(t\Omega^\wedge)\exp(t_3\boldsymbol{j}^\wedge)\exp(t\sigma_2\boldsymbol{k}^\wedge)),$$

where the total rotation Ω *and spins* σ_i *are given by* (2a)–(2h) *in conjunction with* (10a)–(10h). *In this state of motion, the axis of total rotation and the body axes are coplanar. A relative equilibria is stable if either of the following hold:*

- *The total rotation is zero, the total angular momentum is nonzero, and no body has zero spin.*
- *The bodies are bounded away from the overlap configuration by* (25) *and the joint is not at the bodies centers of mass (that is* $\beta \neq 0$*).*

The stability is modulo the product of the group of rotations about the axis of the total angular momentum and the material symmetry group, except if the bodies are rotating about their common axis of symmetry in the opposed or overlapped configurations, in which case the stability is modulo the group of material symmetries.

Bifurcations

Typically, in mechanical systems where symmetry is absent, equilibria are isolated and of finite number. For example, the 7 dimensional manifold M_0 of proposition (1) can serve as an open subset of the Poisson reduced phase space for the system of axially symmetric coupled rigid bodies; a generic symplectic leaf of the reduced space has dimension 4, while the classes of relative equilibria (2e) and (2f) have dimension 3, so transversal intersection at isolated points is expected. Thus, there is the following basic question about the bifurcation of relative equilibria as the values of the momentum changes: *how many equilibria are on each symplectic reduced phase space?* The definition of this question is not affected by singularities in the reduced phase space if we agree to count, on constant levels of the momentum map, equivalence classes of relative equilibria on the original phase space. We continue the analysis of the relative equilibria of two coupled rigid bodies by providing the following answer to this question:

Definition 2. *If* (P, ω, H, G, J) *is a Hamiltonian system with symmetry and* $\mu \in \mathfrak{g}^*$*, denote by* $\#(\mu)$ *the number of equivalence classes of relative equilibria with momentum* μ.

Theorem 5. *Let* $(\mu, \mu_1, \mu_2) \in \mathfrak{so}(3)^* \times \mathbb{R}^2$. *Then the cardinal number* $\#(\mu, \mu_1, \mu_2)$ *may be found as follows:*

- *If* $\mu = 0$ *then* $\#(\mu, \mu_1, \mu_2) = 1$ *if* $(\mu_1, \mu_2) \neq (0,0)$ *and* $\#(\mu, \mu_1, \mu_2)$ *is infinite if* $(\mu_1, \mu_2) = (0,0)$.
- *If* $\mu \neq 0$ *and* $\beta = 0$ *then locate the point* $(\mu_1, \mu_2) \in \mathbb{R}^2$ *in figure* (2).

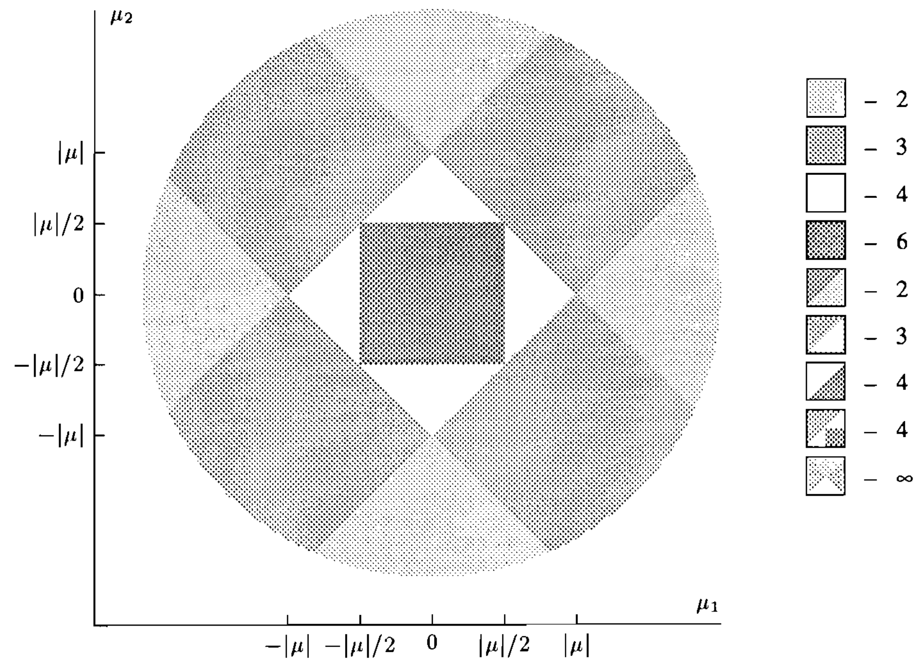


Figure 2: $\#(\mu, \mu_1, \mu_2)$ for $\beta = 0, \mu \neq 0$

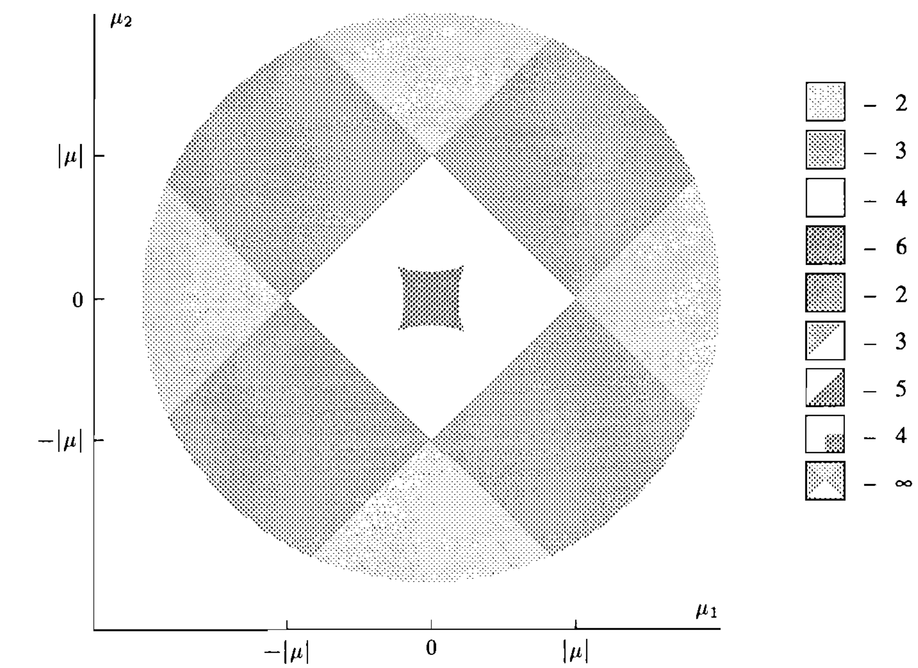


Figure 3: $\#(\mu, \mu_1, \mu_2)$ for $\beta \neq 0, \mu \neq 0$

- *If $\mu \neq 0$ and $\beta \neq 0$ then locate the point $(\mu_1, \mu_2) \in \mathbf{R}^2$ in figure* (3).

Remark. In the $\beta \neq 0$ case, the boundary of the innermost region (with count 6) is the union of a curve ϱ_β (with endpoints included) and the reflection of this same curve about the line $\mu_1 = \mu_2$. The qualitative features of this curve are well represented in the diagram and are described in proposition (20). This innermost region monotonely approaches the square containing the points $(1/2, 1/2)$ and $(-1/2, -1/2)$ as $\beta \to 0$, and monotonely approaches the origin as $\beta \to 1$.

Remark. Each of the diagrams depicts the entire plane $\mathbf{R}^2$, so one may consider the radial scale to be such that the outer bounding circle in each of the diagrams is at infinity.

Remark. As indicated, each of the diagrams is symmetric about the two lines $\mu_1 = \mu_2$ and $\mu_1 = -\mu_2$.

The proof of theorem (5) consists of a tedious plethora of easy lemmas and propositions, the first of which provide two general principles which will simplify the counting chore:

Proposition 11. *Let (P, ω, H, G, J) be a Hamiltonian system with symmetry. If $S \subseteq TQ$ intersects each group orbit exactly once, then $\#(\mu)$ is the cardinality of the set of relative equilibria in S with momenta in the co-adjoint orbit of μ.*

Proof. Let S' be the set of relative equilibria in S with momentum in the co-adjoint orbit of μ and let T be the set of equivalence classes of relative equilibria with momentum μ. For $x \in S'$, choose one $g_x \in G$ such that $\mathrm{CoAd}_{g_x} J(x) = \mu$; then the map $x \mapsto [g_x x]$ is a bijection from S' to T. □

Proposition 12. *Let (Q, G, V) be a simple mechanical system with symmetry. If $V = 0$, and $0 \neq r \in \mathbf{R}$, then $[v] \mapsto [rv]$ is a bijection between those equivalence classes of relative equilibria with momentum μ and those with momentum $r\mu$, and in particular, $\#(\mu) = \#(r\mu)$.*

Proof. By the principle of symmetric criticality (1.1), the map $v \mapsto rv$ takes relative equilibria to relative equilibria, and this map passes to equivalence classes of relative equilibria since the action of G is linear in the fibers of TQ; the resulting map is a bijection since its inverse is similarly generated by the map $v \mapsto v/r$. □

Returning now to the less general context of two coupled rigid bodies, the next proposition rephrases the problem of determining $\#(\mu, \mu_1, \mu_2)$ in terms of the list of relative equilibria provided by proposition (3):

Proposition 13. *$\#(\mu, \mu_1, \mu_2)$ is exactly the cardinality of the set of relative equilibria in the list of proposition* (3) *such that*

$$\mu_1 = \alpha v_3, \quad \mu_2 = \alpha w_3, \quad |\mu| = c(v_1, v_2, w_1, w_2, \theta), \tag{30}$$

where

$$c(v_1, v_2, w_1, w_2, \theta) \stackrel{\text{def}}{=} \Big[(1 - 2\beta\cos\theta + \beta^2)({v_1}^2 + {w_1}^2) + \alpha^2({v_3}^2 + {w_3}^2) + 2(\beta^2\cos\theta - 2\beta + \cos\theta)v_1 w_1 + 2\alpha^2 v_3 w_3 \cos\theta + 2\alpha(v_1 w_3 - w_1 v_3 + \beta v_1 v_3 - \beta w_1 w_3)\sin\theta\Big]^{\frac{1}{2}}.$$

Proof. Let J^1, J^2, and J^3 be the projections of the momentum map onto the factors of $\mathfrak{so}(3)^* \times \mathbb{R}^2$; a straight forward calculation [27] shows that the function c is just $|J^1|$ restricted to M, and of course $J^2 = \alpha v_3$ and $J^3 = \alpha w_3$. The result now follows from proposition (11), since two elements (μ, μ_1, μ_2) and (μ', μ_1', μ_2') of $\mathfrak{so}(3) \times \mathbb{R}^2$ are in the same $SO(3) \times (S^1)^2$ co-adjoint orbit if and only if $\mu_1 = \mu_1'$, $\mu_2 = \mu_2'$ and $|\mu| = |\mu'|$. □

Remark. If M_0 is regarded as a Poisson reduced phase space, then the function c together with the functions αv_3 and αw_3 are three Casimirs whose mutual constant level sets are the symplectic leaves of M_0.

Simple substitution [28] computes the values of the map defined by (30) on the lists of relative equilibria in proposition (3), and for convenience these values are collected here:

Proposition 14. *In correspondence with the list of relative equilibria* (2a)-(2h)*, the evaluations of the map* (30) *on the relative equilibria are*

$$(t_1, t_2) \mapsto \alpha t_1, \alpha t_2, \alpha|t_1 + t_2| \tag{31a}$$

$$(t_1, t_2) \mapsto \alpha t_1, \alpha t_2, \alpha|t_1 - t_2| \tag{31b}$$

$$(t_1, t_2) \mapsto \alpha t_2, \alpha t_2, 2\sqrt{(1-\beta^2){t_1}^2 + \alpha^2 {t_2}^2} \tag{31c}$$

$$(t_1, t_2) \mapsto \alpha t_2, -\alpha t_2, 2\sqrt{(1-\beta^2){t_1}^2 + \alpha^2 {t_2}^2} \tag{31d}$$

$$(t_1, t_2, t_3) \mapsto \alpha t_1, \alpha t_2, \alpha\sqrt{{t_1}^2 + 2t_1 t_2 \cos t_3 + {t_2}^2} \tag{31e}$$

$$(t_1, t_2, t_3) \mapsto -\alpha\kappa_1^{t_1 t_2 t_3}, \alpha\kappa_1^{t_2 t_1 t_3}, \frac{|\beta {t_1}^2 - 2t_1 t_2 + \beta {t_2}^2|\sqrt{{t_1}^2 + 2t_1 t_2 \cos t_3 + {t_2}^2}}{|t_1 t_2|\sin t_3} \tag{31f}$$

$$(t_1, t_2, t_3) \mapsto \alpha t_1, -\alpha t_2 \cos t_3, \alpha|t_1 - t_2| \tag{31g}$$

$$(t_1, t_2, t_3) \mapsto \alpha t_1 \cos t_3, \alpha t_2, \alpha|t_1 + t_2| \tag{31h}$$

Now only the class (2f) when $\mu \neq 0$ gives trouble; the others are easily analyzed by directly solving for the t_1, t_2, t_3 from the equations resulting from setting the values of the maps in (31a)-(31h) to $(\mu_1, \mu_2, |\mu|)$:

Proposition 15.

1. *From the class* (2a)*, there is exactly one relative equilibrium satisfying equations* (30) *if* $\mu_1 + \mu_2 = \pm|\mu|$*, and no such relative equilibria otherwise. For the class* (2b)*, the analogous statement holds if* $\mu_1 + \mu_2 = \pm|\mu|$ *is replaced by* $\mu_1 - \mu_2 = \pm|\mu|$.
2. *From the class* (2c)*, there is exactly 1 relative equilibrium satisfying equations* (30) *if* $\mu_1 = \mu_2$ *and* $|\mu_1| = |\mu_2| < |\mu|/2$*, and no such relative equilibria otherwise. For the class* (2d)*, the analogous statement holds if* $\mu_1 = \mu_2$ *is replaced by* $\mu_1 = -\mu_2$.

3. *From the class* (2e)*, and when* $\mu \neq 0$*, there is exactly one relative equilibrium satisfying equations* (30) *if either*

$$|\mu_1 + \mu_2| < |\mu| \quad \textit{and} \quad |\mu_1 - \mu_2| > |\mu| \tag{32}$$

or

$$|\mu_1 + \mu_2| > |\mu| \quad \textit{and} \quad |\mu_1 - \mu_2| < |\mu| \tag{33}$$

and four infinite sets of relative equilibria (wherein one body is spinning about its axis of symmetry and the other body is stationary) satisfying (30) *if*

$$\mu_1 = \pm|\mu| \quad \textit{and} \quad \mu_2 = 0 \qquad \textit{or} \qquad \mu_2 = \pm|\mu| \quad \textit{and} \quad \mu_1 = 0, \tag{34}$$

and no such relative equilibria otherwise.

4. *From the class* (2e)*, and when* $\mu = 0$*, there is an infinite set of relative equilibria (wherein both bodies are stationary) satisfying equations* (30) *if* $\mu_1 = 0$ *and* $\mu_2 = 0$*, and no such relative equilibria otherwise.*

5. *From the class* (2f)*, and when* $\mu = 0$*, there is exactly 1 relative equilibrium satisfying equations* (30) *if* $\beta \neq 0$ *and* $\mu_1 \neq \pm\mu_2$ *and no such relative equilibria otherwise.*

6. *From the class* (2g) *and* (2h)*, so* $\beta = 0$ *necessarily, and when* $\mu \neq 0$*, there are exactly 4 relative equilibria satisfying equations* (30) *if* $|\mu_1| + |\mu_2| < |\mu|$*, and exactly 1 such relative equilibrium if* $\pm\mu_1 \pm \mu_2 = |\mu|$ *and* $|\mu_1| + |\mu_2| > |\mu|$*, and exactly 2 such relative equilibria otherwise.*

7. *From the class* (2g) *and* (2h)*, so* $\beta = 0$ *necessarily, and when* $\mu = 0$*, there are no relative equilibria satisfying equations* (30) *if* $\mu_1 = \pm\mu_2$ *and exactly 1 such relative equilibrium otherwise.*

Proof. Most of this is easy enough that it needs no elaboration; for example to prove statements (3) and (4), it is required to find all $(t_1, t_2, t) \in \mathbb{R}^2 \times (-1, 1)$ such that (31e) has value $(\mu_1, \mu_2, |\mu|)$. Substituting $t_1 = \mu_1/\alpha$ and $t_2 = \mu_2/\alpha$ into the third component of (31e), and then isolating $t \stackrel{\text{def}}{=} \cos t_3$, one obtains

$$2\mu_1\mu_2 t = |\mu|^2 - {\mu_1}^2 - {\mu_2}^2. \tag{35}$$

Suppose $|\mu| \neq 0$. Then if $\mu_1 = 0$ then $\mu_2 = \pm|\mu|$ and t is arbitrary, and the evaluation of (31e) on any of the infinity of points $(0, \pm|\mu|, t)$ yields $(0, \mu_2, |\mu|)$, and similarly if $\mu_2 = 0$. On the other hand, if $\mu_1 \neq 0$ and $\mu_2 \neq 0$, then equation (35) provides a unique solution for $t \in (-1, 1)$ if and only if

$$-1 < \frac{|\mu|^2 - {\mu_1}^2 - {\mu_2}^2}{2\mu_1\mu_2} < 1,$$

an inequality which is easily seen to be equivalent to the "or" of the two inequalities (32) and (33). Finally, when $\mu = 0$, (31e) implies that $t_1 = t_2 = 0$, since $t \in (-1, 1)$, and then the map (35) simply becomes $(0, 0, t) \mapsto (0, 0, 0)$. □

By proposition (12), $\#(\mu, \mu_1, \mu_2) = \#(r\mu, r\mu_1, r\mu_2)$ for any $r \neq 0$, so we may assume that $|\mu| = 1$ when considering the class (2f) (or any other class) relative equilibria, as long as $\mu \neq 0$. The result is that the problem of counting the class (2f) relative equilibria is reduced to the problem of counting pre-images of a point under a map from an open subset of the plane to the plane. Generally, let us denote by $\#S$ the cardinality of a set S.

Proposition 16. *Define the sets*

$$\begin{aligned} D_0^- &\overset{\text{def}}{=} (-\infty, 0) \times (-1, 1) \\ D_0^+ &\overset{\text{def}}{=} (0, \infty) \times (-1, 1) \end{aligned} \quad \textit{and} \quad D_0 \overset{\text{def}}{=} D_0^- \cup D_0^+,$$

and the sets

$$\begin{aligned} D_\beta^- &\overset{\text{def}}{=} (-\infty, 0) \times (-1, 1) \\ D_\beta^{+-} &\overset{\text{def}}{=} (0, \zeta_1) \times (-1, 1) \\ D_\beta^{+0} &\overset{\text{def}}{=} (\zeta_1, 1/\zeta_1) \times (-1, 1) \\ D_\beta^{++} &\overset{\text{def}}{=} (1/\zeta_1, \infty) \times (-1, 1) \end{aligned} \quad \textit{and} \quad D_\beta \overset{\text{def}}{=} D_\beta^- \cup D_\beta^{+-} \cup D_\beta^{+0} \cup D_\beta^{++},$$

and define the map $\Psi_\beta : D_\beta \to \mathbb{R}^2$ *by*

$$\Psi_\beta(z,t) \overset{\text{def}}{=} \frac{1}{(\beta z^2 - 2z + \beta)\sqrt{z^2 - 2zt + 1}}((zt-1)(\beta - z), z(z-t)(\beta z - 1)).$$

Then from the class (2f) *equilibria there are exactly*

$$\#(\Psi|D_\beta)^{-1}(\mu_1, \mu_2) + \#(\Psi|D_\beta)^{-1}(-\mu_1, -\mu_2) \tag{36}$$

relative equilibria satisfying equations (30).

Proof. Since $t_1 \neq 0$ and $t_2 \neq 0$ in (31f), and since $t_3 \in (0, \pi)$, it is permissible to change dependent variables in (31f) to (z, t, t_2), where $z \overset{\text{def}}{=} t_1/t_2$ and $t \overset{\text{def}}{=} \cos t_3$. Then, after trivial rearrangements, the problem is to count all $z \neq 0$ and $|t| < 1$ such that

$$\frac{-t_2(zt-1)(\beta - z)}{z\sqrt{1-t^2}} = \mu_1, \quad \frac{-t_2(z-t)(\beta z - 1)}{\sqrt{1-t^2}} = \mu_2,$$
$$\frac{|t_2||\beta z^2 - 2z + \beta|\sqrt{z^2 + 2zt + 1}}{|z|\sqrt{1-t^2}} = 1.$$

Solving for t_2, these equations are equivalent to

$$t_2 = \frac{\pm|z|\sqrt{1-t^2}}{|\beta z^2 - 2z + \beta|\sqrt{z^2 + 2zt + 1}}, \tag{37}$$
$$-\operatorname{sgn} t_2 \operatorname{sgn}(z(\beta z^2 - 2z + \beta))\Psi_\beta(z,t) = (\mu_1, \mu_2). \tag{38}$$

Then (36) holds if $\mu_1 \neq 0$ or $\mu_2 \neq 0$, since there is exactly one solution of (37) and (38) for each (z, t) such that $\Psi_\beta(z,t) = (\mu_1, \mu_2)$ and exactly one solution of (37) and (38) for each (z, t) such that $\Psi_\beta(z,t) = (-\mu_1, -\mu_2)$, as follows:

- If $\Psi(z,t) = (\mu_1, \mu_2)$, then if $z(\beta z - 2z + \beta) > 0$, use the negative option in (37) to obtain $t_2 < 0$ and the positive option otherwise.
- If $\Psi(z,t) = (-\mu_1, -\mu_2)$, then if $z(\beta z - 2z + \beta) > 0$, use the positive option in (37) to obtain $t_2 > 0$ and the negative option otherwise.

Finally, (36) holds if $\mu_1 = \mu_2 = 0$ as well, since then the factor $\mathrm{sgn} t_2$ is irrelevant in (38) and either sign in (37) may be used, so there are two solutions for each (z,t) such that $\Psi_\beta(z,t) = (\mu_1, \mu_2)$, since $\mu \neq 0$ and $z \neq 0$. □

Since explicitly computing the (z,t) which map under Ψ_β to (μ_1, μ_2) seems difficult, we are forced to provide minor extensions to the global implicit function theorem as it is described, for example, in [5]. In particular, when finding the image of some map, we need to pay careful attention to where the boundary of the domain maps to, as well as to the critical values.

Definition 3. *If X and Y are first countable topological spaces and $f : A \to Y$ is continuous, define*

$$\Sigma'_f \stackrel{\text{def}}{=} \left\{ \lim_{n\to\infty} f(x_n) \,\middle|\, x_n \to x \text{ and } x \notin \mathrm{Int} A \right\},$$
$$\sigma''_f \stackrel{\text{def}}{=} \left\{ x \,\middle|\, f \text{ is not locally invertible at } x \in \mathrm{Int} A \right\},$$

and also define $\sigma'_f = f^{-1}(\Sigma'_f)$, $\Sigma''_f = f(\sigma''_f)$ and $\Sigma_f = \Sigma'_f \cup \Sigma''_f$.

The following lemma will reduce some results considered below to those in [5].

Lemma 3. *If $A \subseteq X$ is relatively compact, then $f : \mathrm{Int} A \setminus \sigma_f \to Y \setminus \Sigma_f$ is proper.*

Proof. Let $K \subseteq Y \setminus \Sigma_f$ be compact, *and suppose $f^{-1}(K)$ is not compact.* Then some subsequence $x_n \in A \setminus \sigma_f$ has no subsequence converging in $A \setminus \sigma_f$, while the sequence $f(x_n) \in K$ does have a convergent subsequence, as does the sequence x_n, regarded as a sequence in X. Thus, one may assume that $x_n \to x \notin A \setminus \sigma_f$ and $f(x_n) \to y \in K$. If $x \notin \mathrm{Int} A$, then $y \in \Sigma'_f$ by the definition of Σ'_f, *a contradiction*, so $x \in \sigma_f$ necessarily. Then since f is continuous, $f(x) = y$, so $y \in f(\sigma_f) \subseteq \Sigma_f$, and $y \in K \setminus \Sigma_f$, *also a contradiction*. Thus $f^{-1}(K)$ is compact, so $f : A \setminus \sigma_f \to Y \setminus \Sigma'_f$ is proper. □

Proposition 17. *Suppose X and Y are first countable topological spaces, $A \subseteq X$ is relatively compact and $f : A \to Y$ is continuous. Then $\# f^{-1}(y)$ is finite and constant on any connected component of $Y \setminus \Sigma_f$.*

Proof. By proposition (3.4) of [5] and lemma (3), $f^{-1}(y) \cap (A \setminus \sigma_f)$ is finite and constant on connected components of $Y \setminus \Sigma_f$. The proof is completed by noting that since $f(\sigma_f) \subseteq \Sigma_f$, if $y \in Y \setminus \Sigma_f$ then $f^{-1}(y) \cap (A \setminus \sigma'_f) = f^{-1}(y)$. □

Remark. The restriction that A be relatively compact is not burdensome; if not, then replace X with some compactification of X.

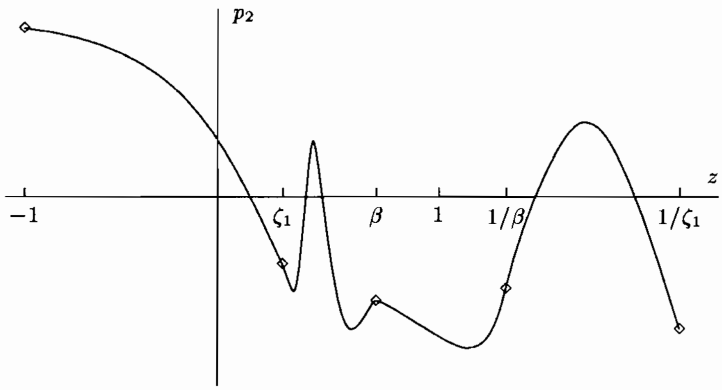


Figure 4: This graph has 6 critical points, so it cannot be the graph of p_2.

Proposition 18. *Suppose X and Y are first countable topological spaces, $A \subseteq X$ is relatively compact and $f : A \to Y$ is continuous. If $Y \setminus \Sigma_f$ is simply connected and $X \setminus \sigma_f$ is connected, then f is a homeomorphism from $A \setminus \sigma_f$ to $Y \setminus \Sigma_f$.*

Proof. Using theorem (3.5) of [5] the proof is identical to that of proposition (17). □

At the lowest level, the analysis of the map Ψ_β will be reduced to the analysis of the roots of various multivariate polynomials of moderate degree. The next lemma is a result of this type and will be used in proposition (19).

Lemma 4. *For $0 < \beta < 1$, the polynomial*

$$\begin{aligned} p_2(z) &= b(5b^2-1)z^6 - 6b^2(3+b^2)z^5 + 3b(11+9b^2)z^4 - 4(4+15b^2+b^4)z^3 \\ &\quad + 3b(11+9b^2)z^2 - 6b^2(3+b^2)z + b(5b^2-1) \end{aligned}$$

is strictly negative on the two open intervals (ζ_1, β) and $(1/\beta, 1/\zeta_1)$.

Proof. Since the set of roots of p_2 are invariant under the map $z \mapsto 1/z$, p_2 has a root in one of the intervals if and only if it has a root in the other. *So suppose p_1 has a root $\bar{z} \in (\zeta_1, \beta)$.* Note that

$$p_2(-1) = 16(1+\beta^4) > 0$$
$$\beta^6 p_2(1/\beta) = p_2(\beta) = -\beta(1-\beta^2)^4 < 0$$
$$p_2(\zeta_1) = {\zeta_1}^6 p_2(1/\zeta_1) = -2\frac{\zeta_1(1-{\zeta_1}^2)^6}{(1+{\zeta_1}^2)^4)} < 0,$$

and furthermore,

$$p_2'(\beta) = 0, \quad p_2''(\beta) = -30\beta(1-\beta^2)^3 < 0, \tag{39}$$

so p_2 has a relative maximum at $z = \beta$. Thus, the graph of p_2 is accurately represented in figure (4), so, the nontrivial polynomial p_2 has degree at most 5 and also has at least 6 critical points, a *contradiction.* □

By proposition (17), to count the number of elements in $\#f^{-1}(y)$ it is enough to consider one y in each connected component of $Y \setminus \Sigma_f$. For the map Ψ_β, the following will suffice:

Proposition 19.

1. *If* $\beta = 0$ *and* $\mu_1 \pm \mu_2 = \pm 1$ *then* $\#(\Psi_\beta | D_\beta) = 0$.

2. *If* $\beta \neq 0$, *and if* $\mu_1 \pm \mu_2 = \pm 1$, *then* $\#(\Psi_\beta | D_\beta)^{-1}(\mu_1, \mu_2) = 0$, *except in the following cases:*

   - $\#(\Psi_\beta | D_\beta)^{-1}(\mu_1, \mu_2) = 1$ *if* $\mu_1 + \mu_2 = -1$ *and* $-1 < \mu_1$.
   - $\#(\Psi_\beta | D_\beta)^{-1}(\mu_1, \mu_2) = 1$ *if* $\mu_1 + \mu_2 = +1$ *and* $0 < \mu_1$.
   - $\#(\Psi_\beta | D_\beta)^{-1}(\mu_1, \mu_2) = 1$ *if* $\mu_1 - \mu_2 = +1$ *and* $\mu_1 \notin (0, 1)$.
   - $\#(\Psi_\beta | D_\beta)^{-1}(\mu_1, \mu_2) = 2$ *if* $\mu_1 - \mu_2 = +1$ *and* $\mu_1 \in (0, 1)$.

   Moreover, if $\Psi_\beta(z, t) = (\mu_1, \mu_2)$ *and* $\mu_1 \pm \mu_2 = \pm 1$, *then* z *is in one of the open intervals* (ζ_1, β) *or* $(1/\zeta_1, 1/\beta)$.

3. *Define*

$$r_{\max} \stackrel{\text{def}}{=} \sqrt{\frac{1-\beta}{2(1+\beta)}}.$$

   Then if $\beta \neq 0$ *and* $\mu_1 = \mu_2$,

   - $\#(\Psi_\beta | D_\beta^{+0}) = 1$ *if* $\mu_1 = \mu_2$ *and* $\mu_1 \notin (-r_{\max}/\sqrt{2}, r_{\max}/\sqrt{2})$,
   - $\#(\Psi_\beta | D_\beta^{+0}) = 2$ *if* $\mu_1 = \mu_2$ *and* $\mu_1 \in (-r_{\max}/\sqrt{2}, r_{\max}/\sqrt{2})$.

Proof. This proof is supported by the `MAPLE` symbolic manipulator [22]. Consider first the case of $\mu_1 + \mu_2 = \pm 1$, an equation which is equivalent to the equation $(\mu_1 + \mu_2)^2 - 1 = 0$. Substituting $(\mu_1, \mu_2) = \Psi_\beta(z, t)$ in the latter yields the following linear equation in t:

$$\begin{aligned} p_3(t) &\stackrel{\text{def}}{=} -z(1+\beta)^2(z-1)^2 t + 2\beta z^4 - (3\beta^2 - 2\beta + 3)z^3 \\ &\qquad - 2(\beta^2 - 4\beta + 1)z^2 - (3\beta^2 - 2\beta + 3)z + 2\beta \\ &= 0 \end{aligned} \tag{40}$$

The value at $z = 1$ is $-8(1-\beta)^2$, so there is no solution when $z = 1$. Assuming $z \neq 1$, $p_2(t)$ is a line with nonzero slope which has the evaluations

$$p_3(-1) = 2(1+z)^2(z-\beta)(\beta z - 1),$$
$$p_3(1) = 2(z^2 - 2\beta z + 1)(\beta - 2z + \beta z^2).$$

Thus (40) has exactly one solution for $t \in (-1, 1)$ if and only if $p_3(-1)$ and $p_3(1)$ have opposite sign, which is never if $\beta = 0$ and exactly when z is in one of the open intervals (ζ_1, β) or $(1/\zeta_1, 1/\beta)$ in the case that $\beta \neq 0$.

When $\beta \neq 0$ and z is in one of those two open intervals, one obtains, after solving (40) for t and substituting the result into $(\mu_1, \mu_2) = \Psi_\beta(z,t)$,

$$\mu_1 = \frac{\operatorname{sgn}(1-z)(z-\beta)(2\beta z^3 - 3(1+\beta^2)z^2 + 6\beta z - (1+\beta^2))}{(z^2-1)(1-\beta^2)(\beta z^2 - 2z + \beta)}, \tag{41}$$

$$\mu_2 = \frac{\operatorname{sgn}(1-z)(1-\beta z)((1+\beta^2)z^3 - 6\beta z^2 + 3(1+\beta^2)z - 2\beta)}{(z^2-1)(1-\beta^2)(\beta z^2 - 2z + \beta)}, \tag{42}$$

and so $\mu_1 + \mu_2 = \operatorname{sgn}(1-z)$. The derivative of (41) with respect to z is

$$\frac{\operatorname{sgn}(1-z)p_2(z)}{(1-z^2)^2(1-\beta^2)(\beta z^2 - 2z + \beta)},$$

which is never zero by lemma (4), and furthermore (41) evaluated at β and ζ_1 is 0 and ∞ respectively. Thus, (41) maps the interval (β, ζ_1) bijectively to the interval $(0, \infty)$ so (41) and (42) together map the first interval bijectively to the line $\mu_1 + \mu_2 = 1$ where $0 < \mu_1$. Similarly the interval $(1/\zeta_1, 1/\beta)$ is mapped bijectively to the part of line $\mu_1 + \mu_2 = -1$ where $-\mu_1 < -1$. Thus there is exactly one solution to the equation $\Psi_\beta(z,t) = (\mu_1, \mu_2)$ on the line $\mu_1 + \mu_2 = -1$ where $-1 < \mu_1 < \infty$ and one solution on the line $\mu_1 + \mu_2 = 1$ where $0 < \mu_1 < \infty$, as required. The case $\mu_1 - \mu_2 = \pm 1$ is similar, resulting in exactly one solution to the equation $\Psi_\beta(z,t) = (\mu_1, \mu_2)$ on the line $\mu_1 - \mu_2 = 1$ where $\mu_1 \notin (0,1)$, two solutions on the same line if $\mu_1 \in (0,1)$, and exactly no solutions for any (μ_1, μ_2) on the line $\mu_1 - \mu_2 = -1$.

Finally, to compute $\#(\Psi_\beta | D_\beta^{+0})$ along the line $\mu_1 = \mu_2$, note that $\mu_1 = \mu_2$ is equivalent to

$$-(1+z)(\beta z^2 - (1 + \beta - (1-\beta)t)z + \beta) = 0.$$

Since $z = -1$ is inconsistent with $(z,t) \in D_\beta^{+0}$, this is equivalent to

$$t = -\frac{\beta z^2 - (1+\beta)z + \beta}{(1-\beta)z}. \tag{43}$$

It is easily verified that (43) yields $t \in (-1,1)$ if and only if $z \in (\zeta_1, 1)$ or $z \in (1, 1/\zeta_1)$. Substituting (43) into the first component of Ψ_β gives

$$\mu_1 = -\operatorname{sgn}(z-1)\frac{(\beta z - 1)(z-\beta)}{(\beta - 2z + \beta z^2)\sqrt{1-\beta^2}} \tag{44}$$

and the derivative of (44) is zero only at $z = \pm 1$. Then by checking endpoints, one verifies that (44) maps the interval $(\zeta_1, 1)$ bijectively to the interval $(-r_{\max}/\sqrt{2}, \infty)$ and the interval $(1, 1/\zeta_1)$ bijectively to the interval $(-\infty, r_{\max}/\sqrt{2})$. Therefore, the points on the line $\mu_1 = \mu_2$ with $\mu_1 \in (-r_{\max}/\sqrt{2}, r_{\max}/\sqrt{2})$ have two pre-images under $\Psi_\beta | D_\beta$, and the others only one, as required. □

The following lemma is needed in proposition (20), but is placed here because it illustrates a technique that can be used to show that two multivariate polynomials do not have a common zero, a problem that will occur in lemma (6). From here on our dealings with the map Ψ_β are supported by the `MAPLE` symbolic manipulator through [23].

Lemma 5. *If $0 < \beta < 1$, the polynomial*

$$\begin{aligned} p_4(z) \stackrel{\text{def}}{=} & \beta((1-\beta^2)^2-2)z^6 + 12\beta^2 z^5 - 3\beta(\beta^4+6\beta^2+3)z^4 + 4(5\beta^4+4\beta^2+1)z^3 \\ & - 3\beta(\beta^4+6\beta^2+3)z^2 + 12\beta^2 z + \beta((1-\beta^2)^2-2) \end{aligned} \tag{45}$$

is strictly positive on the interval $[\zeta_2, 1/\zeta_2]$, where ζ_2 is the single real root of the polynomial p_1, as in lemma (2).

Proof. One easily checks that

$$\frac{1}{z^3}p_4 = \beta((1-\beta^2)^2-2)x^3 + 12\beta^2 x^2 + 20\beta^4 - 8\beta^2 + 4 \tag{46}$$

where $x = z + 1/z$. Now the cubic (46) has exactly one root in the interval $(2, \infty)$: Indeed, the derivative of (46) is a concave down quadratic polynomial with vertical axis

$$x = \frac{4\beta}{2-(1-\beta^2)^2}, \tag{47}$$

which is easily checked to be in the interval $(0, 2)$, and the evaluation of (46) at $x = 2$ is strictly negative. Therefore, p_4 has at most two roots on the interval $(0, \infty)$, since the function $z \mapsto z + 1/z$ takes this interval at most two to one to the interval $[2, \infty)$. Then since

$$p_4(1) = 4(1-\beta)^5 > 0, \quad p_4(\beta) = \beta^6 p_4(1/\beta) = -\beta(1-\beta)^2 < 0, \tag{48}$$

it is obvious that p_4 has exactly two roots in the interval $(\beta, 1/\beta)$.

Therefore, if $p_4(\zeta_2) > 0$, then p_4 will have one root in the interval (β, ζ_2) and one root in the interval $(1/\zeta_2, 1/\beta)$, and hence no roots in $[\zeta_2, 1/\zeta_2]$, as required. For the moment, assume that $p_4(\zeta_2) \neq 0$ for all $0 < \beta < 1$. Then since ζ_2 is a smooth function of β, $p_4(\zeta_2)$ is either strictly positive or strictly negative for all $0 < \beta < 1$, so it is enough to check that $p_4(\zeta_2) > 0$ for one specific β, say $\beta = 1/2$. But simple evaluation shows that if $\beta = 1/2$ then

$$p_1(3/4) = -\frac{11}{256} < 0, \quad p_4(3/4) = \frac{337}{131072} > 0. \tag{49}$$

Thus, if $\beta = 1/2$ then $3/4 < \zeta_2$ from the first evaluation, so if $p_4(\zeta_2) < 0$ then there would be one root of p_4 in the interval $(3/4, \zeta_2)$ and another in $(\beta, 3/4)$, a contradiction. Thus, $p_4(\zeta_2) > 0$ at $\beta = 1/2$, and hence for all β, as required.

It remains to be seen that $p_4(\zeta_2) \neq 0$; that is that p_1 and p_4 have no common root. One general way to establish necessary conditions under which two such multivariate polynomials a_1 and a_2 have a common root is as follows: suppose the leading coefficient of a_1 cannot be zero. Then if a_1 and a_2 have a common zero, so do a_2 and a_3, where a_3 is the remainder of the division of a_1 into a_2, regarding a_1 and a_2 as polynomials in just one of the indeterminants. If the leading coefficient of a_2 might be zero, then this same argument applies after deleting the leading coefficient of a_2. Iterating this argument yields a list of sets of polynomials in one less indeterminant, one set of which must have a common zero if a_1 and a_2 have a common zero. When applied to polynomials in one indeterminant, this argument amounts to the Euclidean algorithm for the computation of the greatest common divisor and the assertion that two relatively prime polynomials cannot have a common root.

Applied to the polynomials p_1 and p_4, the above procedure produces the following list of polynomials in β, brought to you almost unedited for your viewing enjoyment:

$$\{\, 2-\beta^2 \neq 0, 3\beta(\beta^6+19\beta^4+20\beta^2-16)(1-\beta^2)^3 \neq 0,$$
$$8(\beta^6+2\beta^4+17\beta^2+4)(\beta^2-2)^4 \neq 0, (\beta^6+19\beta^4+20\beta^2-16)^2(1-\beta^2)^6\beta = 0\,\}$$
$$\{\, 2-\beta^2 = 0, 3\beta \neq 0, 9(\beta+1)(\beta-1)(3\beta^8-16\beta^6+29\beta^4-22\beta^2+4) \neq 0,$$
$$(-5264\beta^2+24484\beta^4+729\beta^{16}-56680\beta^6+74273\beta^8-\beta^{10}+26838\beta^{12}$$
$$+432-6804\beta^{14})\beta = 0\,\}$$
$$\{\, 2-\beta^2 = 0, 3\beta = 0, 3\beta^2 \neq 0, 729\beta^10-1701\beta^8+1242\beta^6-458\beta^4+61\beta^2-1 = 0\,\}$$
$$\{\, 2-\beta^2 = 0, 3\beta = 0, 3\beta^2 = 0, \beta = 0\,\}$$
$$\{\, 2-\beta^2 = 0, 3\beta \neq 0, 9(1-\beta^2)(3\beta^8-16\beta^6+29\beta^4-22\beta^2+4) = 0,$$
$$\beta(91\beta^4-56\beta^2+9\beta^8+8-54\beta^6) = 0\,\}$$
$$\{\, 2-\beta^2 \neq 0, 3\beta(\beta^6+19\beta^4+20\beta^2-16)(1-\beta^2)^3 = 0,$$
$$6\beta^2(-1+5\beta^2)(\beta^2+4)(1-\beta^2)^3 \neq 0, 2019\beta^{16}+33175\beta^{14}-268784\beta^{12}-617904\beta^{10}$$
$$+943808\beta^8-539008\beta^6+29184\beta^4+38912\beta^2+\beta^{20}-2843\beta^{18}+8192 = 0\,\}$$
$$\{\, 2-\beta^2 \neq 0, 3\beta(\beta^6+19\beta^4+20\beta^2-16)(1-\beta^2)^3 = 0,$$
$$6\beta^2(-1+5\beta^2)(\beta^2+4)(1-\beta^2)^3 = 0, \beta(\beta^6-17\beta^4-16\beta^2-16)(1-\beta^2)^3 = 0\,\}$$
$$\{\, 2-\beta^2 \neq 0, 3\beta(\beta^6+19\beta^4+20\beta^2-16)(1-\beta^2)^3 \neq 0,$$
$$8(\beta^6+2\beta^4+17\beta^2+4)(\beta^2-2)^4 = 0, 16\beta(1+\beta^2)(\beta^2+5)(\beta^2-2)^4 = 0\}$$

The first of these sets of equations is manifestly inconsistent, while the equations in the second, third, fourth, fifth and eighth set cannot all vanish, since $\beta \neq \sqrt{2}$, and in the sixth and the seventh, one may compute that two equations in each are relatively prime. Thus, none of the sets of equations have a common root for $0 < \beta < 1$, so p_1 and p_4 do not have a common root, as required. □

To compute $\#\Psi_\beta^{-1}(\mu_1, \mu_2)$ for $(\mu_1, \mu_2) \in \Sigma''_{\Psi_\beta}$ also seems difficult. Fortunately, the bifurcations that occur at these points can be completely analyzed by the following results of H. Whitney **[31]**:

Definition 4. *Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be C^∞. Then a point $p = (x, y)$ is good if p is not a critical point of f or if $dJ_f(p) = 0$, where $J_f \stackrel{\text{def}}{=} \det Df$, and f is called good if every point of U is good . A good critical point p is a fold point if $\dot{J}_f(p) \neq 0$, where*

$$\dot{J}_f(p) \stackrel{\text{def}}{=} Df(p) \begin{bmatrix} -\partial J_f/\partial y \\ \partial J_f/\partial x \end{bmatrix},$$

and is a cusp point if it is not a fold point and $\ddot{J}_f(p) \neq 0$, where

$$\ddot{J}_f(p) \stackrel{\text{def}}{=} D\dot{J}_f(p) \begin{bmatrix} -\partial J_f/\partial y \\ \partial J_f/\partial x \end{bmatrix}.$$

The map f is called excellent if it is good and if every critical point is either a fold point or a cusp point.

Theorem 6. *Suppose $f : U \subseteq \mathbf{R}^2 \to \mathbf{R}^2$ is C^∞ and $p \in U$ is a fold point [respectively cusp point]. Then f is locally conjugate by local C^∞ diffeomorphisms to the map $(x,y) \mapsto (x^2, y)$ [respectively $(x,y) \mapsto (xy - x^3, y)$].*

We now turn to the problem of determining the critical points and critical values of the map Ψ_β.

Lemma 6.

1. *The set of critical points of the map $\Psi_\beta | D_\beta$ is exactly the zero set of the polynomial p^Δ of lemma* (2). *Thus, there are no critical points if $\beta = 0$, and if $\beta \neq 0$ the critical points form one of the curves in figure* (1).

2. *The map $\Psi_\beta | D_\beta$ is excellent for all $0 \leq \beta < 1$, and if $\beta \neq 0$ then all critical points are fold points, except $z = 1$, $t = (3\beta - 1)/(\beta + 1)$, which is a cusp point having critical value*
$$\mu_1 = -\mu_2 = -\frac{1}{2}\sqrt{\frac{1-\beta}{1+\beta}}.$$

3. *If $\beta \neq 0$ and $(z,t) \in D_\beta$ is a critical point with $(\mu_1, \mu_2) = \Psi_\beta(z,t)$ on the line through the origin with slope m, then $z \in [\zeta_2, 1/\zeta_2)$ and*
$$m^2 = -\frac{p_1(z)(\beta z - 1)^3}{p_1(1/z)(z-\beta)^3 z^3}, \tag{50}$$
and also
$$\begin{aligned} r^2 &\stackrel{\text{def}}{=} {\mu_1}^2 + {\mu_2}^2 \\ &= \frac{-1}{(1-\beta^2)(\beta - 2z + \beta z^2)^3}\Big(\beta(\beta^4 - 2\beta^2 - 1)z^6 + 12\beta^2 z^5 \\ &\quad - 3\beta(\beta^4 + 6\beta^2 + 3)z^4 + 4(5\beta^4 + 4\beta^2 + 1)z^3 \\ &\qquad - 3\beta(\beta^4 + 6\beta^2 + 3)z^2 + 12\beta^2 z + \beta(\beta^4 - 2\beta^2 - 1)\Big). \end{aligned} \tag{51}$$
where the polynomial p_1 and its single real root ζ_2 are as in lemma (2). *Conversely, for any $m \neq 1$, there is exactly one $z \in [\zeta_2, 1/\zeta_2)$, say $z_{m,\beta}$, satisfying* (50), *and if one sets*
$$t_{m,\beta} = \begin{cases} \dfrac{\beta z_{m,\beta}^3 - z_{m,\beta}^2 - m z_{m,\beta} + m\beta}{z_{m,\beta}(\beta z_{m,\beta} - m z_{m,\beta} + m\beta - 1)} & \text{if } m \neq -1 \\ \dfrac{3\beta - 1}{\beta + 1} & \text{if } m = -1 \end{cases} \tag{52}$$
then $t_{m,\beta} \in (-1, 1)$, and $(z_{m,\beta}, t_{m,\beta})$ is a critical point of Ψ_β with $\Psi_\beta(z_{m,\beta}, t_{m,\beta})$ on the line through the origin with slope m. Moreover, the map $(m, \beta) \mapsto z_{m,\beta}$ is C^∞ away from $m = -1$, and if $m^2 < 1$ is fixed, then
$$z_{m,\beta} = \left(\frac{\beta}{2(1-m^2)}\right)^{1/3} + \mathcal{O}(\beta). \tag{53}$$

Proof. The determinant of the derivative of Ψ_β at (z,t) is

$$J_{\Psi_\beta}(z,t) = -\frac{zp^\Delta(z,t)}{(z^2-2zt+1)^2(\beta z^2-2z+\beta)},$$

which vanishes exactly when p^Δ vanishes. This completes the proof of the first statement.

If $\beta = 0$, then Ψ_β is obviously excellent, since it has no critical points by lemma (2), and because the third statement refers only to the case $\beta \neq 0$, one may assume $\beta \neq 0$ in the remainder of the proof. To show that every critical point is good, one is required to calculate the derivative of J_{Ψ_β} and show that it is not zero at any critical point. In fact, the partial derivative of Ψ_β with respect to t is

$$\begin{aligned}\frac{dJ_{\Psi_\beta}}{dt} &= [\beta(\beta^2-3)z^4 + 2(3\beta^2+1)z^3 - 6\beta(1+\beta^2)z^2 + 2(3\beta^2+1)z \\ &\quad + \beta(\beta^2-3)]zt + \beta z^6 - 4\beta^2 z^5 + \beta(2\beta^2+5)z^4 \\ &\quad - 4(1+\beta^2)z^3 + \beta(2\beta^2+5)z^2 - 4\beta^2 z + \beta. \end{aligned} \tag{54}$$

Now if at some critical point (z,t) both the coefficient of t in (54) vanished and (54) itself vanished, then J_{Ψ_β} would not depend on t at that z. Since J_{Ψ_β} is zero at any critical point, this would imply that J_{Ψ_β} vanish for some z and all $t \in (-1,1)$, contradicting lemma (2). Thus the coefficient of t in (54) does not vanish, so (54) may be solved for t as a function of z. The result of putting that solution into p^Δ yields just three possibilities: $z=1$, $z=\zeta_2$, or $z = 1/\zeta_2$. If $z=1$, then $t=(3\beta-1)/(\beta+1)$, and the value of (54) at this critical point is not zero by direct substitution. If $z=\zeta_2$ then $t=z$ necessarily, and putting $t=z$ into the partial derivative of J_{Ψ_β} with respect to z implies that

$$\begin{aligned}&\beta^2(\beta^2-2)z^6 + 4\beta(-1+2\beta^2)z^5 - (14\beta^4-5\beta^2-4)z^4 - 8\beta(1-\beta^2)z^3 \\ &\quad + \beta^2(9\beta^2-4)z^2 - 4\beta(-1+2\beta^2)z + \beta^2 = 0.\end{aligned}$$

Thus, this polynomial and p_1 have a common root, namely ζ_2, and now a proof similar to that used in lemma (5) shows this is impossible. The case of $z = 1/\zeta_2$ is similar, and these arguments show that Ψ_β is good. That $z=1$ and $t=(3\beta-1)/(\beta+1)$ is a cusp point with the indicated critical value is simply a matter of substitution into $\dot{J}_\Psi$ and $\ddot{J}_\Psi$. The proof that every other critical point is a fold point can be accomplished by yet more applications of the method of lemma (5) to the various polynomials obtained by computing $\dot{J}_{\Psi_\beta}$.

For the third statement, suppose (z,t) is a critical point and $(\mu_1,\mu_2) = \Psi_\beta(z,t)$ is on the line with slope m. Then

$$0 = \mu_2 - m\mu_2 = \frac{(\beta z - mz + m\beta - 1)zt - (\beta z^3 - z^2 - mz + m\beta)}{(\beta z^2 - 2z + \beta)\sqrt{z^2-2zt+1}} = 0, \tag{55}$$

which implies one of the following cases:

- Equation (55) may be solved for t. Then

$$t = \frac{\beta z^3 - z^2 - mz + m\beta}{z(\beta z - mz + m\beta - 1)}, \tag{56}$$

  and substituting this value of t into p^Δ, and solving the resulting linear equation in m^2 and r^2, yields (50) and (51).

- $z = 1$, $m = -1$, $t \in (-1, 1)$. Then substituting $z = 1$ into the left side of (50) and $m = -1$ into the right shows (50). Similarly, since (z, t) is a zero of p^{Δ}, $t = (3\beta - 1)(\beta + 1)$ by lemma (2), and (51) follows by direct evaluation of the left and the right side.

- $z = -1$, $m = 1$, $t \in (-1, 1)$. Since (z, t) is a critical point, $\zeta_2 \leq z \leq 1/\zeta_2$, and as $z = -1$ is not in this interval by lemma (2), this case is impossible.

- $z = 1/\beta$, $m = 0$, $t \in (-1, 1)$. As $1/\beta \notin [\zeta_2, 1/\zeta_2]$ this case is also impossible.

Therefore (50) and (51) hold in any case, as required.

For the converse statement, the derivative of the left side of (50) is

$$\frac{dm^2}{dz} = \frac{6\beta(\beta z - 1)^2(z^2 - 1)^2(1 - \beta)^2(z^2 - 2\beta z + 1)}{z^6 p(1/z)^2(z - \beta)^4} \tag{57}$$

which is zero on the interval $[\zeta_2, 1/\zeta_2)$ only at $z = 1$, and since evaluating the left side of (50) at $z = \zeta_2$ and $z = 1$ yields 0 and 1 respectively, and since the limit of the left side of (50) as z approaches $1/\zeta_2$ from the right is ∞, it follows that the right side of (50) maps the interval $[\zeta_2, 1/\zeta_2)$ bijectively to the interval $[0, \infty)$. Therefore, for any m, and so for any $m \neq -1$, there is exactly one z, say $z_{m,\beta}$, satisfying (50), as required. For later use, note that this argument also implies that $\zeta_2 \leq z_{m,\beta} < 1$ if $m^2 < 1$.

Now assume $m \neq \pm 1$. *Suppose the denominator of the fraction in the $m \neq -1$ case of (52) is zero at $z_{m,\beta}$.* Then

$$z_{m,\beta} = \frac{1 - m\beta}{\beta - m}, \tag{58}$$

since $z_{m,\beta} \neq 0$ and $\beta \neq 1$. As $z_{m,\beta}$ satisfies (50), substituting (58) into (50) yields

$$\frac{m^2(1 - m^2)(\beta m^2 - 2m + \beta)}{2m^3 - 3m^2\beta + \beta} = 0.$$

If $m = 0$ then (58) shows $z = 1/\beta$, *contradicting* $z \in [\zeta_1, 1/\zeta_1)$, so one must have $\beta m^2 - 2m + \beta = 0$. But then $m = \zeta_1$ or $m = 1/\zeta_1$, and (58) shows $z = 1/m$ by putting $\beta = 2m/(1 + m^2)$, so $z = \zeta_1$ or $z = 1/\zeta_1$, *also contradicting* $z \in [\zeta_2, 1/\zeta_2)$. Therefore, the case $m \neq -1$ in (52) may be used to compute t by evaluation at $z = z_{m,\beta}$, the result being, say $t_{m,\beta}$. Since (50) was obtained by substituting (56) into $p^{\Delta} = 0$, the resulting $(z_{m,\beta}, t_{m,\beta})$ is a zero of p^{Δ}, so $t_{m,\beta} \in (-1, 1)$ and $(z_{m,\beta}, t_{m,\beta})$ is a critical point of Ψ_{β}. Furthermore, as (55) is equivalent to $\mu_2 - m\mu_1 = 0$, the (μ_1, μ_2) obtained by computing $\Psi_{\beta}(z_{m,\beta}, t_{m,\beta})$ is on the line through the origin with slope m. Also, if $m = -1$ then (50) is equivalent to

$$\frac{\beta(z^2 - 1)^3(1 - \beta^2)^3}{z^3 p(1/z)(z - \beta)^3} = 0,$$

so that $z_{m,\beta} = 1$. Thus computing t by the $m = -1$ case of (52) yields a $t_{m,\beta}$ such that $(z_{m,\beta}, t_{m,\beta})$ is a critical point of Ψ_{β} and $t \in (-1, 1)$, as required.

To see (53), note first that (50) may be rewritten as a polynomial equation in z, namely

$$m^2 z^3 p_1(1/z)(z - \beta)^3 + p_1(z)(\beta z - 1)^3 = 0. \tag{59}$$

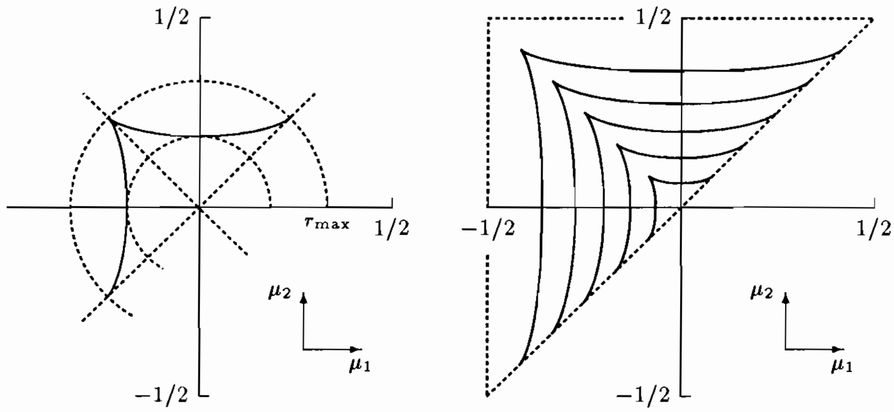


Figure 5: The curves of critical values.

Now $z_{m,\beta}$ is a solution to (59), and for $|m| < 1$, $z_{m,\beta} \to 0$ as $\beta \to 0$. Indeed, *if not* then $0 < z_{m,\beta} < 1$ since $|m^2| < 1$. Thus, there is some sequence $\beta_n \to 0$ such that $z_{m,\beta_n} \to \tilde{z}_m > 0$ as $n \to \infty$, and so $z = \tilde{z}_m$, $\beta = 0$ is a solution to (59). But if $\beta = 0$ then (59) becomes $2m^2z^3 = 0$, so $\tilde{z}_m = 0$, a *contradiction*. Now the derivative of (59) with respect to β at $\beta = 0$, $z = 0$ is -1, so the implicit function theorem implies that (59) may be solved uniquely for β as a C^∞ function of z and m near $z = 0$. Then by substitution of β as a power series in z into (59) and after equating coefficients, it is easy to obtain

$$\beta = 2(1 - m^2)z^3 + \mathcal{O}(z^5),$$

and then (53) follows easily.

Finally, the derivative of (50) with respect to z is (57), and this is not zero unless $z = 1$, so $z_{m,\beta}$ is C^∞ in m and β away from $m = -1$. □

Proposition 20. *The set of critical values of Ψ_β is a continuous curve in the (μ_1, μ_2) plane. That curve is accurately represented by figure (5), in that:*

1. *The curve of critical values meets each radial line exactly once, except $\mu_1 = \mu_2$, and therefore defines a curve in polar coordinates $r = \varrho_\beta(\theta)$, $\theta \in (\pi/4, 5\pi/4)$. Moreover, the curve of critical values lies above the line $\mu_1 = \mu_2$, so that $\varrho_\beta > 0$.*

2. *The function $\varrho_\beta(\theta)$ is C^∞ in θ except at $\theta = 3\pi/4$, where the curve ϱ_β is tangent to the line $\mu_1 = -\mu_2$. The function $\beta \mapsto \varrho_\beta(\theta)$ is C^∞ for all θ.*

3. *The curve $r = \varrho_\beta(\theta)$ is symmetric about the line $\mu_1 = -\mu_2$, the portion of the curve with $\theta \in (\pi/4, 3\pi/4)$ is symmetric about the vertical axis, and the portion of the curve with $\theta \in (3\pi/4, 5\pi/4)$ is symmetric about the horizontal axis.*

4. *The radius ϱ_β decreases monotonely as θ increases on the interval $(\pi/4, \pi/2)$ and increases monotonely as θ increases on the interval $(\pi/2, 3\pi/4)$.*

5. *The curve of critical values of Ψ_β is entirely contained in the circle of radius $r_{\max}$ centered at the origin, where*

$$r_{\max} \stackrel{\text{def}}{=} \sqrt{\frac{1-\beta}{2(1+\beta)}}.$$

Moreover, the endpoints of the curve ϱ_β are $\theta = \pi/4$ or $\theta = 5\pi/4$ and $r = r_{\max}$, and the curve approaches these endpoints tangentially to the line $\mu_1 = \mu_2$, and $\varrho_\beta(\theta) < r_{\max}$ with the single exception of $\theta = 3\pi/4$.

6. *For fixed θ, the image of $0 < \beta < 1$ under $\varrho_\beta(\theta)$ is the open radial line segment joining the boundary of the triangle*

$$\{ (\mu_1, (\mu_2) \mid |\mu_1| < 1/2,\ |\mu_2| < 1/2,\ \mu_2 > \mu_1 \} \tag{60}$$

to the origin, and furthermore, the radius ϱ_β decreases monotonely as β increases on the interval $(0,1)$. The curve of critical values of Ψ_β shrinks monotonely to the origin as β increases to 1, approaches the boundary of (60) *as β decreases to 0, and the set of curves of critical values fills up the triangle* (60) *as β varies over the interval $(0,1)$, as shown in figure* (5).

Proof. Just by the definition of Ψ_β, the critical point $(\zeta_2, 1/\zeta_2)$ has critical value on the vertical line $\mu_1 = 0$. Moreover, given $m \neq 1$, the critical point $(z_{m,\beta}, t_{m,\beta})$ from lemma (50) gives one critical point with critical value on the line with slope m. Thus, there is at least one critical value on every radial line that is not $\mu_1 = \mu_2$; it must be shown that there is exactly one. If (z,t) is a critical point on the line with slope $m = 0$, then $z_{m,\beta} = \zeta_2$ from (51), while if (z,t) is on the vertical line, then $z = 1/\zeta_2$, since all other choices of z give a finite slope by (51). As there is exactly one critical point on the vertical lines above ζ_2 and $1/\zeta_2$ by lemma (2), the proof is complete for the special case of horizontal and vertical lines. Now suppose $m \neq 0$ and *suppose there is another such critical value, say $\Psi_\beta(\tilde{z}, \tilde{t})$, on the radial line with slope m.* Then $\tilde{z} = z_{m,\beta}$, since $\tilde{z}$ and $z_{m,\beta}$ both satisfy (51). Similarly, $(z_{-m,\beta}, t_{-m,\beta})$ is another critical point, and $z_{m,\beta} = z_{-m,\beta}$. Since these three critical points are distinct, there are three distinct critical points above the vertical line $z = z_{m,\beta}$, a *contradiction.* Therefore, the curve of critical values meets each radial line except $\mu_1 = \mu_2$ exactly once.

Since the curve of critical values is the image under Ψ_β of the continuous curve of critical points, the curve of critical values is a continuous curve that never meets the line $\mu_1 = \mu_2$, and therefore is on one side of this line or the other. As the image of the critical point $(1, (3\beta-1)/(\beta+1))$ is in the upper half of the line, so the entire curve of critical values is in the upper half of this line as well. This completes the proof of item (1).

If $\theta \neq \pi/2$, then the value of $\varrho_\beta(\theta)$ may be calculated as follows: set $m = \tan\theta$, solve (50) for $z_{m,\beta}$ and calculate $\varrho_\beta(\theta) = \sqrt{r}$ by putting $z_{m,\beta}$ into the left side of (51). Thus, the function $(\theta,\beta) \mapsto \varrho_\beta(\theta)$ is C^∞ except at $\theta = 3\pi/4$ and $\theta = \pi/2$. Moreover the three symmetry statements of item (3) are seen now to be consequences of the following symmetries of the two equations (50) and (51) respectively: 1) If z is replaced by $1/z$ and m is replaced by $1/m$ in (50) and (51) then the same equations result. 2) If m is replaced by $-m$ in (50) and (51) then the same equations result. The special case $\varrho_\beta(0) = \varrho_\beta(\pi/2)$ of item (3) follows since ϱ_β is continuous, thus completing the proof of item (3). By the local representation of a cusp point in theorem (6), the curve of critical values is tangent at $\theta = 3\pi/4$ to some line segment, and by symmetry this line must be the radial line $\theta = 3\pi/4$. By symmetry, the function $(\theta,\beta) \mapsto \varrho_\beta(\theta)$ is C^∞ at $\theta = \pi/2$, since it is C^∞ at $\theta = 0$, and furthermore, $\varrho_\beta(3\pi/4) = r_{\mathbf{max}}$, so $\beta \mapsto \varrho_\beta(3\pi/4)$ is C^∞. This completes the proof of item (2).

For item (4), by symmetry there is no loss in generality if θ is restricted to the interval $(\pi/4, 3\pi/2)$, an interval over which $z_{m,\beta} \in (1, 1/\zeta_2)$ if $m = \tan\theta$. By implicit differentiation, one calculates

$$\frac{d\varrho_\beta}{d\theta} = \frac{\varrho_\beta(1+m^2)}{m}\frac{z^6(z-\beta)^4 p_1(1/z)^2((1+\beta^2)z^2 - 4\beta z + 1 + \beta^2)}{(1-z^2)(1-\beta^2)^2(\beta z - 1)^2(\beta - 2z + \beta z^2)^4}, \tag{61}$$

which is strictly negative for $z \in (1, 1/\zeta_2)$. Thus $\varrho_\beta(\theta)$ monotonely decreases as θ increases on the interval $(\pi/4, 3\pi/2)$, as required.

Item (4) shows that ϱ_β achieves its maximum $r_{\mathbf{max}}$ at the cusp $\theta = 3\pi/4$. Since the curve approaches the line $\theta = 3\pi/4$ tangentially, it approaches its endpoints at $\theta = \pi/4$ and $\theta = 5\pi/4$ tangentially as well, by symmetry. This completes the proof of item (5).

By implicit differentiation,

$$\frac{\partial\varrho_\beta}{\partial\beta} = -\frac{2(z_{m,\beta}^2 - 2\beta z_{m,\beta} + 1)p_4(z_{m,\beta})}{(1-\beta^2)^2(\beta - 2z_{m,\beta} + \beta z_{m,\beta}^2)^4},$$

which is strictly negative by lemma (5). Thus, the value $\varrho_\beta(\theta)$ decreases monotonely as β increases to 1 for $\theta \neq 3/\pi/4$ and $\theta \neq \pi/2$. The case $\theta = \pi/2$ follows by symmetry, and the case $\theta = 3\pi/4$ is immediate, since $\varrho_\beta(3\pi/4) = r_{\mathbf{max}}$.

As the curve is contained in the circle of radius $r_{\mathbf{max}}$, and this radius tends to 0 as β tends to 1, one endpoint of the radial line segment $r = \varrho_\beta(\theta)$ is at the origin. To show that the other end is on the boundary of the stated triangle, first fix $|\theta| < \pi/4$, so that $|m| < 1$, where $m = \tan\theta$, and note that a point (r,θ) in polar coordinates is on one of the lines $\mu_1 = \pm 1/2$ if and only if

$$4r^2 - (1+m^2) = 0. \tag{62}$$

Then using the estimate (53) of lemma (6), and substituting this estimate, (50) and (51) into (62), one computes that the limit as β tends to zero of (62) with ϱ_β replacing r is zero. Thus, the other endpoint of the radial line segment $r = \varrho_\beta(\theta)$ is on one of the lines $\mu_1 = \pm 1/2$. By symmetry, the same statement is true if $|\theta| > \pi/4$ and the line $\mu_1 = \pm 1/2$ is replaced by the line $\mu_2 = \pm 1/2$. The remaining case of $\theta = 3\pi/4$ is immediate since $r_{\mathbf{max}}$ has the value $1/\sqrt{2}$ when $\beta = 0$. □

The analysis of the set of critical points thusly complete, we turn to the points of Σ_{Ψ_β} obtained from the boundary of the set D_β.

Proposition 21. *Define open sets E_0^-, E_0^+ and, for $\beta \neq 0$, E_β^-, E_β^{+-}, E_β^{+0}, E_β^{++}, as in figure (21). Then $\Sigma'_{\Psi_\beta|D} \subseteq \partial E$, where D is one of the open sets D_0^-, D_0^+, D_β^-, D_β^{+-}, D_β^{+0}, D_β^{++} and E is the corresponding open set E_0^-, E_0^+, E_β^-, E_β^{+-}, E_β^{+0}, E_β^{++}.*

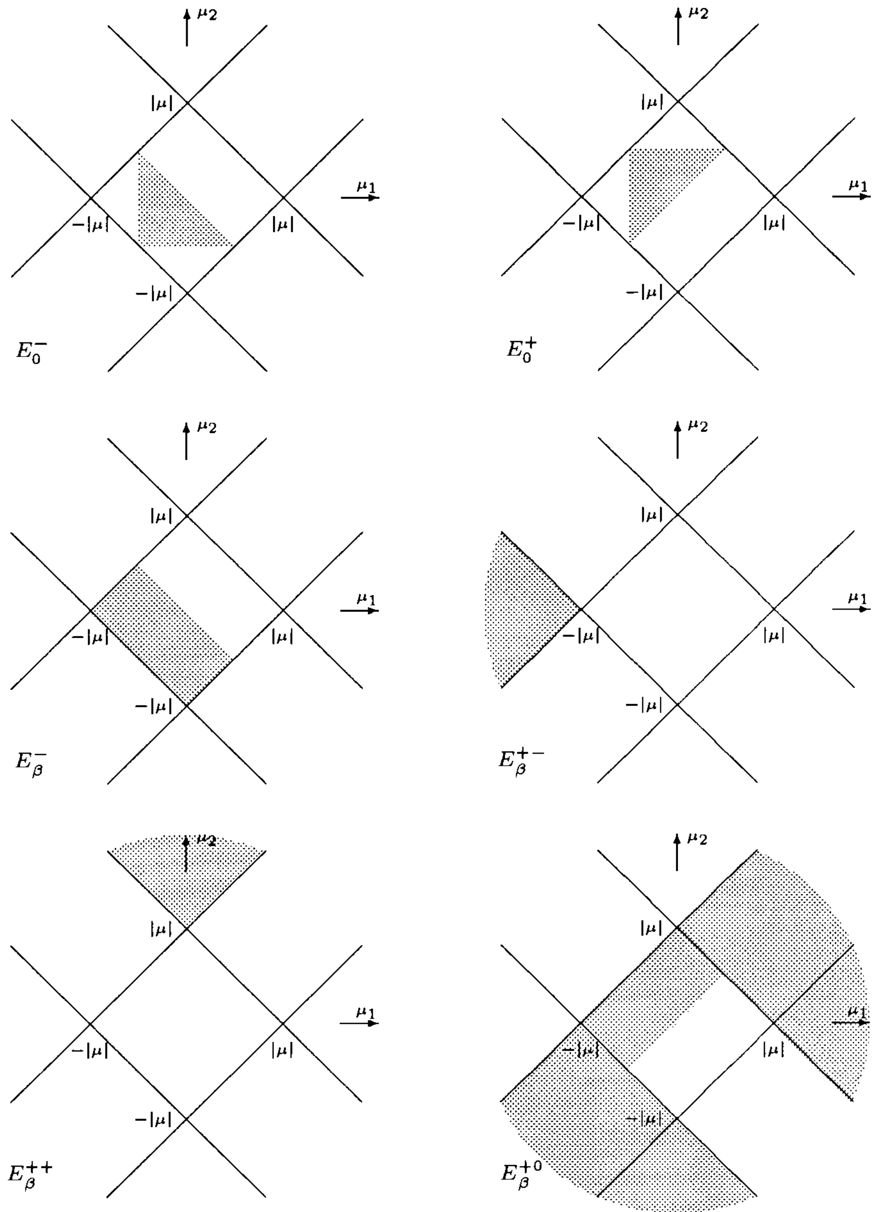


Figure 6: The open sets E_0^-, E_0^+,E_β^-, E_β^{+-}, E_β^{+0},E_β^{++}.

Proof. Consider first the statement $\Sigma'_{\Psi|D_0^-} \subseteq \partial E_0^-$. Note that

$$\Psi_0(z,t) = \frac{1}{2\sqrt{z^2-2zt+1}}(zt-1, z-t), \tag{63}$$

and regard D_0^- as a subset of $[-\infty, 0] \times \mathbf{R}$, so that D_0^- is relatively compact. Suppose that $(z_n, t_n) \to (z,t) \in [-\infty, 0] \times \mathbf{R}$ where $(z,t) \notin D_0^-$ and that $\Psi_0(z_n, t_n) \to (\mu_1, \mu_2)$ as $n \to \infty$. Then there are three cases:

- (z_n, t_n) has limit on one of the lines $t = 1$, $t = -1$, or $z = 0$, and this limit is not the point $(-1,-1)$: Then since (63) serves to extend $\Psi_0|D_0^-$ to the continuous function Ψ_0, the domain of which is on an open set containing these lines and excluding the point $(-1,-1)$, one may compute (μ_1, μ_2) by finding the limit of the sequence $\Psi_0(z_n, t_n)$ as $n \to \infty$. Putting $t = 1$, $t = -1$, and $z = 0$ in (63), one computes that (μ_1, μ_2) is

  $$-(1,1)/2, \quad \operatorname{sgn}(z+1)(-1,1)/2, \quad \{\, -(1,t)/2 \mid |t| \le 1 \,\}$$

  respectively. This completes that proof in this case, since all these points are contained in the set ∂E_0^-.

- (z_n, t_n) has limit $(-\infty, t)$, where $|t| \le 1$. Then

  $$\begin{aligned} \lim_{n\to-\infty} \Psi_0(z_n, t_n) &= \lim_{n\to-\infty} \frac{1}{2|z_n|\sqrt{z_n^2 - 2z_n t_n + 1}} z_n(t_n - 1/z_n, t - t_n/z_n) \\ &= (-t/2, 1/2), \end{aligned}$$

  points which are contained in the set ∂E_0^-.

- $(z_n, t_n) \to (-1,-1)$: Then

  $$\mu_1 = \lim_{n\to-\infty} \frac{z_n t_n - 1}{2\sqrt{z_n^2 - 2z_n t_n + 1}}, \quad \mu_2 = \lim_{n\to-\infty} \frac{z_n - t_n}{2\sqrt{z_n^2 - 2z_n t_n + 1}},$$

  and so

  $$\begin{aligned} |\mu_1 + \mu_2| &= \frac{1}{2} \lim_{n\to-\infty} \frac{|z_n t_n - 1 + z_n - t_n|}{2\sqrt{z_n^2 - 2z_n t_n + 1}} \\ &= \frac{1}{2} \lim_{n\to-\infty} \frac{|z_n - 1||1 + t_n|}{2\sqrt{z_n^2 - 2z_n t_n + 1}} \\ &= \lim_{n\to-\infty} \left(\frac{|1+t_n|}{{z_n}^2(1-{t_n}^2) + (1 - z_n t_n)^2} \right)^{1/2} \sqrt{1+t_n} \\ &\le \lim_{n\to-\infty} \frac{\sqrt{1+t_n}}{|z_n|(1-t_n)} \\ &= 0, \end{aligned}$$

so $\mu_1 = -\mu_2$. Moreover,

$$\mu_1 = \frac{1}{2}\lim_{n\to-\infty}\frac{|1-z_nt_n|}{\sqrt{z_n^2(1-t_n^2)+(1-z_nt_n)^2}} \le \frac{1}{2}$$
$$\mu_2 = \frac{1}{2}\lim_{n\to-\infty}\frac{|z_n-t_n|}{\sqrt{(z_n-t_n)^2+(1-t_n^2)}} \le \frac{1}{2},$$

so (μ_1,μ_2) is contained in the line segment joining $(1/2,-1/2)$ to $(-1/2,1/2)$, and this line segment is contained in the set ∂E_0^-.

Therefore, $(\mu_1,\mu_2) \in \partial E_0^-$ in any case, as required. The proof of the statement $\Sigma'_{\Psi|D_0^+} \subseteq \partial E_0^+$ is similar.

When $\beta \neq 0$, note that

$$\Psi_\beta(z,t) = \frac{1}{\beta z^2 - 2z + \beta}(\beta - z, z(\beta z - 1))\Psi_0(z,t),$$

where the multiplication is componentwise. Thus, if $(z_n,t_n) \to (-1,-1)$ and $\Psi_0(z_n,t_n) \to (\mu_1,\mu_2)$ as $n \to \infty$, then using the results just obtained for Ψ_0,

$$\begin{aligned}(\mu_1,\mu_2) &= \left(\frac{2}{\beta z^2 - 2z + \beta}\right)\Big|_{z=-1,t=-1}(\mu_1,-\mu_1)\\ &= (\mu_1,-\mu_1),\end{aligned}$$

points which are all contained in the set E_β^-. The case where $(z_n,t_n) \to (1,1)$ is similar, and the cases arising from (z_n,t_n) approaching other boundary points over which Ψ_β has a continuous extension or from $z_n \to \pm\infty$ are similar to these same cases when $\beta = 0$. □

The following then completes our analysis of the map Ψ_β, and therefore completes the proof of theorem (5).

Proposition 22.

1. *The map Ψ_0 is a diffeomorphism from D_0^- [respectively D_0^+] to E_0^- [respectively E_0^+].*
2. *When $\beta \neq 0$, the map Ψ_β is a diffeomorphism from E_β^- [respectively E_β^{+-}, E_β^{++}] to D_β^- [respectively D_β^{+-}, D_β^{++}].*
3. *When $\beta \neq 0$, the cardinal number $\#(\Psi_\beta|D_\beta^{+0})^{-1}(\mu_1,\mu_2)$ may be found by locating the point (μ_1,μ_2) in figure (7).*

Proof. For the first statement, by lemma (6), the map Ψ_0 has no critical points, and so is a local diffeomorphism. Thus, by proposition (21),

$$\Sigma_{\Psi_0} = \Sigma'_{\Psi_0} = \partial E_\beta^-,$$

and so by proposition (17) and proposition (19), $\#(\Psi_0|D_0^-)^{-1}(\mu_1,\mu_2) = 0$ if $(\mu_1,\mu_2) \notin \mathrm{Cl}E_0^-$. Then regarding $\Psi_0|E_0$ as a map into $\mathrm{Cl}E_0^-$, one sees that proposition (18) can be

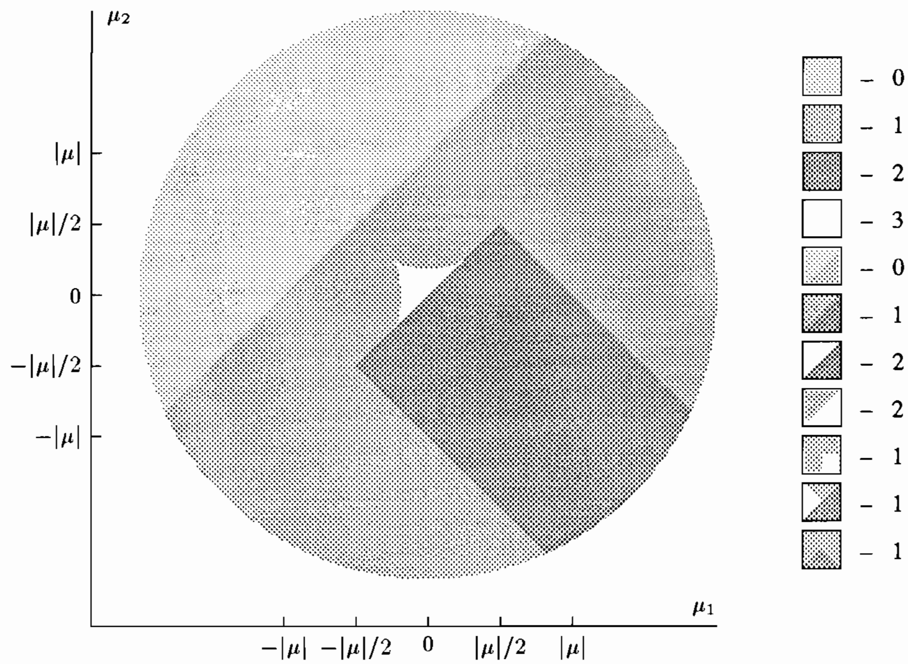


Figure 7: $\#(\Psi_\beta|D_\beta^{+0})^{-1}(\mu_1, \mu_2)$ for $\beta \neq 0$.

applied to conclude that Ψ_0 is a diffeomorphism from D_0^- to E_0^-. This argument also suffices for the proof in the case that D_0^+ and E_0^+ replaces D_0^- and E_0^-, as well as for the entire proof of the second statement.

For the third statement, putting proposition (20) with proposition (21), one obtains

$$\Sigma_{\Psi_\beta} = \partial E_\beta^{+0} \cup \text{Image}\varrho_\beta,$$

and then proposition (19) determines the quantity $\#(\Psi_\beta|D_\beta^{+0})^{-1}(\mu_1, \mu_2)$ everywhere except on the curve ϱ_β and in the open region, say E_{ϱ_β}, bounded by the curve ϱ_β and the line joining this curve's two endpoints. Let (μ_1, μ_2) be a point on the curve ϱ_β which is not the cusp on the curve ϱ_β. Then (μ_1, μ_2) is the image of exactly one critical point, and Ψ_β is locally conjugate to the map $(x, y) \mapsto (x^2, y)$, by theorem (6). It follows that the the quantity $\#(\Psi_\beta|D_\beta^{+0})^{-1}(\mu_1, \mu_2)$ must increase or decrease by 1 exactly at (μ_1, μ_2) and must increase or decrease by 2 as (μ_1, μ_2) enters E_{ϱ_β}. Since this quantity is 1 on the region adjoining E_{ϱ_β} and obviously must be positive, it cannot decrease by 2, so then $\#(\Psi_\beta|D_\beta^{+0})^{-1}(\mu_1, \mu_2) = 3$ in E_{ϱ_β} and $\#(\Psi_\beta|D_\beta^{+0})^{-1}(\mu_1, \mu_2) = 2$ at a point on ϱ_β and not at the cusp point of this curve. A similar argument, using of course the different local representation, determines that $\#(\Psi_\beta|D_\beta^{+0})^{-1}(\mu_1, \mu_2) = 1$ if (μ_1, μ_2) is exactly the cusp of the curve ϱ_β. □

Chapter 3

A Momentum Preserving Symplectic Integrator

When investigating the phase portrait of a complicated Hamiltonian system with symmetry, like the system of two coupled rigid bodies, a numerical integrator is of obvious utility. Recently, attention has been attracted towards the class of symplectic integrators, which are algorithms that are discrete flows obtained by iterating a symplectic mapping of phase space that approximates the exact time–Δt flow for small Δt. (See [6] and the references therein.) In the context of a simple mechanical system with symmetry (Q, G, V), one way to construct such an algorithm is as follows: A function f on Q^2 generates a symplectic map $\alpha_1 \mapsto \alpha_2$ of T^*Q by first solving for q_2 from the equation

$$\alpha_1 = -\boldsymbol{d}_1 f(q_1, q_2), \tag{1}$$

where α_1 has base point q_1, and then setting

$$\alpha_2 = \boldsymbol{d}_2 f(q_1, q_2). \tag{2}$$

Furthermore, a classical result [3] states that this procedure generates the Hamiltonian evolution if f is taken to be the function S_t defined by

$$S_t(q_1, q_2) \stackrel{\text{def}}{=} \int_\gamma L(q, \dot{q})\, dt, \tag{3}$$

where the integration is performed along the evolution curve γ joining q_1 to q_2 in time t. A function $\tilde{S}_t$ which agrees with S_t to arbitrarily high orders may be found from equation (3) by using the Lagrangian vector field to generate Taylor expansions of the evolution; the discrete flow which constitutes the algorithm is obtained by choosing t small and then by using equations (1) and (2) with $f = \tilde{S}_t$ (equation (1) is solved numerically.) It is an observation of Ge and Marsden [7] that if f is invariant under the diagonal action of G on Q^2, then the symplectic map generated by f preserves the momentum map J, and so if $\tilde{S}_t$ is G invariant then the algorithm generated by $\tilde{S}_t$ will be symplectic *and* momentum preserving. This chapter gives an exposition of the theory behind these ideas and outlines a practical application to the system of two coupled rigid bodies.

We begin by showing that equations (1) and (2) do indeed generate momentum preserving symplectic maps when the generating function is invariant. Here and below, since it is clear from context, we avoid pedantically specifying which action of G is intended, be it the usual lifts of the action on Q to TQ and T^*Q, the diagonal action on Q^2, or the actions on the products of these spaces with $\mathbf{R}$ obtained by the product of the action on the space and the trivial action on $\mathbf{R}$.

Proposition 1. *Suppose $\Psi : U \subseteq T^*Q \to T^*Q$ admits an invariant function f with domain containing the set*

$$\{ (\tau_Q^*(\alpha), \tau_Q^* \circ \Psi(\alpha)) \mid \alpha \in U \}$$

such that for all $\alpha \in U$,

$$\alpha = -\boldsymbol{d}_1 f(\tau_Q^*(\alpha), \tau_Q^* \circ \Psi(\alpha)), \quad \Psi(\alpha) = \boldsymbol{d}_2 f(\tau_Q^*(\alpha), \tau_Q^* \circ \Psi(\alpha)), \tag{4}$$

where $\tau_Q^* : T^*Q \to Q$ is the canonical projection. Then Ψ is symplectic and $J \circ \Psi = J$.

Proof. Given (4), and using the definition of the canonical one form θ_0, a simple calculation shows that

$$\Psi^*\theta_0 - \theta_0 = \boldsymbol{d}\left(\alpha \mapsto f(\tau_Q^*(\alpha), \tau_Q^* \circ \Psi(\alpha))\right).$$

Taking the exterior derivative of this expression shows that Ψ is symplectic. That $J_\xi \circ \Psi = J_\xi$ for $\xi \in \mathfrak{g}$ follows from differentiating the equation

$$f(\tau_Q^*(\alpha), \tau_Q^* \circ \Psi(\alpha)) = f(\exp(t\xi) \cdot \tau_Q^*(\alpha), \exp(t\xi) \cdot \tau_Q^* \circ \Psi(\alpha))$$

with respect to t at $t = 0$.□

Remark. If Ψ is the unique map that satisfies equations (4), then it is similarly easy to show that Ψ is G equivariant.

Remark. In coordinates ${q_1}^i$ for the first factor of Q^2 and ${q_2}^i$ for the second, and letting the corresponding conjugate coordinates be ${p^1}_i$ and ${p^2}_i$ respectively, equations (4) become the familiar

$${p^1}_i = -\frac{\partial f}{\partial {q_1}^i}({q_1}^i, {q_2}^i), \quad {p^2}_i = \frac{\partial f}{\partial {q_2}^i}({q_1}^i, {q_2}^i).$$

Given a particular approximation $\tilde{S}_t$ to S_t, we are interested in determining the order with which the corresponding symplectic maps agree at $t = 0$, since this order will determine the order of the symplectic integrator constructed by $\tilde{S}_t$. However, as $t \to 0$ with q_1 and q_2 fixed, the Lagrangian evolution curve joining q_1 to q_2 in time t (if such a curve exists) must start at q_1 with arbitrarily high initial velocity, so the generating function S_t defined by equation (3) has a singularity at $t = 0$; for precise work, an exact determination of this singularity is necessary. For example, in the case of a single free particle, where $Q = \mathbf{R}$ and $V = 0$, it is trivial to verify that $S_t = (q - q_0)^2/2t$, so tS_t is a C^∞ function on $Q^2 \times \mathbf{R}$. This $1/t$ behavior as $t \to 0$ is generally true. The idea of the proof is as follows:

Let F_t be the flow of the Lagrangian vector field $\boldsymbol{X}_E$. Then the generating function S_t is the composition of the C^∞ function

$$(v_q, t) \mapsto \int_0^t L \circ F_s(v_q)\, ds, \tag{5}$$

and the inverse of the map

$$(v_q, t) \mapsto (q, \tau_Q^* \circ F_t(v_q), t). \tag{6}$$

Thus the singularity of S_t at $t = 0$ is caused by the poor behavior of the inverse of the map (6). It turns out, though, that this inverse followed by multiplication by t in its first component—that is, the inverse of the map

$$(v_q, t) \mapsto (q, \tau_Q^* \circ F_t(v_q/t), t), \tag{7}$$

is C^∞, and furthermore that the function

$$(v_q, t) \mapsto t \int_0^t L(F_s(v_q/t))\, ds \tag{8}$$

is C^∞ as well. This displays tS_t as the composition of the C^∞ function (8) and the C^∞ inverse of the map (7), so tS_t is C^∞.

To make the above argument precise, we prove the following two lemmas:

Lemma 1. *Let* exp *be the exponential map of the kinetic energy metric. The map* Ξ *defined by*

$$\Xi(v_q, \lambda) \stackrel{\text{def}}{=} \begin{cases} (q, \tau_Q^* \circ F_\lambda(v_q/\lambda), \lambda) & \text{if } \lambda \neq 0 \\ (q, \exp v_q, \lambda) & \text{if } \lambda = 0 \end{cases}$$

is a C^∞ *equivariant diffeomorphism from an open neighborhood of* $Z_{TQ} \times \{0\} \subseteq TQ \times \mathbb{R}$ *to an open neighborhood of* $\Delta_{Q^2} \times \{0\} \subseteq Q^2 \times \mathbb{R}$, *where* Z_{TQ} *is the zero section to* TQ *and* Δ_{Q^2} *is the diagonal of* Q^2.

Proof. For $\lambda \in \mathbb{R}$, consider the Lagrangian L_λ defined by

$$L_\lambda(v_q) \stackrel{\text{def}}{=} \frac{1}{2}\langle v_q, v_q\rangle - \lambda^2 V(q), \tag{9}$$

and by abuse of notation, let M_λ denote the scalar multiplication map on both TQ and T^*Q, so that

$$M_\lambda(v_q) \stackrel{\text{def}}{=} \lambda v_q, \quad M_\lambda(\alpha_q) \stackrel{\text{def}}{=} \lambda \alpha_q.$$

Then the Lagrangian vector field $\boldsymbol{X}_{E_\lambda}$ of L_λ, which is obviously a smoothly parameterized vector field on TQ, is given by

$$\boldsymbol{X}_{E_\lambda}(v_q) = \begin{cases} \lambda TM_\lambda \boldsymbol{X}_E(v_q/\lambda) & \text{if } \lambda \neq 0 \\ X_{E_0}(v_q) & \text{if } \lambda = 0 \end{cases} \tag{10}$$

Indeed, for $\lambda = 0$ this statement has no content, while otherwise one easily verifies that

$$M_\lambda \circ FL = FL \circ M_\lambda, \quad FL_\lambda = FL, \quad M_\lambda^* \theta_0 = \lambda \theta_0,$$

and thereby that $M_\lambda^* \omega_{L_\lambda} = \lambda \omega_L$. Thus, using $\langle \cdot, \cdot \rangle$ for the contraction of a vector field and a one form,

$$\begin{aligned} \langle \lambda M_{\lambda *} \boldsymbol{X}_E, \omega_{L_\lambda} \rangle &= \lambda \langle \boldsymbol{X}_E, M_\lambda^* \omega_{L_\lambda} \rangle \\ &= \lambda \langle \boldsymbol{X}_E, \lambda \omega_L \rangle \\ &= \lambda^2 M_{\lambda *} \boldsymbol{d} E \\ &= \lambda^2 \boldsymbol{d} \left(v_q \mapsto \frac{1}{2\lambda^2} \langle v_q, v_q \rangle + V(q) \right) \\ &= \boldsymbol{d} E_\lambda, \end{aligned}$$

and $\lambda M_{\lambda *} \boldsymbol{X}_E(v_q) = \lambda T M_\lambda \boldsymbol{X}_E(v_q/\lambda)$, so equation **(10)** is verified. Now the flow F_t^λ of $\boldsymbol{X}_{E_\lambda}$ is given by

$$F_t^\lambda(v_q) = \begin{cases} \lambda F_{\lambda t}(v_q/\lambda) & \text{if } \lambda \neq 0 \\ F_t^0(v_q) & \text{if } \lambda = 0 \end{cases} \tag{11}$$

Indeed, for $\lambda = 0$ this statement has no content, while otherwise

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \lambda F_{\lambda t}(v_q/\lambda) &= TM_\lambda \left(\lambda \boldsymbol{X}_E(F_{\lambda t}(v_q/\lambda)) \right) \\ &= \lambda TM_\lambda \boldsymbol{X}_E \left(\frac{\lambda F_\lambda(v_q/\lambda)}{\lambda} \right) \\ &= \boldsymbol{X}_{E_\lambda}(F_t^\lambda(v_q)), \end{aligned}$$

as required. Thus the map $(v_q, t, \lambda) \mapsto F_t^\lambda(v_q)$ is well defined and C^∞ on some open subset $\tilde{U}$ of $TQ \times \mathbf{R}^2$, since F_t^λ is the flow of the smoothly parameterized vector field $\boldsymbol{X}_{E_\lambda}$. Moreover, $TQ \times \mathbf{R} \times \{0\} \subseteq \tilde{U}$, since the geodesic flow F_t^0 is complete. Thus the set

$$U \stackrel{\text{def}}{=} \{ (v_q, \lambda) \mid (v_q, 1, \lambda) \in D \}$$

is open and contains $TQ \times \{0\}$, and since

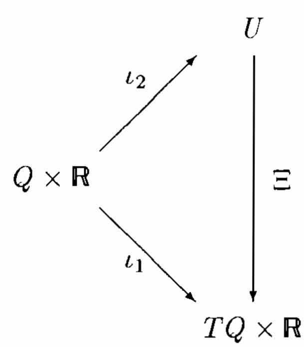
$$\Xi(v_q, \lambda) = (q, \tau_Q^* \circ F_1^\lambda(v_q), \lambda),$$

it follows that Ξ is well defined and C^∞ on U.

Obviously $\Xi(0_q, 0) = (q, q, 0)$, so that if ι_1 and ι_2 are the embeddings

$$\begin{aligned} Q \times \mathbf{R} \to Q^2 \times \mathbf{R} &: \quad \iota_1(q,t) = (q,q,t), \\ Q \times \mathbf{R} \to TQ \times \mathbf{R} &: \quad \iota_2(q,t) = (0_q, t), \end{aligned}$$

then the diagram



commutes. Furthermore, $T\Xi$ is invertible at points in the image of ι_2: indeed, if $w \in T_{0_q}(TQ)$ has horizontal and vertical components $w^v, w^h \in T_qQ$, then $T(\exp)w = w^v$, so that

$$T\Xi(w,(0,\tilde{t})) = (w^h, w^v, (0,\tilde{t})).$$

In this general situation, and since $Q \times \mathbf{R}$ is paracompact, local inverses of Ξ may be glued together along the image of ι_1 [12:99], thereby showing that Ξ is a diffeomorphism from an open subset containing the image of ι_1 to an open subset containing the image of ι_2. □

Corrollary 1. *Consider a simple mechanical system with Lagrangian evolution F_t, and let* exp *denote the exponential map for the kinetic energy metric. Then, for all $v_q \in TQ$, the Lagrangian evolution curve starting at v_q/λ lasts for time at least λ, for λ in some open interval of* 0, *and*

$$\lim_{\lambda \to 0} \tau_Q^* \circ F_\lambda(v_q/\lambda) = \exp v_q$$

Lemma 2. *The function*

$$\tilde{S}(v_q,\lambda) = \begin{cases} \lambda \displaystyle\int_0^\lambda L \circ F_t(v_q/\lambda)\, dt & \text{if } \lambda \neq 0, \\ \dfrac{1}{2}|v_q|^2 & \text{if } \lambda = 0, \end{cases}$$

is well defined, invariant and C^∞ in some neighborhood of $TQ \times \{0\}$.

Proof. Suppose $\lambda \neq 0$, and compute as follows:

$$\begin{aligned} \tilde{S}(v_q,\lambda) &= \lambda \int_0^\lambda L \circ F_t(v_q/\lambda)\, dt \\ &= \lambda \int_0^\lambda (E - 2V \circ \tau_Q^*) \circ F_t(v_q/\lambda)\, dt \\ &= \lambda^2 E(v_q/\lambda) - 2\lambda \int_0^\lambda V \circ \tau_Q^* \circ F_t(v_q/\lambda)\, dt \\ &= \lambda^2 E(v_q/\lambda) - 2\lambda^2 \int_0^1 V \circ \tau_Q^* \circ \lambda F_{\lambda t}(v_q/\lambda)\, dt \\ &= \frac{1}{2}|v_q|^2 + \lambda^2 V(q) - 2\lambda^2 \int_0^1 V \circ \tau_Q^* \circ F_t^\lambda(v_q)\, dt \end{aligned}$$

where F_t^λ is the C^∞ map (11) of lemma (1). □

As promised, we now combine lemma (1) and lemma (2), thereby obtaining detailed information about the generating function S_t and about the map which tells what the initial velocity is for the evolution joining two base points in a specified time.

Theorem 1. *Consider a simple mechanical system with symmetry (Q,V,G) where the Lagrangian vector field $\boldsymbol{X}_E$ has flow F_t. Then there is an invariant open neighborhood U of $\Delta_{Q^2} \times \{0\} \subseteq Q^2 \times \mathbf{R}$ with the following properties:*

1. *There exists a* C^∞ *equivariant map* $\Delta : U\backslash Q^2\times\{0\} \to TQ$ *such that, for* $(q_1, q_2, t) \in U$ *with* $t \neq 0$, *the flow* F_s *starting at* $\Delta(q_1, q_2, t)$ *exists for time at least* t *and*

$$\tau_Q^* \circ \Delta(q_1, q_2, t) = q_1, \quad \tau_Q^* \circ F_t \circ \Delta(q_1, q_2, t) = q_2.$$

   Thus, $\Delta(q_1, q_2, t)$ *gives an initial velocity at* q_1 *whose base integral curve is at* q_2 *after time* t.

2. *The square of the distance function* $d(q_1, q_2)^2$ *of the kinetic energy metric is* C^∞ *and invariant on the invariant open neighborhood* U_0 *of* Δ_{Q^2}, *where*

$$U_0 \stackrel{\text{def}}{=} \{\, (q_1, q_2) \mid (q_1, q_2, s) \in U \text{ for some } s \in \mathbf{R} \,\},$$

   and there exists an invariant C^∞ *function* $f^1 : U \to \mathbf{R}$ *such that the function*

$$S(q_1, q_2, t) = \frac{d(q_1, q_2)^2}{2t} + f^1(q_1, q_2, t) \tag{12}$$

   satisfies the equation

$$S(q_1, q_2, t) = \int_0^t L \circ F_s \circ \Delta(q_1, q_2, t)\, ds \tag{13}$$

   for $t \neq 0$.

3. *The set* U *is invariant under interchange of* q_1 *and* q_2 *and invariant under time reversal, so that*

$$(q_1, q_2, t) \in U \Leftrightarrow (q_2, q_1, t) \in U \Leftrightarrow (q_1, q_2, -t) \in U.$$

Moreover,

$$\Delta(q_1, q_2, t) = -\Delta(q_1, q_2, -t), \tag{14}$$

and

$$S(q_1, q_2, t) = S(q_2, q_1, t), \quad S(q_1, q_2, t) = -S(q_1, q_2, -t),$$

and the various partial derivatives of S *are*

$$\boldsymbol{d}_1 S = -FL \circ \Delta, \quad \boldsymbol{d}_2 S = FL \circ F_t \circ \Delta, \quad \frac{\partial S}{\partial t} = E \circ \Delta.$$

Proof. For the first statement, let

$$\Delta(q_1, q_2, t) \stackrel{\text{def}}{=} \frac{\pi_1 \circ \Xi^{-1}(q_1, q_2, t)}{t},$$

where Ξ is the map defined by lemma (1) and $\pi_1 : TQ \times \mathbf{R} \to TQ$ is the natural projection. Defining

$$S(q_1, q_2, t) \stackrel{\text{def}}{=} \frac{\tilde{S} \circ \Xi^{-1}}{t},$$

where $\tilde{S}$ is the function defined by lemma (2), it is apparent that the proof of (12) and (13) will be complete if

$$\tilde{S} \circ \Xi^{-1}(q_1, q_2, 0) = \frac{1}{2} d(q_1, q_2)^2.$$

Indeed, since Ξ is a local diffeomorphism, q_2 is contained in an open neighborhood of q_1 on which $\exp_{q_1}$ is a local diffeomorphism. By consequence $d(q_1, q_2) = |\Xi^{-1}(q_1, q_2, 0)|$, since conjugate points need not be considered, and since $\exp(\Xi^{-1}(q_1, q_2, 0)) = q_2$. By intersecting finitely many sets, one can shrink U to ensure it is invariant under $q_1 \leftrightarrow q_2$ and $t \leftrightarrow -t$, and then from the proof of lemma (1)—take $\lambda = -1$—one has $F_t(v_q) = -F_{-t}(-v_q)$, and (14) as well as the invariance properties of S follow easily from this.

There remains the computation of the derivative of S, which seems to be most easily accomplished in—wretch—local coordinates. Set in local coordinates, one has

$$S(q_1, q_2, t) = \int_0^t L(q^i(q_1, q_2, t, s), \dot{q}^i(q_1, q_2, t, s))\, ds,$$

where the Lagrangian evolution is

$$s \mapsto (q^i(q_1, q_2, t, s), \dot{q}^i(q_1, q_2, t, s)),$$

and that evolution satisfies the boundary conditions

$$q^i(q_1, q_2, t, 0) = {q_1}^i, \quad q^i(q_1, q_2, t, t) = {q_2}^i. \tag{15}$$

Thus,

$$\begin{aligned}
\frac{\partial S}{\partial {q_2}^i} &= \int_0^t \frac{\partial L}{\partial q^j} \frac{\partial q^j}{\partial {q_2}^i} + \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial {q_2}^i}\, ds \\
&= \int_0^t \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}^j}\right) \frac{\partial q^j}{\partial {q_2}^i} + \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial {q_2}^i}\, ds \\
&= \left.\frac{\partial L}{\partial \dot{q}^j} \frac{\partial q^j}{\partial {q_2}^i}\right|_{s=0}^{s=t} + \int_0^t -\frac{\partial L}{\partial \dot{q}^j} \frac{d}{ds}\left(\frac{\partial q^j}{\partial {q_2}^i}\right) + \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial {q_2}^i}\, ds \\
&= \frac{\partial L}{\partial \dot{q}^j} \frac{\partial}{\partial {q_2}^i}\left(q^j(q_1, q_2, t, t)\right) - \frac{\partial L}{\partial \dot{q}^j} \frac{\partial}{\partial {q_2}^i}\left(q^j(q_1, q_2, t, 0)\right) \\
&= \frac{\partial L}{\partial \dot{q}^j} \frac{\partial}{\partial {q_2}^i}\left({q_2}^j\right) - \frac{\partial L}{\partial \dot{q}^j} \frac{\partial}{\partial {q_2}^i}\left({q_1}^j\right) \\
&= \frac{\partial L}{\partial \dot{q}^i}({q_2}^j, \dot{q}^j(q_1, q_2, t, t)),
\end{aligned}$$

as required (here the Euler-Lagrange equations and the boundary conditions (15) have been used in conjunction with an integration by parts). Similarly,

$$\frac{\partial S}{\partial {q_1}^i} = -\frac{\partial L}{\partial \dot{q}^i}({q_1}^j, \dot{q}^j(q_1, q_2, t, 0)),$$

and

$$\begin{aligned}\frac{\partial S}{\partial t} &= L\Big|_{s=t} + \int_0^t \frac{\partial L}{\partial q^j}\frac{\partial q^j}{\partial t} + \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial t}\, ds \\ &= L({q_2}^j, \dot{q}^j(q_1, q_2, t, t)) + \frac{\partial L}{\partial \dot{q}^j}\frac{\partial q^j}{\partial t}\Big|_{s=0}^{s=t}.\end{aligned}$$

Now one cannot interchange the evaluation at $s = t$ and the derivative with respect to t in the last term, but there is the following calculation:

$$\begin{aligned}0 &= \frac{\partial}{\partial t}\left(q^j(q_1, q_2, t, s)\Big|_{s=t}\right) \\ &= \frac{\partial q^j}{\partial t}\Big|_{s=t} + \frac{\partial q^j}{\partial s}\Big|_{s=t} \\ &= \frac{\partial q^j}{\partial t}\Big|_{s=t} + \dot{q}^j(q_1, q_2, t, t).\end{aligned}$$

Thus, solving this for $\partial q^j/\partial t$, and recalling that the action is defined by

$$A = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i$$

one obtains

$$\begin{aligned}\frac{\partial S}{\partial t} &= (L - A)({q_2}^j, \dot{q}^j(q_1, q_2, t, t)) \\ &= E({q_2}^j, \dot{q}^j(q_1, q_2, t, t)),\end{aligned}$$

as required. □

Corrollary 2. *The generating function S_t defined by theorem* (1) *satisfies the Hamilton-Jacobi equation*

$$\frac{\partial S_t}{\partial t} + H \circ \mathbf{d}_2 S_t = 0.$$

Proof. From theorem (1),

$$\begin{aligned}\frac{\partial S_t}{\partial t} + H \circ \mathbf{d}_2 S_t &= -E \circ \Delta + H \circ FL \circ F_t \circ \Delta \\ &= -E \circ \Delta + E \circ F_t \circ \Delta \\ &= 0. \square\end{aligned}$$

The generating function S_t and its approximations share some elementary properties, which for convenience we collect here in the following definition:

Definition 1. *A regular type 1 generating function is a C^∞ function $f_t : U \setminus (Q^2 \times \{0\}) \to \mathbb{R}$ of the form*

$$f_t(q_1, q_2) = \frac{f^0(q_1, q_2)}{t} + f^1(q_1, q_2, t),$$

where $U \subseteq Q^2 \times \mathbb{R}$ is open, U_0 is open and $f^0 : U_0 \to \mathbb{R}$ is C^∞, $f^1 : U \to \mathbb{R}$ is C^∞, and

1. $\boldsymbol{d}f^0(q,q) = 0$ *for all* $(q,q) \in U_0$,

2. $f^1(q,q,0) = 0$ *for all* $(q,q,0) \in U$,

3. *the bilinear form* $\boldsymbol{d}_1\boldsymbol{d}_2 f^0$ *on* T_qQ *is nonsingular.*

Proposition 2. *The function* S_t *defined in theorem* (1) *is a regular type 1 generating function.*

Proof. Using the kinetic energy Lagrangian L_0 and its associated generating function S^0, one obtains

$$d(q_1,q_2)^2 = 2tS^0(q_1,q_2,t)$$

near (q,q) for small enough t, and since the evolution curve for the L_0 starting at q with velocity zero remains there for all time, so $\Delta^0(q,q,t) = 0_q$, where Δ^0 refers to the Lagrangian L_0. Thus

$$\boldsymbol{d}_1 S^0(q,q,t) = FL_0 \circ F_t^0 \circ \Delta^0(q,q,t) = 0,$$

and similarly with $\boldsymbol{d}_2 S^0$, so that if $v_1, v_2 \in T_qQ$,

$$\begin{aligned} \boldsymbol{d}\left((q_1,q_2) \mapsto d(q_1,q_2)^2\right)(v_1,v_2) &= 2t(\boldsymbol{d}_1 S_t^0(q,q)v_1 + \boldsymbol{d}_2 S_t^0(q,q)v_2) \\ &= 0, \end{aligned}$$

which is the first of the properties in definition (1).

This same idea also shows the third of those properties by computing the mixed second partials of the square of the distance function: let $s \mapsto c(s)$ be a curve tangent to v_2 at $s = 0$. Then

$$\begin{aligned} \boldsymbol{d}_1\boldsymbol{d}_2 S_t^0(q,q)(v_1,v_2) &= \left.\frac{d}{ds}\right|_{s=0} \langle \boldsymbol{d}_1 S_t^0(q,c(s)), v_1\rangle \\ &= -\left.\frac{d}{ds}\right|_{s=0} \langle FL \circ \Delta^0(q,c(s),t), v_1\rangle \\ &= -\langle v_1, T_q\Delta_{q,t} v_2\rangle, \end{aligned}$$

where the last pairing is the kinetic energy metric. By definition of Δ, for fixed t and q_1, the map $q_2 \mapsto \Delta(q_1,q_2,t)$ is locally invertible, so $T_q\Delta_{q,t}$ is a linear isomorphism, and thus $\boldsymbol{d}_1\boldsymbol{d}_2 S_t(q_1,q_2)$ is nonsingular.

Finally, using the function $\tilde{S}_t$ of lemma (2),

$$\begin{aligned} f^1(q,q,0) &= \lim_{\lambda\to 0} \frac{\tilde{S}_t(0_q,\lambda)}{\lambda} \\ &= \lim_{\lambda\to 0} \int_0^\lambda L \circ F_t(0_q)\, dt \\ &= 0 \end{aligned}$$

which is the last of the properties in definition (1). □

We now prove that an invariant regular type 1 generating function defines a momentum preserving symplectic integrator—that is, defines a symplectic, momentum preserving equivariant map on any given relatively compact open set D. One imagines, then, starting with an initial condition in D and iterating the symplectic map for as long as the orbit thereby generated remains in D. Of course, the existence of the symplectic integrator just amounts to an application of the implicit function theorem, the only nuance being the avoidance of the singularity of f_t at $t = 0$ by solving the equation

$$t\alpha_1 = -\boldsymbol{d}_1(tf_t)(\tau_Q^*(\alpha_1), \tau_Q^*(\alpha_2))$$

instead of this equation without the multiples of t.

Theorem 2. *Let $U \subseteq Q^2 \times \mathbb{R}$ be a invariant open set, $f_t : U \setminus (Q^2 \times \{0\}) \to \mathbb{R}$ be an invariant regular type 1 generating function and let $D \subseteq T^*Q$ be an invariant open set with compact closure. Then there exists an $\epsilon > 0$ and a C^∞ map $\Psi : D \times (-\epsilon, \epsilon) \to T^*Q$ such that:*

1. *$\Psi(\alpha, 0) = \alpha$ for all $\alpha \in D$.*
2. *For all $\alpha \in T^*Q$, the curve $c(t) = \Psi_t(\alpha)$ satisfies*

$$\alpha = -\boldsymbol{d}_1 f_t(\tau_Q^*(\alpha), \tau_Q^* \circ c(t)), \quad c(t) = \boldsymbol{d}_2 f_t(\tau_Q^*(\alpha), \tau_Q^* \circ c(t)). \tag{16}$$

3. *For all $\alpha \in T^*Q$, any other curve $c(t)$ which satisfies* (16) *and $c(0) = \alpha$ also satisfies $c(t) = \Psi_t(\alpha)$ for all t in the domain of c such that $(\alpha, t) \in D$.*
4. *For all $t \in (-\epsilon, \epsilon)$, the map $\Psi_t : D \to T^*Q$ is symplectic, momentum preserving and equivariant.*
5. *There exists a unique C^∞ function $H : D \times (-\epsilon, \epsilon) \to \mathbb{R}$ such that f and H satisfy the Hamilton-Jacobi equation*

$$\frac{\partial f_t}{\partial t} + H_t \circ \boldsymbol{d}_2 f_t = 0.$$

Moreover, H_t is invariant, and for all α in D, the curve $t \mapsto \Psi(\alpha, t)$ is the integral curve of time dependent vector field $\boldsymbol{X}_{H_t}$ starting at α at time $t = 0$.

Proof. The first step in the proof is to obtain local existence of the Ψ_t near some arbitrary $\bar{\alpha} \in D$ with base point $\bar{q}$. In local coordinates, and after multiplying by t, the first of equations (16) becomes the following equation for ${q_2}^i({q_1}^j, {p^1}_j, t)$:

$$t{p^1}_i = A_i({q_1}^j, {q_2}^j, t), \tag{17}$$

where

$$A_i = -\frac{\partial}{\partial {q_1}^i}(f^0 + tf^1). \tag{18}$$

Since the bilinear form $\boldsymbol{d}_1\boldsymbol{d}_2 f(\bar{q}, \bar{q}, 0)$ is nonsingular, the array

$$\frac{\partial A_i}{\partial {q_1}^i}(\bar{q}^j, \bar{q}^j, 0) = \frac{\partial^2 f^0}{\partial {q_1}^i \partial {q_2}^j}(\bar{q}^j, \bar{q}^j, 0)$$

is nonsingular, and $t = 0, q_1{}^j = q_2{}^j = \bar{q}^j$ are solutions to (17) since $\boldsymbol{d}f^0$ vanishes on Δ_{Q^2}. Thus, the implicit function theorem finds an open sets U and V of $\bar{\alpha}$, an $\epsilon > 0$, and a unique solution $q_2{}^i(q_1{}^j, p^1{}_j, t) \in V$ of equation (17) defined on $U \times (-\epsilon, \epsilon)$ which is C^∞. Moreover, since

$$t = 0, \quad q_1{}^j = q_2{}^j$$

are all solutions to equation (17), it follows that $q_2{}^i(q_1{}^j, p^1{}_j, 0) = q_1{}^i$.

The projection to Q of this (local) Ψ_t found, the "vertical" part $p^2{}_i(q_1{}^j, p^1{}_j, t)$ of Ψ_t is

$$p^2{}_i(q_1{}^j, p^1{}_j, t) = \begin{cases} \dfrac{\partial f_t}{\partial q_2{}^i}(q_1{}^j, q_2{}^j(q_1{}^k, p^1{}_k, t)) & \text{if } t \neq 0, \\ p^1{}_i & \text{if } t = 0. \end{cases} \tag{19}$$

It remains to be shown that $p^2{}_i$ so defined is C^∞: Indeed, since the derivative of f^0 vanishes on Δ_{Q^2}, there exists a C^∞ function a and C^∞ functions a_{ij} such that

$$f^0(q_1{}^i, q_2{}^i) = a(q_1{}^i) + \frac{1}{2} a_{jk}(q_1{}^i, q_2{}^i)\,(q_2{}^j - q_1{}^j)(q_2{}^k - q_1{}^k),$$

and since $q_2{}^i(q_1{}^j, p^1{}_j, 0) = q_1{}^i$, there exists C^∞ functions $v^i(q_1{}^j, p^1{}_j, t)$ such that

$$q_2{}^i(q_1{}^j, p^1{}_j, t) = q_1{}^i + t v^i(q_1{}^j, p^1{}_j, t).$$

Thus, by substitution into equation (19),

$$p^2{}_i = a_{ij} v^j + \frac{t}{2} \frac{\partial a_{jk}}{\partial q_2{}^i} v^j v^k + \frac{\partial f^1}{\partial q_2{}^i}, \tag{20}$$

which is clearly C^∞. Thus $p^2{}_i$ defined by (19) will be C^∞ if the evaluation of equation (20) at $t = 0$ is $p^1{}_i$. This evaluation is

$$\begin{aligned} p^2{}_i(q_1{}^j, p^1{}_j, 0) &= a_{ij}(q_1{}^k, q_1{}^k) v^j(q_1{}^k, p^1{}_k, 0) + \frac{\partial f^1}{\partial q_2{}^i}(q_1{}^k, q_1{}^k, 0) \\ &= \frac{\partial^2 f^0}{\partial q_2{}^i q_2{}^j}(q_1{}^k, q_1{}^k) \frac{dq_2{}^i}{dt}(q_1{}^k, p^1{}_k, 0) + \frac{\partial f^1}{\partial q_2{}^i}(q_1{}^k, q_1{}^k, 0), \end{aligned} \tag{21}$$

by definition of the a_{ij} and v^j. But then differentiating the equation

$$t p_1{}^i = -\left(\frac{\partial f^0}{\partial q_1{}^i} + t \frac{\partial f_1^t}{\partial q_1{}^i} \right) (q_1{}^j, q_2{}^j(q_1{}^k, p^1{}_k, t))$$

at $t = 0$—that is differentiating equation (17) at $t = 0$—yields

$$p_1{}^i = \frac{\partial^2 f^0}{\partial q_2{}^i q_2{}^j}(q_1{}^k, q_1{}^k) \frac{dq_2{}^i}{dt}(q_1{}^k, p^1{}_k, 0) - \frac{\partial f^1}{\partial q_1{}^i}(q_1{}^k, q_1{}^k, 0),$$

the right side of which is precisely the left side of equation (21), since $f^1(q^i, q^i, 0) = 0$ and therefore

$$\frac{\partial f^1}{\partial q_1{}^i}(q_1{}^k, q_1{}^k, 0) = -\frac{\partial f^1}{\partial q_2{}^i}(q_1{}^k, q_1{}^k, 0)$$

Now let $c(t)$ be another solution to (16) defined on an open interval I containing 0, and consider the set

$$A = \{\, t \in I \cap (-\epsilon, \epsilon) \mid [\tau_Q^* \circ c(t)]^i = q_2{}^i(\bar{q}_1{}^i, \bar{p}^1{}_i, t) \,\}.$$

Then A is clearly a closed set of $I \cap (-\epsilon, \epsilon)$. But uniqueness of the solution $q_2{}^i(q_1{}^j, p^1{}_j, t)$ found above by the implicit function theorem implies that

$$[\tau_Q^* \circ c(t)]^i = q_2{}^i(\bar{q}_1{}^i, \bar{p}^1{}_i, t)$$

for all $t \in I \cap (-\epsilon, \epsilon)$ such that $c(t) \in V$. This fact shows that A is an open subset of $I \cap (-\epsilon, \epsilon)$, since if $\tilde{t} \in A$, then $c(\tilde{t}) \in V$, so for t in some neighborhood of $\tilde{t}$, $c(t) \in V$ and hence $t \in A$. Thus $A = I \cap (-\epsilon, \epsilon)$, and hence the curve $c(t)$ and the local solution constructed above agree on the intersection of their domains. Since D is relatively compact, it may be covered by a finite number of such U_a having associated ϵ_a, V_a, and maps $\Psi^a : U_a \times (-\epsilon_a, \epsilon_a) \to V_a$. The local uniqueness just shown then validates the following construction of a map $\Psi : D \times (-\epsilon, \epsilon) \to \mathbf{R}$ satisfying properties (1)–(4) of the theorem: let ϵ be the minimum of the ϵ_a and let Ψ be the union of the G translates of the maps Ψ_a.

To show property (5), note that the function

$$(q_1, q_2) \mapsto -f_t(q_2, q_1) \tag{22}$$

is also an invariant regular type 1 generating function. Then, possibly shrinking ϵ, the argument up to this point may be applied to (22) and thereby constructs a map $\acute{\Psi}_t : D \times (-\epsilon, \epsilon) \to T^*Q$ such that, for all $\alpha \in D$ and $t \in (-\epsilon, \epsilon)$,

$$\alpha = \boldsymbol{d}_2 f_t(\tau_Q^* \circ \acute{\Psi}_t(\alpha), \tau_Q^*(\alpha)), \quad \acute{\Psi}_t(\alpha) = -\boldsymbol{d}_1 f_t(\tau_Q^* \circ \acute{\Psi}_t(\alpha), \tau_Q^*(\alpha)). \tag{23}$$

Thus if H_t and f_t satisfy the Hamilton-Jacobi equation, then

$$\begin{aligned} H_t(\alpha) &= H_t \circ \boldsymbol{d}_2 f_t(\tau_Q^* \circ \acute{\Psi}_t(\alpha), \alpha) \\ &= -\frac{\partial S_t}{\partial t}(\tau_Q^* \circ \acute{\Psi}_t(\alpha), \alpha) \end{aligned} \tag{24}$$

so H_t is unique, and moreover, given f_t, this equation may be used to define H_t.

Finally, the proof of the theorem will be complete if $t \mapsto \Psi_t(\alpha)$ is an integral curve of $\boldsymbol{X}_{H_t}$. Half of this is shown in theorem (5.2.18) of [1:390]; that reference reduces one to showing that, for all $\alpha \in D$,

$$\frac{d}{dt}\tau_Q^* \circ \Psi_t(\alpha) = T\tau_Q^* \circ \boldsymbol{X}_{H_t} \circ \boldsymbol{d}_2 f_t(p), \tag{25}$$

where $p \stackrel{\text{def}}{=} (\tau_Q^*(\alpha), \tau_Q^* \circ \Psi_t(\alpha))$ for temporary convenience. To show (25), let $v \in TQ$ have the same base point as α, and then differentiate the equation

$$\langle \alpha, v \rangle = -\langle \boldsymbol{d}_1 f_t(\tau_Q^*(\alpha), \tau_Q^* \circ \Psi_t(\alpha)), v \rangle$$

—that is, differentiate the first of equations (1)—with respect to t:

$$0 = -\langle \boldsymbol{d}_1 f_t'(p), v \rangle - \left\langle \boldsymbol{d}_1 \boldsymbol{d}_2 f_t(p), (v, \frac{d}{dt}\tau_Q^* \circ \Psi_t(\alpha)) \right\rangle, \tag{26}$$

where $f_t' \stackrel{\text{def}}{=} \partial f / \partial t$. Also, use the identity $T\tau_Q^* \circ \boldsymbol{X}_H = FH_t$, where FH_t is the fiber derivative of H_t, to differentiate the Hamilton-Jacobi equation in the variable q_1 at p:

$$\begin{aligned} 0 &= \langle \boldsymbol{d}_1 f_t'(p), v \rangle + \langle FH_t \circ \boldsymbol{d}_2 f_t(p), \boldsymbol{d}_1 \boldsymbol{d}_2(p) v \rangle \\ &= \langle \boldsymbol{d}_1 f_t'(p), v \rangle + \langle \boldsymbol{d}_1 \boldsymbol{d}_2(p), T\tau_Q^* \boldsymbol{X}_{H_t} \circ \boldsymbol{d}_2 f_t(p) \rangle. \end{aligned} \tag{27}$$

Subtracting (26) and (27) gives

$$\boldsymbol{d}_1 \boldsymbol{d}_2(p) \cdot \left(v, \frac{d}{dt} \tau_Q^* \circ \Psi_t(\alpha) - T\tau_Q^* \circ \boldsymbol{X}_{H_t} \circ \boldsymbol{d}_2 f_t(p) \right) = 0$$

for all v. This completes the proof, since $\boldsymbol{d}_1 \boldsymbol{d}_2 f^0$ is nonsingular on the diagonal of Q^2, so that, possibly by shrinking ϵ, one may assume that $\boldsymbol{d}_1 \boldsymbol{d}_2 f_t(\tau_Q^*(\alpha), \tau_Q^* \circ \Psi_t(\alpha))$ is nonsingular for all $t \in (-\epsilon, \epsilon)$ and $\alpha \in D$. □

Before proceeding to the next theorem, it may be useful to consider an example. Specifically, the simple harmonic oscillator with Hamiltonian $H \stackrel{\text{def}}{=} p^2/2 + q^2/2$ has flow

$$\begin{aligned} q(t) &= q_0 \cos t + p^0 \sin t, \\ p(t) &= -q_0 \sin t + p^0 \cos t, \end{aligned}$$

and a simple calculation gives

$$S_t(q_1, q_2) = \frac{(q_2)^2 \cos t - 2q_1 q_2 + (q_1)^2 \cos t}{2 \sin t}.$$

Here the metric coefficient is constant and S_t is quadratic in q_1 and q_2, so we perform the change of coordinates $q = \ln x$ to avoid specialities. Then renaming the variable x to q, the Hamiltonian becomes

$$H \stackrel{\text{def}}{=} \frac{1}{2} q^2 p^2 + \frac{1}{2} \ln^2 q$$

with generating function

$$S_t(q_1, q_2) = \frac{\ln^2 q_2 \cos t - 2 \ln q_1 \ln q_2 + \ln^2 q_1 \cos t}{2 \sin t}$$

and flow

$$\begin{aligned} q(t) &= \exp(\ln q_0 \cos t + q_0 p_0 \sin t), \\ p(t) &= \frac{- \ln q_0 \sin t + q_0 p_0 \cos t}{t q(t)}. \end{aligned}$$

Now we construct an approximation $\tilde{S}_t$ to S_t by expanding S_t in a Taylor series in q_2, t about the point $q_2 = q_1, t = 0$, retaining terms of the form $\Delta q^k t^l$ where $k + l \leq 2$ and $\Delta q \stackrel{\text{def}}{=} q_2 - q_1$:

$$\tilde{S}_t(q_1, q_2) \stackrel{\text{def}}{=} \frac{1}{2t} \left(\frac{\Delta q^2}{(q_1)^2} - \frac{\Delta q^3}{(q_1)^3} \right) - \frac{t \ln q_1}{2} \left(\ln q_1 + \frac{\Delta q}{q_1} \right).$$

After a small exercise in the manipulation of Taylor series [24], this approximate generating function gives curves $\tilde{q}(t)$ and $\tilde{p}(t)$ such that

$$\tilde{q}(t) = q(t) + \frac{(q_0)^2 p_0}{6}[11(p_0 q_0)^2 + 3\ln q_0 - 2]t^3 + \mathcal{O}(t^4),$$
$$\tilde{p}(t) = p(t) + \frac{1}{6q_0}[(p_0 q_0)^4 + 2(3\ln q_0 - 2)(p_0 q_0)^2 + 3\ln^2 q_0 - \ln q_0]t^3 + \mathcal{O}(t^4),$$

and the associated time dependent Hamiltonian is

$$\tilde{H}_t = H + \frac{1}{8}[11p^4 q^4 + 2(3\ln q - 2)p^2 q^2 - \ln^2 q]t^2 + \mathcal{O}(t^3).$$

Thus, the approximate evolution generated by $\tilde{S}_t$ is precisely a t^3 perturbation of the exact evolution, while the approximate Hamiltonian is precisely a t^2 perturbation of the exact Hamiltonian. The second order agreement between $p(t)$ and $\tilde{p}(t)$ is really quite remarkable, since the the coefficient of t^2 in $\tilde{p}$ is generated in part by the coefficient of t^3 in $\tilde{q}(t)$, the latter quantity over which, ostensibly, there is little control. But the degree of agreement between the approximate and exact solutions, as well as between the Hamiltonians, is generally true, and that is the subject of the next theorem:

Theorem 3. *Let $f : U \setminus Q^2 \times \{0\} \to \mathbb{R}$ and $\tilde{f} : U \setminus Q^2 \times \{0\} \to \mathbb{R}$ be two functions of the form*

$$f(q_1, q_2, t) = \frac{f^0(q_1, q_2)}{t} + f^1(q_1, q_2, t),$$
$$\tilde{f}(q_1, q_2, t) = \frac{\tilde{f}^0(q_1, q_2)}{t} + \tilde{f}^1(q_1, q_2, t),$$

where U is open, $f^0 : U^0 \to \mathbb{R}$ is C^∞ on an open subset $U^0 \subseteq Q^2$, $f^1 : U \to \mathbb{R}$ is C^∞ and similarly for $\tilde{f}^0$ and $\tilde{f}^1$. Suppose that for some $r \geq 1$ the functions f^0 and $\tilde{f}^0$ have $(r+1)$-contact at (q,q) for all $(q,q) \in U^0$ and the functions f^1 and $\tilde{f}^1$ have r-contact at $(q,q,0)$ for all $(q,q,0) \in U$. Then if f is a regular type 1 generating function, so is $\tilde{f}$, and if an application of theorem (2) *to f and $\tilde{f}$ generates maps*

$$\Psi_t : D \times (-\epsilon, \epsilon) \to T^*Q, \quad \tilde{\Psi}_t : \tilde{D} \times (-\epsilon, \epsilon) \to T^*Q,$$

then the curves

$$t \mapsto \Psi_t(\alpha), \quad t \mapsto \tilde{\Psi}_t(\alpha) \tag{28}$$

have r-contact at $t = 0$ for all $\alpha \in D \cap \tilde{D}$. Moreover, the time dependent Hamiltonians H_t and $\tilde{H}_t$ associated by theorem (2) *to f and $\tilde{f}$ have $(r-1)$-contact at $t = 0$, so that*

$$\tilde{H}_t = H_t + \mathcal{O}(t^r).$$

Proof. It is immediate that $\tilde{f}$ is a regular generating function of type 1, since $r \geq 1$, so f^0 and $\tilde{f}^0$ have 2-contact and f^1 and $\tilde{f}^1$ have 1-contact on the diagonal of Q^2.

Now the proof that the "horizontal" part of the two curves (28) have r-contact at $t = 0$ is obtained by examining the proof of the local part of theorem (2). Indeed, the functions f and $\tilde{f}$ have r-contact at $q_1 = q_2 = \bar{q}$, $t = 0$, so that the two results of multiplying t into one derivative with respect to q_1 of these two functions have r-contact at these points as well. Therefore, the A_i defined by (18) and the similarly defined $\tilde{A}_i$ have r-contact at $q_1 = q_2 = \bar{q}$, $t = 0$, so by implicit differentiation, the curves

$$t \mapsto {q_2}^i(\bar{q}^j, \bar{p}_j, t), \quad t \mapsto {\tilde{q}_2}{}^i(\bar{q}^j, \bar{p}_j, t), \tag{29}$$

have r-contact at $t = 0$.

The "vertical" parts of $\Psi_t(\bar{\alpha})$ and $\tilde{\Psi}_t(\bar{\alpha})$ may be computed by

$$p^2{}_i(\bar{q}^k, \bar{p}_k, t) = \frac{1}{t}\left[\frac{\partial(tf_t)}{\partial {q_2}^i}(\bar{q}_1{}^j, {q_2}^j(\bar{q}^k, \bar{p}_k, t))\right], \tag{30}$$

$$\tilde{p}^2{}_i(\bar{q}^k, \bar{p}_k, t) = \frac{1}{t}\left[\frac{\partial(t\tilde{f}_t)}{\partial {q_2}^i}(\bar{q}_1{}^j, {\tilde{q}_2}{}^j(\bar{q}^k, \bar{p}_k, t))\right]. \tag{31}$$

In the same way as above, the functions

$$({q_1}^j, {q_2}^j, t) \mapsto \frac{\partial(tf_t)}{\partial {q_2}^i}({q_1}^j, {q_2}^j, t), \quad ({q_1}^j, {q_2}^j, t) \mapsto \frac{\partial(t\tilde{f}_t)}{\partial {q_2}^i}({q_1}^j, {q_2}^j, t)$$

have r-contact at $q_1 = q_2 = \bar{q}$, $t = 0$. Thus, the square bracket part of (30) and (31) have r-contact at $t = 0$; the division by t reduces this contact by 1, and so the curves (28) have $(r-1)$-contact at $t = 0$. And this is where the matter would lie, were it not for the fact that the order of contact between (30) and (31) is r, not $r - 1$. To see this, compute as follows: using the chain rule, and the information that the curves (29) have r-contact at $t = 0$, and the fact that tf and $t\tilde{f}$ have r-contact at $q_1 = q_2 = \bar{q}$, $t = 0$, one computes that

$$\begin{aligned} &\left.\frac{d^{r+1}}{dt^{r+1}}\right|_{t=0} \left(tp^2{}_i(\bar{q}^k, \bar{p}_k, t) - t\tilde{p}^2{}_i(\bar{q}^k, \bar{p}_k, t)\right) \\ &= \sum_{k=0}^{r+1} \binom{r+1}{k} \frac{\partial^{r+2}(tf_t - t\tilde{f}_t)}{\partial t^k \partial {q_2}^{z_k} \partial {q_2}^i} \left(\frac{dq_2}{dt}\right)^{z_k} + \frac{\partial^2 f^0}{\partial {q_2}^i \partial {q_2}^j}\left(\frac{d^{r+1}{q_2}^j}{dt^{r+1}} - \frac{d^{r+1}{\tilde{q}_2}{}^j}{dt^{r+1}}\right), \end{aligned} \tag{32}$$

where z_k is a multi-index of length $r + 1 - k$, and the functions on the right are evaluated at $q_1 = q_2 = \bar{q}$, $t = 0$. On the other hand, by the definition of the curves (29), one has

$$\frac{\partial(tf_t)}{\partial {q_1}^i}({q_1}^j, {q_2}^j(\bar{q}^k, \bar{p}_k, t)) - \frac{\partial(t\tilde{f}_t)}{\partial {q_1}^i}({q_1}^j, {\tilde{q}_2}{}^j(\bar{q}^k, \bar{p}_k, t)) = t\bar{p}_k - t\bar{p}_k = 0$$

and differentiating this $r + 1$ times with respect to t yields

$$\sum_{k=0}^{r+1} \binom{r+1}{k} \frac{\partial^{r+2}(tf_t - t\tilde{f}_t)}{\partial t^k \partial {q_2}^{z_k} \partial {q_1}^i} \left(\frac{dq_2}{dt}\right)^{z_k} + \frac{\partial^2 f^0}{\partial {q_1}^i \partial {q_2}^j}\left(\frac{d^{r+1}{q_2}^j}{dt^{r+1}} - \frac{d^{r+1}{\tilde{q}_2}{}^j}{dt^{r+1}}\right) = 0, \tag{33}$$

the left side of which is the right side of (32) if only ${q_2}^i$ would appear wherever ${q_1}^i$ does. In fact the replacement of ${q_1}^i$ with ${q_2}^i$ in (33), up to an irrelevant sign change, is valid: for the second term of (33), since f_t is a regular type 1 generating function, $\boldsymbol{d}f^0(q, q) = 0$, so that

$$\frac{\partial f^0}{\partial {q_1}^j}(q^k, q^k) = \frac{\partial f^0}{\partial {q_2}^j}(q^k, q^k) = 0,$$

and differentiating the second of these equations yields

$$\frac{\partial^2 f^0}{\partial {q_1}^i \partial {q_2}^j}(q^k, q^k) = -\frac{\partial^2 f^0}{\partial {q_2}^i \partial {q_2}^j}(q^k, q^k).$$

The replacement of ${q_1}^i$ with ${q_2}^i$ in the first term of (33) is accomplished a similar way, since tf and $t\tilde{f}$ have at least $(r+1)$-contact at all $(q^i, q^i, 0)$, so

$$\frac{\partial^{r+1}(f^0 - \tilde{f}^0)}{\partial t^k \partial {q_2}^{z_k}}(q^j, q^j, 0) = 0$$

for any multi-index z_k of length $r + 1 - k$.

Finally, since the time dependent Hamiltonians H_t and $\tilde{H}_t$ are defined by equation (24) with S_t replaced by f_t and $\tilde{f}_t$ respectively, it is obvious that H_t and $\tilde{H}_t$ have $(r-1)$-contact in t at $t = 0$: The argument to this point shows that the curves arising from $\acute{\Psi}$ and its $\tilde{\Psi}$ counterpart have r-contact at $t = 0$, while the two functions H_t and $\tilde{H}_t$ are the compositions of these with the partial t derivatives of f_t and $\tilde{f}_t$, which have $(r-1)$-contact at $t = 0$. □

Having determined what terms to keep in the approximation $\tilde{S}_t$ to S so that a given order algorithm is generated, we may now give a formula for $\tilde{S}_t$ that suffices to generate algorithms of order 1,2 and 3.

Theorem 4. *In local coordinates, the expansion of S_t leading to an order 3 algorithm is*

$$\begin{aligned}\tilde{S}_t &= \frac{1}{2t}\left[g_{ij}\Delta q^i \Delta q^j + \frac{1}{2} g_{ij;k}\Delta q^i \Delta q^j \Delta q^k + \frac{1}{12}\left(-g_{ab}\Gamma^a_{ij}\Gamma^b_{kl} + 2g_{ij;kl}\right)\Delta q^i \Delta q^j \Delta q^k \Delta q^l\right] \\ &\quad - t\left[V + \frac{1}{2}V_{;i}\Delta q^i + \frac{1}{12}\left(\Gamma^a_{ij}V_{;a} + 2V_{;ij}\right)\Delta q^i \Delta q^j\right] \\ &\quad - \frac{t^3}{24} g^{ab} V_{;a} V_{;b},\end{aligned}$$

where the functions on the right are evaluated at q_1.

Proof. The generating function S_t conveniently satisfies the equation

$$\begin{aligned} S_t(q_1, \tau_Q^* \circ F_t(v)) &= \int_0^t L \circ F_s(v)\, ds \\ &= \int_0^t (E - 2V) \circ F_s(v)\, ds \\ &= tE(v) - 2\int_0^t V \circ F_s(v)\, ds \end{aligned} \tag{34}$$

where this particular form has been chosen to avoid expanding the functions g_{ij}. From the Euler-Lagrange equations,

$$\begin{aligned}[\tau_Q^* \circ F_t(v)]^i &= {q_1}^i + v^i t - \frac{t^2}{2}\left[g^{ia}V_{;a} + \Gamma^i_{ab} v^a v^b\right] \\ &\quad \frac{t^3}{6}\left[\left[(2g^{bc}\Gamma^i_{ab} - g^{ic}_{;a})V_{;c} - g^{ib}V_{;ab}\right]v^a + (2\Gamma^i_{ad}\Gamma^d_{bc} - \Gamma^i_{ab;c})v^a v^b v^c\right] + \mathcal{O}(t^4)\end{aligned}$$

where the functions of the right are evaluated at q_1. Then this may be inserted into (34), along with

$$V(q_2) = V + V_{;a}\Delta q^a + \frac{1}{2}V_{;ab}\Delta q^a \Delta q^b$$

and

$$\begin{aligned} S_t &= \frac{1}{2t}\left[\frac{1}{2}\kappa^{02}_{ab}\Delta q^a\Delta q^b + \frac{1}{6}\kappa^{03}_{abc}\Delta q^a\Delta q^b\Delta q^c + \frac{1}{24}\kappa^{04}_{abcd}\Delta q^a\Delta q^b\Delta q^c\Delta q^d\right] \\ & t\left[\kappa^{20} + \kappa^{21}_a\Delta q^a + \frac{1}{2}\kappa^{22}_{ab}\Delta q^a\Delta q^b\right] + t^3\kappa^{40}, \end{aligned}$$

and the formula for $\tilde{S}_t$ results from solving for the κ's by comparing coefficients of powers of v^i and t [25]. □

Remark. If one uses an r-order approximation $\tilde{S}_t$ to S_t, then theorem (3) guarantees that the time step map of the symplectic integrator is the flow of a time dependent Hamiltonian $\tilde{H} = H + \mathcal{O}(t^{r-1})$. Thus, iterating the map $\tilde{\Psi}_t$ defined by $\tilde{S}_t$ is equivalent to performing a discrete sample of the flow of the time dependent Hamiltonian obtained by restricting $\tilde{H}$ to the interval $[0, t)$ and then extending that to a periodic function on all of $\mathbb{R}$. In this way, the algorithm is realized as a small, high frequency perturbation of the exact system, a fact which suggests that averaging techniques of Hamiltonian systems theory may serve as a vehicle to prove that the approximate phase portrait retains features of the exact one.

If the local coordinates in which S_t is approximated are not adapted to the action of the symmetry group, then the group invariance property of S_t may be lost in the truncation process used to obtain the approximation $\tilde{S}_t$. This is an important issue, since if $\tilde{S}_t$ is not invariant, then the algorithm it produces need not be momentum preserving nor equivariant, and the entire purpose of our development would be stymied. The following lemma shows that this is not a problem when the action of the symmetry group is affine in the local coordinates used to expand S_t.

Lemma 3. *Let $U \subseteq E$ and $V \subseteq F$ be open, E and F be Banach spaces, $f : U^2 \times V \to \mathbb{R}$ be C^∞, $A : E \to F$ be linear, $b \in E$, and suppose that*

$$f(Aq_1 + b, Aq_2 + b, t) = f(q_1, q_2, t) \tag{35}$$

for all $(q_1, q_2, t) \in U^2 \times V$ such that $(Aq_1 + b, Aq_2 + b, t) \in U^2 \times V$. Then any function $\tilde{f}$ obtained by retaining finitely many terms in the Taylor expansion of f in the variables q_2 and t about $q_2 = q_1$, $t = 0$ also satisfies the invariance property (35).

Proof. Every term in the Taylor expansion is a constant multiple of a function $T(q_1, q_2, t)$ of the form

$$T(q_1, q_2, t) = \mathcal{D} \cdot f\left(q_1, q_1 + \sum_{i=1}^{n}\alpha_i\Delta q, \sum_{i=1}^{m}\beta_i t\right),$$

where $\mathcal{D}$ is the differential operator

$$\mathcal{D} \stackrel{\text{def}}{=} \left.\frac{d^{n+m}}{d\alpha_1\cdots d\alpha_n d\beta_1\cdots d\beta_m}\right|_{\substack{\alpha_1=\cdots=\alpha_n=0\\ \beta_1=\cdots=\beta_m=0}},$$

and the proof is completed by noting that any such function satisfies the same invariance property as f does:

$$\begin{aligned} T(Aq_1+b, aq_2+b, t) &= \mathcal{D}\cdot f\left(Aq_1+b, Aq_1+b+\sum_{i=1}^{n}\alpha_i A\Delta q, \sum_{i=1}^{m}\beta_i t\right) \\ &= \mathcal{D}\cdot f\left(Aq_1+b, A\left[q_1+\sum_{i=1}^{n}\alpha_i\Delta q\right]+b, \sum_{i=1}^{m}\beta_i t\right) \\ &= \mathcal{D}\cdot f\left(q_1, q_1+\sum_{i=1}^{n}\alpha_i\Delta q, \sum_{i=1}^{m}\beta_i t\right) \\ &= T(q_1,q_2,t).\square \end{aligned}$$

Unfortunately, the act of invoking local coordinates usually complicates the action of the symmetry group to such a degree that it is not affine (consider for example the manifestation in Euler angle coordinates of the action of $SO(3)$ on itself by left or right multiplication), so a method is required that will modify the approximate generating function $\tilde{S}_t$ to an invariant function. Such a scheme is provided by assuming the existence of a section to the action of the symmetry group within the domain of the coordinate chart—that is, assuming that the coordinate chart reduces us to the configuration space $U \subseteq \mathbb{R}^n$ which admits a submanifold $\mathcal{S}$ and a C^∞ map $\Theta : U \to G$ such that:

1. $\mathcal{S}$ intersects each G orbit in U exactly once.
2. For all $x \in U$, $\Theta(x)$ is the unique element of G such that $\Theta(x)\cdot x \in \mathcal{S}$.

Sections, for example, exist near points with nontrivial isotropy, and furthermore, it is typically easy to find them explicitly, while isotropy at a point will typically preclude the existence of a section through it. Given a section, an invariant approximation $\hat{S}_t$ to S_t may be constructed by decreeing that

$$\hat{S}_t(q_1,q_2) \stackrel{\text{def}}{=} \tilde{S}_t(\Theta(q_1)\cdot q_1, \Theta(q_1)\cdot q_2).$$

Indeed, $\hat{S}_t$ is invariant, since if $g\in G$ then $g\cdot q_1$ and q_1 belong to the same G orbit, so that

$$(\Theta(gq_1))\cdot(g\cdot q_1) = (\Theta(gq_1)g)\cdot q_1 = \Theta(q_1)\cdot q_1$$

and thereby $\Theta(gq_1) = \Theta(q_1)g^{-1}$. Thus,

$$\hat{S}_t(gq_1, gq_2) = \tilde{S}_t(\Theta(gq_1)\cdot gq_1, \Theta(gq_1)\cdot gq_2) = \hat{S}_t(q_1,q_2).$$

Furthermore, since S_t is invariant, $\hat{S}_t$ has the same order contact with S_t as $\tilde{S}_t$ has with S_t.

Since $\hat{S}_t$ so constructed is invariant, the symplectic map Ψ_t that it generates is equivariant, an observation that implies a fact essential to the construction of an integrator that makes effective use of the symmetry of the system: *it suffices to be able to compute $\Psi_t(\alpha)$ for $\alpha \in T^*U$ with base point in $\mathcal{S}$.* Indeed, if $\alpha_q \in T^*U$ with $q \notin \mathcal{S}$, then we may compute $\Psi_t(\alpha_q)$ using the equation

$$\Psi_t(\alpha_q) = \Theta(q)^{-1}\cdot\Psi_t(\Theta(q)\cdot\alpha_q).$$

In practice, the initial state of the system is represented as $g_T\alpha_q$, where $q \in \mathcal{S}$ and $g_T \in G$. Then equations (1) and (2) with f replaced by $\hat{S}_t$ are used to generate an intermediary $\alpha'_{q'} \stackrel{\text{def}}{=} \Psi_t(\alpha)$, where q' is not necessarily in $\mathcal{S}$, and then α and g_T are updated by the replacements

$$\alpha \leftarrow \Theta(q')\cdot\alpha, \quad g_T \leftarrow g_T\Theta(q')^{-1}.$$

This procedure can then be iterated until $\alpha' \notin U$, at which time a transformation must be made to another chart.

The final theoretical element of the algorithm concerns the computation of $\Psi_t(\alpha_q)$ when $q \in \mathcal{S}$, and here matters are expedited if we arrange that

$$\mathcal{S} = \{\,(q^1,\cdots,q^m,0,\cdots,0) \mid q^1,\cdots,q^m \in \mathbf{R}\,\} \cap U.$$

Then the map

$$(q_1,q_2) \mapsto (\Theta(q_1)\cdot q_1, \Theta(q_1)\cdot q_2) \stackrel{\text{def}}{=} (\tilde{q}_1{}^\gamma(q_1,q_2), \tilde{q}_2{}^i(q_1,q_2))$$

yields m functions $\tilde{q}_1{}^\gamma$ of q_1 and q_2 and $n \stackrel{\text{def}}{=} \dim Q$ functions $\tilde{q}_2{}^i$ of q_1 and q_2 (here and immediately below, Greek indices will run from 1 to m and Latin indices from 1 to n). Then the equation (1) becomes, after multiplying by t,

$$tp_1{}^i = -\frac{\partial(t\tilde{S}_t)}{\partial\tilde{q}_1{}^\gamma}\frac{\partial\tilde{q}_1{}^\gamma}{\partial q_1{}^i} - \frac{\partial(t\tilde{S}_t)}{\partial\tilde{q}_2{}^j}\frac{\partial\tilde{q}_2{}^j}{\partial q_1{}^i},$$

so that the derivatives of $\tilde{S}$ need only be computed at $(q_1,q_2) \in \mathcal{S}\times U$. Equation (2) simply becomes

$$p_2{}^i = \frac{\partial\tilde{S}_t}{\partial\tilde{q}_2{}^i},$$

since if $q_1 \in \mathcal{S}$, then $\Theta(q_1) = \mathrm{Id}$.

We close this chapter by giving coordinate charts and sections that have been used in an implementation of the above algorithm to the system of two axially symmetric identical coupled rigid bodies. One chart is suggested by proposition (2.1): the coordinate chart on $SO(3)^2$ implied by the map

$$(q^1,\cdots,q^6) \mapsto (\exp(q^2\boldsymbol{k}^\wedge)\exp(q^3\boldsymbol{j}^\wedge)\exp(q^4\boldsymbol{k}^\wedge), \exp(q^5\boldsymbol{k}^\wedge)\exp(q^1\boldsymbol{j}^\wedge)\exp(q^6\boldsymbol{k}^\wedge)), \tag{36}$$

where $0 < q^1 < \pi$. Strictly speaking, this is not a chart unless the coordinates $q^2,\cdots,q^6$ are further restricted, but looseness here has no adverse effects as long as these angles are prevented from becoming too large by suitable addition or subtraction of multiples of 2π. The set M_0' of proposition (2.1) is a section to the action of $SO(3)\times(S^1)^2$, which in these coordinates becomes simply

$$\mathcal{S}_1 \stackrel{\text{def}}{=} \{\,(q^1,0,\cdots,0) \mid q^1 \in \mathbf{R}\,\}$$

and the map $\Theta(q^i)$ is given by first finding ϕ_1, ϕ_2 and ϕ_3 such that

$$\exp(\phi_1\boldsymbol{k}^\wedge)\exp(\phi_2\boldsymbol{j}^\wedge)\exp(\phi_3\boldsymbol{k}^\wedge) = A_1{}^tA_2,$$

where A_1 and A_2 are the first and second components of the right side of (36), and then setting

$$\Theta(q^i) \stackrel{\text{def}}{=} (\exp(-\phi_1 \boldsymbol{k}^\wedge) {A_1}^t, -\phi_1, -\phi_2).$$

In practice, a point in $SO(3)^2$ is deemed inside this coordinate chart when its projection to the section $\mathcal{S}_1$ gives q^1 inside the interval $(a_1, \pi - a_1)$ for some small and somewhat arbitrary positive choice of $a_1 < \pi/2$. Algorithms up to order 3 constructed using this section have been observed to have the expected momentum preserving properties of their design purpose, in addition to the excellent long term stability inherent in symplectic integration algorithms in general.

Charts covering the remaining part of $SO(3)^2$ must deal with the two points with nontrivial isotropy, namely $(\mathrm{Id}, \mathrm{Id})$ and $(\mathrm{Id}, \exp(\pi \boldsymbol{j}^\wedge))$, and so the results here are somewhat less satisfactory. One way to proceed is to use the chart implied by the map

$$(q^1, \cdots, q^6) \mapsto (\exp(q^4 \boldsymbol{i}^\wedge) \exp(q^5 \boldsymbol{j}^\wedge) \exp(q^6 \boldsymbol{k}^\wedge), \exp(q^1 \boldsymbol{i}^\wedge) \exp(q^2 \boldsymbol{j}^\wedge) \exp(q^3 \boldsymbol{k}^\wedge)), \qquad (37)$$

along with the section, *to the action of $SO(3)$ only*, defined by

$$\mathcal{S}_2 \stackrel{\text{def}}{=} \{ (q^1, q^2, q^3, 0, 0, 0) \mid q^1, q^2, q^3 \in \mathbb{R} \}$$

and then $\Theta(q^i) \stackrel{\text{def}}{=} {A_1}^t$, where A_1 is the first component of the right side of (37). The practice here is that a point in $SO(3)^2$ is deemed inside this chart when its projection to the section $\mathcal{S}_2$ gives q^2 outside the interval $(-a_2, a_2)$ for some choice of positive $a_2 < \pi/2$. Without the use of the section, an integrator built just using the chart defined by (37) would preserve the moments associated to the action of $(S^1)^2$, since that action in this chart is just translation in the variables q^6 and q^3. Using the section destroys this affine nature in the variable q^3 while retaining it in the variable q^6, so an algorithm based on the use of the section conserves all but one of the moments.

Finally, if $a_1 + a_2 < \pi/2$, then the two charts above cover $SO(3)^2$. Indeed, after multiplying some matrices, one finds that the domain of the first chart is

$$\{ (A_1, A_2) \in SO(3)^2 \mid a_1 < \arccos\langle {A_1}^t A_2 \boldsymbol{k}, \boldsymbol{k} \rangle < \pi - a_1 \},$$

while that of the second is

$$\{ (A_1, A_2) \in SO(3)^2 \mid a_1 < |\arcsin\langle {A_1}^t A_2 \boldsymbol{k}, \boldsymbol{k} \rangle| < \pi - a_1 \},$$

and the union of these two sets is all of $SO(3)^2$, since the first consists of the $(A_1, A_2) \in SO(3)^2$ such that ${A_1}^{-1} A_2 \boldsymbol{k}$ is outside a cone about the $\boldsymbol{k}$ axis having opening angle $2a_1$, while the second set consists of the $(A_1, A_2) \in SO(3)^2$ such that ${A_1}^{-1} A_2 \boldsymbol{k}$ is outside a cone about the $\boldsymbol{i}$ axis having opening angle $2a_2$.

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