

Geometric Discrete Analogues of Tangent Bundles and Constrained Lagrangian Systems

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Abstract

Discretizing variational principles, as opposed to discretizing differential equations, leads to discrete-time analogues of mechanics, and, systematically, to geometric numerical integrators. The phase space of such variational discretizations is often the set of configuration pairs, analogously corresponding to initial and terminal points of a tangent vector. We develop alternative discrete analogues of tangent bundles, by extending tangent vectors to finite curve segments, one curve segment for each tangent vector. Towards flexible, high order numerical integrators, we use these discrete tangent bundles as phase spaces for discretizations of the variational principles of Lagrangian systems, up to the generality of nonholonomic mechanical systems with nonlinear constraints. We obtain a self-contained and transparent development, where regularity, equations of motion, symmetry and momentum, and structure preservation, all have natural expressions.

1 Introduction

A discretization of a Lagrangian system $L: TQ \rightarrow \mathbb{R}$ consists of

1. a time step $h > 0$;
2. the discrete phase space $Q \times Q = \{(q^+, q^-)\}$, thought of as a discrete version of the tangent bundle TQ ;
3. a discrete Lagrangian $L_h: Q \times Q \rightarrow \mathbb{R}$, obtained by approximately integrating L over an appropriate interpolation from q^- to q^+ .

The discrete Lagrangian L_h and the discrete phase space $Q \times Q$ together define a *discrete Lagrangian system*. Evolutions are sequences $q_k \in Q$, $k = 1, \dots, N$ that are critical points of the *discrete action* S , defined by

$$S \equiv \sum_{k=1}^N L_h(q_k, q_{k-1}),$$

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subject to the constraint q_0 and q_N constant. As is easily shown, q_k is an evolution if and only if it satisfies the *discrete Euler-Lagrange equation*

$$\frac{\partial L_h}{\partial q^+}(q_k, q_{k-1}) + \frac{\partial L_h}{\partial q^-}(q_{k+1}, q_k) = 0. \quad (1)$$

Lagrangian discretizations lead to (implicit, symplectic, and momentum conserving) numerical methods because Equation (1) can be used to advance through time h by stepping states (q_k, q_{k-1}) to (q_{k+1}, q_k) .

Such Lagrangian discretizations are towards discrete Lagrangian models that reflect physical reality so well that they have a stature with continuous Lagrangian models. States of a continuous model — such as values of the independent variables of a differential equation — and states of a discrete model — such as the pairs of configurations used in the map implied by (1), can both serve as abstract representations of system states. Most important is not the particular representation of the states, but whether the states evolve as the physical system does. If errors, in either a discrete or a continuous model, are below measurement errors of a physical system, and neither the discrete nor the continuous model violates fundamental physical principles to that accuracy, then both models have a similar stature. Lagrangian discretizations also provide variational discrete analogues of continuous Lagrangian systems. They are of interest in themselves, and they provide a framework for the analysis, understanding, and development of geometric integration algorithms [12] for Lagrangian systems as purely mathematical objects. For more details, see [7, 10, 11, 16, 17, 25].

In this article, we further develop discretizations of Lagrangian systems. We refer to the discretizations outlined above as *Moser-Veselov (MV) discretizations*. Our discrete Lagrangian systems replace the MV discrete phase space $Q \times Q$ with a discrete phase space \mathcal{V} consisting of curve segments in Q which one-to-one correspond with elements of TQ . To any such discrete Lagrangian system, there is naturally associated an isomorphic MV discrete Lagrangian system, obtained by identifying our curve segments with their boundaries. Viewing tangent bundles as curve segments is generally consistent with viewing discretizations in general as attaching to a manifold finite rather than infinitesimal objects [5].

Systematically using curve segments provides theoretical flexibility and geometric clarity. For example, our curve segments can be naturally shrunk, and this helps to analyze limits where the time step tends to zero. The interpolating curves of MV discretizations are obtained implicitly from boundary value problems, with boundary values the two configurations of the MV discrete states. We finesse this implicit dependence by directly using the interpolating curves. We achieve a self contained variational theory, which does not depend on discrete versions of the Legendre transform, nor on any canonical formalism on the cotangent bundle. We show that the entire development extends to nonholonomic systems with nonlinear constraints. We extend the curvature conditions for holonomic subsystems of nonholonomic systems [19] to the discrete case, and prove the nonholonomic momentum equation [4, 7] in our context. As well, we show how our discrete Lagrangian systems specialize to discrete *holonomic* systems.

Some notations: Unless otherwise noted, objects are sufficiently smooth to permit the required operations. If \mathcal{M} is a manifold and $v_m, w_m \in T\mathcal{M}$, then define

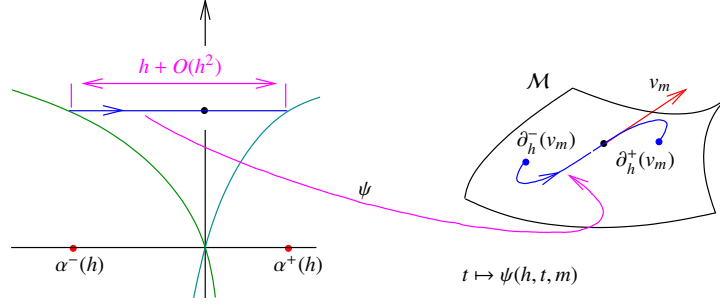
$$\text{vert}_{v_q} w_q \equiv \left. \frac{d}{dt} \right|_{t=0} (v_q + tw_q).$$

If $\pi: E \rightarrow \mathcal{M}$ is a vector bundle, and $z \in T_{0_m}E$ i.e. if z is a tangent vector at the zero section, then we denote the horizontal and vertical parts of z by $\text{hor } z \in T_m\mathcal{M}$ and $\text{vert } z \in E_m$, respectively. We denote the fiber dimension of a fiber bundle by fdim and the fiber codimension of a subbundle by cofdim . To reduce double subscripts, we sometimes use the functional notation $x(k)$ instead of x_k for a sequence. If A is a set, then we will use the notation $A[M, N]$ for the sequences in $v(k) \in A$, $k = M, \dots, N$. If \mathcal{G} acts smoothly on a manifold \mathcal{M} then we denote the Lie algebra of \mathcal{G} by \mathfrak{g} , and the *infinitesimal generator* of $\xi \in \mathfrak{g}$ at $m \in \mathcal{M}$ by

$$\xi m \equiv \left. \frac{d}{dt} \right|_{t=0} \exp(\xi t)m.$$

Assembling these into a vector field gives $\xi_{\mathcal{M}}(m) \equiv \xi m$.

2 Discretizations of Tangent Bundles



Let \mathcal{M} be a manifold and $m \in \mathcal{M}$. Two curves $c: (a, b) \rightarrow \mathcal{M}$ and $\tilde{c}: (\tilde{a}, \tilde{b}) \rightarrow \mathcal{M}$ with $0 \in (a, b)$ and $0 \in (\tilde{a}, \tilde{b})$ are *tangent at m* if 1) $c(0) = \tilde{c}(0) = m$; and 2) $\phi c(t) - \phi \tilde{c}(t) = O(t^2)$ in any chart ϕ of \mathcal{M} with domain including m . Tangency at m is an equivalence relation, and the *tangent space* $T_m\mathcal{M}$ at $m \in \mathcal{M}$ may be defined [2] as the set of equivalence classes of curves at m . Our discretizations of Lagrangian systems depend on the development of a discretization of a tangent bundle $T\mathcal{M}$ as assignments of curve segments in \mathcal{M} to tangent vectors of \mathcal{M} . We will require a parameter h such that $T\mathcal{M}$ is obtained in the limit $h \rightarrow 0^+$. So, we posit a map $\psi(h, t, m)$, with values in \mathcal{M} , and obtain the curve segments $t \mapsto \psi(h, t, m)$:

Definition 1. A C^k discretization of $T\mathcal{M}$, $k \geq 1$, is a tuple $(\psi, \alpha^+, \alpha^-)$, where

$$\psi: U \subseteq \mathbb{R}^2 \times \mathcal{M} \rightarrow \mathcal{M}, \quad \alpha^+: [0, a) \rightarrow \mathbb{R}_{\geq 0}, \quad \alpha^-: [0, a) \rightarrow \mathbb{R}_{\leq 0},$$

are such that

1. ψ is continuous, U is open, and $\{0\} \times \{0\} \times \mathcal{M} \subseteq U$;
2. α^+, α^- are C^1 , and $\alpha^+(h) - \alpha^-(h) = h$;
3. $\psi(h, 0, v_m) = m$, and $\frac{\partial \psi}{\partial t}(h, 0, v_m) = v_m$;
4. the boundary maps defined by

$$\partial_h^-(v_m) \equiv \psi(h, \alpha^-(h), v_m), \quad \partial_h^+(v_m) \equiv \psi(h, \alpha^+(h), v_m), \quad (2)$$

are C^k in (h, v_m) , and

$$\frac{d}{dh} \Big|_{h=0} \partial_h^+(v_m) = \dot{\alpha}^+ v_m, \quad \frac{d}{dh} \Big|_{h=0} \partial_h^-(v_m) = \dot{\alpha}^- v_m, \quad (3)$$

where

$$\dot{\alpha}^+ \equiv \frac{d\alpha^+}{dh}(0), \quad \dot{\alpha}^- \equiv \frac{d\alpha^-}{dh}(0).$$

Remark 1. Putting $h = 0$ in $\alpha^+(h) - \alpha^-(h) = h$ gives $\alpha^+(0) = \alpha^-(0) = 0$ because $\alpha^+ \geq 0$ and $\alpha^- \leq 0$. Differentiating $\alpha^+(h) - \alpha^-(h) = h$ at $h = 0$ gives $\dot{\alpha}^+ - \dot{\alpha}^- = 1$. If ψ is C^1 , then Assumptions (3) are superfluous, since

$$\frac{d}{dh} \Big|_{h=0} \partial_h^+(v_m) = \frac{d}{dh} \Big|_{h=0} \psi(h, \alpha^+(h), v_m) = \frac{d}{dh} \Big|_{h=0} \psi(h, 0, v_m) + \frac{d}{dh} \Big|_{h=0} \psi(0, \alpha^+(h), v_m) = \dot{\alpha}^+ v_m,$$

and similarly with ∂_h^- . The definition allows ψ to be only piecewise smooth in h, t .

For all $v_m \in \mathcal{M}$, the set $\{(h, t) : (h, t, v_m) \in U\}$ is open and contains $h = 0, t = 0$, and so contains the set $\{(h, t) : h \in [0, b), \alpha^-(h) \leq t \leq \alpha^+(h)\}$ for some $0 < b < a$. So assigned to every $v_m \in \mathcal{TM}$ and small enough $h > 0$ is the curve segment

$$t \mapsto \psi(h, t, v_m), \quad \alpha^-(h) \leq t \leq \alpha^+(h),$$

which, since $\psi(h, 0, v_m) = m$ and $\frac{\partial \psi}{\partial t}(h, 0, v_m) = v_m$, is a curve at m which is tangent to v_m . A *discretization* $(\psi, \alpha^+, \alpha^-)$ assigns to every v_m a curve segment that can be thought of as a translational step like $h v_m$.

Example 1. Let $\mathcal{M} \equiv \mathbb{R}^N$, $0 \leq \gamma \leq 1$, $\alpha^+(h) \equiv \gamma h$, $\alpha^-(h) \equiv -(1 - \gamma)h$, and $\psi(t, h, v_m) \equiv m + t v$. More generally, let X be any second order vector field on \mathcal{M} , $0 \leq \gamma \leq 1$ and define $\psi^X(h, t, v_q) \equiv \tau_{\mathcal{M}} F_t^X(v_q)$ where F_t^X is the flow of X and $\tau_{\mathcal{M}}: \mathcal{TM} \rightarrow \mathcal{M}$ is the canonical projection. ψ^X and α^+, α^- is the X -discretization with bias (α^-, α^+) .

Example 2. Let X be a vector field on a manifold \mathcal{M} . A *one-step numerical method* for X is a map $\varphi: U \subseteq [0, \infty) \times \mathcal{M} \rightarrow \mathcal{M}$ such that

1. $\{0\} \times \mathcal{M} \subseteq U$;
2. $\varphi(0, m) = m$ for all $m \in \mathcal{M}$;
3. $\frac{d}{dt} \Big|_{t=0} \varphi(t, m) = X(m)$.

If φ is a one-step numerical method for a second order vector field on \mathcal{TM} then $\psi(h, t, v_q) \equiv \tau_{\mathcal{M}} \varphi(t, v_q)$ is a discretization of \mathcal{TM} .

Generally, when we speak of a *discretization* we mean a *family of discrete objects* parametrized by $h \in \mathbb{R}$, such that a continuous target is approached as $h \rightarrow 0$. A *discrete object* is an *instance of a discretization*, obtained by fixing h to a particular value and possibly dropping data not required to make operational the discrete representation of the continuous target. For tangent bundles, we choose the transition from discretization to discrete as the juncture at which we drop the curve segments in discretizations of tangent bundles, retaining only their endpoints:

Definition 2. Let \mathcal{M} be a manifold. A discrete tangent bundle of \mathcal{M} is a tuple $(\mathcal{V}, \partial^+, \partial^-)$, where \mathcal{V} is a manifold, $\dim \mathcal{V} = 2 \dim \mathcal{M}$ and $\partial^+: \mathcal{V} \rightarrow \mathcal{M}$ and $\partial^-: \mathcal{V} \rightarrow \mathcal{M}$ satisfy

1. ∂^+ and ∂^- are submersions such that $\ker T\partial^+ \cap \ker T\partial^- = 0$; and
2. for all $m \in \mathcal{M}$, the backward fiber $\mathcal{V}_m^+ \equiv (\partial^+)^{-1}(m)$ and the forward fiber $\mathcal{V}_m^- \equiv (\partial^-)^{-1}(m)$ meet in exactly one point, denoted 0_m .

The discrete zero section is $0_{\mathcal{V}} \equiv (\partial^\pm)^{-1} \Delta(\mathcal{M} \times \mathcal{M})$, where $\Delta(\mathcal{M} \times \mathcal{M})$ is the diagonal of $\mathcal{M} \times \mathcal{M}$.

Remark 2. Let $\partial^\pm: \mathcal{V} \rightarrow \mathcal{M} \times \mathcal{M}$ be defined by $\partial^\pm(p) \equiv (\partial^+(p), \partial^-(p))$. Item 1 of Definition 2 implies that $T_v \partial^\pm$ is a linear isomorphism for all $v \in \mathcal{V}$, and therefore ∂^\pm is a local diffeomorphism. Also, Item 2 of Definition 2 implies that ∂^\pm is bijective from $0_{\mathcal{V}}$ to $\Delta(\mathcal{M} \times \mathcal{M})$ so that the local diffeomorphism ∂^\pm is a diffeomorphism of $0_{\mathcal{V}}$ to $\Delta(\mathcal{M} \times \mathcal{M})$. $0_{\mathcal{V}}$ is a closed submanifold of \mathcal{V} because $\Delta(\mathcal{M} \times \mathcal{M})$ is a closed submanifold of $\mathcal{M} \times \mathcal{M}$, and Theorem 1 (semiglobal inverse function theorem) of [8] provides open neighborhoods U of $0_{\mathcal{V}}$ and V of $\Delta(\mathcal{M} \times \mathcal{M})$ for which ∂^\pm is a diffeomorphism.

Remark 3. Our discrete tangent bundles are similar to the groupoid based constructions of discrete phase spaces for Lagrangian systems in [24]. Indeed, to any discrete tangent bundle \mathcal{V} there is an associated Lie groupoid consisting of 1) sequences v_k in \mathcal{V} which satisfy $\partial^+(v_k) = \partial^-(v_{k+1})$, and the reverses of these, 2) units the elements 0_m , and 3) source and target maps ∂^+ and ∂^- . Some of the constructions below are the same as those found in the groupoid context. One can regard discrete tangent bundles of \mathcal{M} to be groupoids over \mathcal{M} for which the set of irreducible elements \mathcal{V} is a manifold which satisfies Item 1 of Definition 2. To the extent of this article, the algebraic structure of the groupoid seems to generate more ambiguity than it does clarity. For example, starting as in Definition 2 with a tangent vector $v \in \mathcal{V}$, one might include the formal reverse of v in order to have its groupoid inverse. There will generally be another, different, element in \mathcal{V} with the same source and target as that formal reverse. And, the product of that with the original v is a two element sequence that starts and ends at the same place of \mathcal{M} , but it is not the same as the discrete zero vector.

Remark 4. Given any manifold \mathcal{M} , the tuple $(\mathcal{M} \times \mathcal{M}, \partial_{\mathcal{M} \times \mathcal{M}}^+, \partial_{\mathcal{M} \times \mathcal{M}}^-)$ is a discrete tangent bundle, where

$$\mathcal{M} \times \mathcal{M} \equiv \{(m^+, m^-)\}, \quad \partial_{\mathcal{M} \times \mathcal{M}}^-(m^+, m^-) \equiv m^-, \quad \partial_{\mathcal{M} \times \mathcal{M}}^+(m^+, m^-) \equiv m^+.$$

Thus the usual MV discretizations of the tangent bundle are special cases of Definition 2. For a discrete tangent bundle $(\mathcal{V}, \partial^+, \partial^-)$, the map ∂^\pm is typically a diffeomorphism in the region of interest, so one can in principle, using ∂^\pm , replace any discrete tangent bundle $(\mathcal{M}, \partial^+, \partial^-)$ as in Definition 2 by its image in $\mathcal{M} \times \mathcal{M}$, dispense with the maps ∂^+ and ∂^- , and use the corresponding MV discrete tangent bundle:

$$\begin{array}{ccc} & \mathcal{M} & \\ \partial^+ \nearrow & & \nwarrow \partial_{\mathcal{M} \times \mathcal{M}}^+ \\ \mathcal{V} & \xrightarrow{\partial^\pm} & \mathcal{M} \times \mathcal{M} \\ \partial^- \searrow & & \swarrow \partial_{\mathcal{M} \times \mathcal{M}}^- \\ & \mathcal{M} & \end{array}$$

However, the freedom of including ∂^+ and ∂^- in the definition of a discrete tangent bundle, and the abstraction of the discrete tangent vectors as elements of a manifold \mathcal{V} , is helpful and clarifying.

Since *discretizations* are to be families of *discrete analogues*, it is necessary to show that a discretization of a tangent bundle gives discrete tangent bundles for sufficiently small h (Proposition 1 below). This is not immediate because there is a singularity at $h = 0$. To see the problem, consider the example $\mathcal{M} \equiv \mathbb{R}^N$, with discretization $\psi(h, t, (m, v)) \equiv m + tv + O(h^2)$. To show Item 2 of Definition 2, it is sufficient to show the ∂_h^\pm is a diffeomorphism near $h = 0$ i.e. that the equations

$$m^- = m + \alpha^-(h)v + O(h^2), \quad m^+ = m + \alpha^+(h)v + O(h^2), \quad (4)$$

can be solved uniquely and smoothly for (m, v) in terms of (m^+, m^-) . Because there is no a priori knowledge of the details of the $O(h^2)$ term, the proof of that has to be perturbative from $h = 0$. However, when $h = 0$, Equations (4) cannot be solved at for v because they reduce to $m^- = m, m^+ = m$. Replacing v with $\tilde{v} \equiv v/h$ would solve the problem

but would require, at later stages, smoothness assumptions in v/h , and would make the subsequent development awkward and less general. But, replacing (m^+, m^-) with the new variables (\bar{m}, z) defined by

$$\bar{m} \equiv \frac{m^+ + m^-}{2}, \quad z \equiv \frac{m^+ - m^-}{h}, \quad (5)$$

converts Equations (4) to

$$\bar{m} = m + \frac{\alpha^+(h) + \alpha^-(h)}{2}v + O(h^2), \quad z = v + O(h). \quad (6)$$

We remark that a division by h has reduced the order of the trailing term in the second equation. At $h = 0$, Equations (6) are

$$\bar{m} = m, \quad z = v,$$

which has solution $m = \bar{m}$, $v = z$. By the implicit function theorem, Equation (6) may be solved for m, v in terms of \bar{m}, z for sufficiently small h , and hence, through Equation (5), for m, v in terms of (m^+, m^-) , as required. To obtain a result near $h = 0$ which is valid for m, v near the whole zero section $0(TM)$ i.e. local along h but global along $0(TM)$, we again make use of Theorem 1 (semiglobal inverse function theorem) of [8].

Proposition 1. *Let $(\psi, \alpha^+, \alpha^-)$ be a discretization of the tangent bundle of \mathcal{M} and let $\mathcal{M}_0 \subset \mathcal{M}$ be a relatively compact open set. Then there is an $a > 0$ such that, for all $h \in (0, a)$ there is an open set $\mathcal{V}_h \subseteq TM$ such that*

1. *the tuple $(\mathcal{V}_h, \partial_h^+, \partial_h^-)$ is a discrete tangent bundle of \mathcal{M}_0 ;*
2. *∂_h^\pm is a diffeomorphism from \mathcal{V}_h to an open neighborhood of $\Delta(\mathcal{M}_0 \times \mathcal{M}_0)$.*

Moreover, for all $v_m \in T_m\mathcal{M}_0$, there is a sufficiently small h and \mathcal{V}_h such that $v_m \in \mathcal{V}_h$.

Given a discretization of a tangent bundle, one can obtain, by choosing h small enough, a discrete tangent bundle which provides discrete analogues of arbitrarily large tangent vectors. In physical contexts, this implies that arbitrarily high velocities can be accommodated in the discrete systems by using sufficiently small time steps.

Proof of Proposition 1. In the generic manifold context the construct $m^+ - m^-$ of Equation (5) is unavailable, but it can be replaced by the fibers of a tubular neighborhood of the diagonal $\Delta(\mathcal{M} \times \mathcal{M})$. The vector bundle $E \equiv \{(v_m, -v_m) : v_m \in TM\}$ is a normal bundle to the diagonal $\Delta(\mathcal{M} \times \mathcal{M})$ of $\mathcal{M} \times \mathcal{M}$, so there is a tubular neighborhood $\zeta: W^E \subset E \rightarrow W^{\mathcal{M} \times \mathcal{M}} \subset \mathcal{M} \times \mathcal{M}$.

The diffeomorphism ζ may be chosen so that $T\zeta$ is the identity on the zero section $0(E)$ with respect to the horizontal-vertical decomposition i.e. for all $w \in T_{0(m,m)}E$,

$$T\zeta(w) = \text{hor } w + \text{vert } w. \quad (7)$$

Any $(v_m^+, v_m^-) \in T_{(m,m)}(\mathcal{M} \times \mathcal{M})$ may be decomposed as

$$(v_m^+, v_m^-) = \left(\frac{1}{2}(v_m^+ + v_m^-), \frac{1}{2}(v_m^+ + v_m^-) \right) + \left(\frac{1}{2}(v_m^+ - v_m^-), \frac{1}{2}(v_m^- - v_m^+) \right).$$

If $T\zeta^{-1}(v_m^+, v_m^-) = w$ then this is the unique decomposition of $T\zeta(w)$ according to the direct sum $T_{(m,m)}(\mathcal{M} \times \mathcal{M}) \oplus E_{(m,m)}$. By Equation (7), $\text{hor } w + \text{vert } w$ is also this decomposition of $T\zeta(w)$, so

$$\text{vert } T\zeta^{-1}(v_m^+, v_m^-) = \left(\frac{1}{2}(v_m^+ - v_m^-), \frac{1}{2}(v_m^- - v_m^+) \right). \quad (8)$$

Consider the map $\varphi : \{(h, v_m) : \partial_h^\pm(v_m) \in W^{\mathcal{M} \times \mathcal{M}}\} \rightarrow \mathbb{R} \times E$ by

$$\varphi(h, v_m) \equiv \begin{cases} \left(h, \frac{1}{h} \zeta^{-1} \partial_h^\pm(v_m)\right), & h > 0, \\ \left(0, \left(\frac{v_m}{2}, -\frac{v_m}{2}\right)\right), & h = 0. \end{cases}$$

Using Equations (3), and (8),

$$\text{vert } \frac{d}{dh} \Big|_{h=0} \zeta^{-1} \partial_h^\pm(v_m) = \left(\frac{v_m}{2}, -\frac{v_m}{2}\right).$$

so φ is smooth by [8], Proposition 1. φ is a local diffeomorphism at any $(0, v_m)$ since the derivative of φ is nonsingular there, and φ is a diffeomorphism from $\{0\} \times \mathcal{T}\mathcal{M}$ to $\{0\} \times E$, so φ is a diffeomorphism from some open neighborhood of $\{0\} \times \mathcal{T}\mathcal{M}$ to some open neighborhood of $\{0\} \times E$ ([8], Theorem 1). The domain of the map

$$\tilde{\varphi}(h, (v_m, -v_m)) \equiv (h, \zeta(hv_m, -hv_m))$$

includes $\{0\} \times E$, $\tilde{\varphi}$ is a diffeomorphism except at $h = 0$, and

$$(h, \partial^\pm(v_m)) = \tilde{\varphi} \varphi(h, v_m).$$

Thus there are open neighborhoods $U \supseteq \{0\} \times \mathcal{T}\mathcal{M}$ and $W \supseteq \{0\} \times \Delta(\mathcal{M} \times \mathcal{M})$ such that $(h, v_m) \mapsto (h, \partial_h^\pm(v_m))$ is a diffeomorphism from $U \setminus (\{0\} \times \mathcal{T}\mathcal{M})$ to $W \setminus (\{0\} \times (\mathcal{M} \times \mathcal{M}))$.

Given a relatively compact open $\mathcal{M}_0 \subseteq \mathcal{M}$, choose $a > 0$ such that $[0, a) \times \Delta(\mathcal{M}_0 \times \mathcal{M}_0) \subseteq W$. Assume $0 < h < a$ and define

$$\mathcal{V}_h \equiv \{v_m : (h, v_m) \in U\} \cap (\partial_h^+)^{-1}(\mathcal{M}_0) \cap (\partial_h^-)^{-1}(\mathcal{M}_0).$$

∂_h^\pm is a diffeomorphism on \mathcal{V}_h since $(h, v_m) \mapsto (h, \partial_h^\pm(v_m))$ is a diffeomorphism. Also, $(h, (m_0, m_0)) \in W$ for any $m_0 \in \mathcal{M}_0$, so $\delta_h^+(v_m) = m_0$ and $\delta_h^-(v_m) = m_0$ where v_m is defined by $\tilde{\varphi}(h, v_m) = (h, (m_0, m_0))$, hence ∂_h^+ and ∂_h^- are onto \mathcal{M}_0 . By continuity, given any $v_m \in \mathcal{T}\mathcal{M}_0$, h can be chosen so small that $\partial_h^+(v_m) \in \mathcal{M}_0$ and $\partial_h^-(v_m) \in \mathcal{M}_0$, and so small that $(h, v_m) \in U$. Thus h can be chosen so small that $v_m \in \mathcal{V}_h$. \square

As an aside, we get the following Corollary, which, given a second order vector field, is obtained by applying the proof of Proposition 1 (particularly the construction of $\tilde{\varphi}$) to the X discretization with bias $\alpha^-(h) = 0$, $\alpha^+(h) = h$.

Corollary 2. *Let X be a C^k second order vector field on $\mathcal{T}\mathcal{M}$, $k \geq 2$. Then there are open neighborhoods $U \supseteq \{0\} \times \mathcal{T}\mathcal{M}$ and $W \supseteq \{0\} \times \Delta(\mathcal{M} \times \mathcal{M})$ such that $(t, v) \mapsto (t, \tau_{\mathcal{M}} F_t^X(v), \tau_{\mathcal{M}}(v))$ is a C^{k-1} diffeomorphism from $U \setminus \{0\} \times \mathcal{T}\mathcal{M}$ to $W \setminus \{0\} \times \Delta(\mathcal{M} \times \mathcal{M})$.*

Corollary 2 is important when constructing classical generating functions of type 1 for Lagrangian systems $L : \mathcal{Q} \rightarrow \mathbb{R}$. These are functions on $\mathcal{Q} \times \mathcal{Q}$ which are defined as integrals of the Lagrangian L over solutions with specified endpoints i.e. the classical action as a function of endpoints. To well define the generating function using the flow of the Lagrangian vector field, one should construct the map $\Delta_t(q_2, q_1)$ from $\mathcal{Q} \times \mathcal{Q}$ to $\mathcal{T}\mathcal{Q}$ that returns the initial velocity at q_1 that evolves to q_2 over time interval t . The generating function is then

$$S_t(q_2, q_1) \equiv \int_0^t F_s^{X_E}(\Delta_s(q_2, q_1)) ds$$

where X_E is the Euler-Lagrange vector field and F^{X_E} is its flow. The map $\Delta(q_2, q_1)$ cannot be straightforwardly constructed using the implicit function theorem (as is attempted for example in [2]) because of a singularity at $t = 0$: infinite velocity is required to traverse from q_1 to q_2 in zero time. But the map $\Delta(q_2, q_1)$ is easily extracted from Corollary 2. The MV discrete ‘exact’ Lagrangian [16] is the same as the the type 1 generating function and both suffer the same singularity.

In the context of $Q \equiv \mathbb{R}^N$ with $\psi(h, t, v_x) \equiv x + tv$ and bias $\alpha^-(h) \equiv 0, \alpha^+(h) \equiv h$, one has

$$(\partial_h^\pm)^{-1}(m^-, m^+) = (m^-, (m^+ - m^-)/h).$$

Thus the inverse of ∂_h^\pm may be given the interpretation of a difference quotient. This is used in Definition 3 to define the discrete derivative of a sequence in \mathcal{M} . That is important because it gives the definition of a discrete first order sequence in $T\mathcal{M}$, which is crucial to the discrete variation principle LdA'_d of Section 3.2.

Definition 3.

1. If m_k is a sequence in \mathcal{M} , then a discrete derivative of m_k is a sequence $m'_k \in \mathcal{V}$ such that $\partial^\pm(m'_k) = (m_{k+1}, m_k)$.
2. A sequence $v_k \in \mathcal{V}$ is called first order if $v_k = m'_k$ for some sequence $m_k \in \mathcal{M}$.

A sequence m_k is first order if and only if

$$\partial^+(v_k) = \partial^-(v_{k+1}), \tag{9}$$

because the derivative of every sequence m_k satisfies Equation (9), and every sequence v_k satisfying Equation (9) is the derivative of

$$m_0 \equiv \partial^-(v_0), \quad m_1 \equiv \partial^-(v_1), \dots, m_k \equiv \partial^-(v_k), \dots, m_{N-1} \equiv \partial^-(v_{N-1}), \quad m_N \equiv \partial^+(v_{N-1}).$$

By Remark 2, or Proposition 1 if the maps ∂^+, ∂^- arise from a discretization, the discrete derivative of m_k is unique as long as the pairs (m_{k+1}, m_k) lie sufficiently close to the diagonal and the sequence values v_k are restricted to be sufficiently near the zero section.

Let $(\mathcal{V}, \partial^+, \partial^-)$ be a discrete tangent bundle of \mathcal{M} . Define the *backward vertical bundle* by

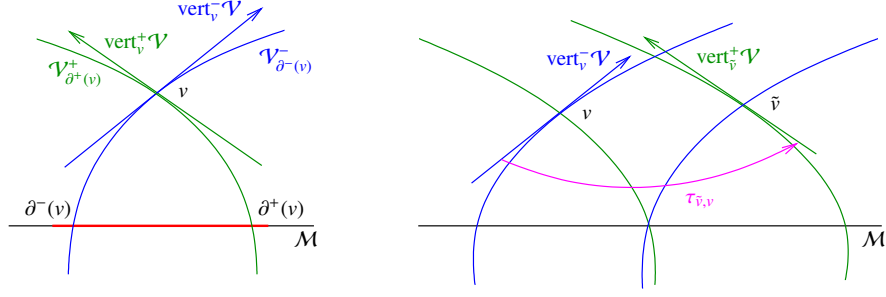
$$\text{vert}_v^+ \mathcal{V} \equiv \ker T_v \partial^+, \quad \text{vert}^+ \mathcal{V} \equiv \ker T \partial^+,$$

and the *forward vertical bundle* by

$$\text{vert}_v^- \mathcal{V} \equiv \ker T_v \partial^-, \quad \text{vert}^- \mathcal{V} \equiv \ker T \partial^-.$$

The fibers of the forward [backward] vertical bundles are the tangent spaces to the forward [backward] fibers of the discrete tangent bundle. Item 1 of Definition 2 gives $T\mathcal{V} = \text{vert}^+ \mathcal{V} \oplus \text{vert}^- \mathcal{V}$ so that every $\delta v \in T_v \mathcal{V}$ decomposes uniquely as $\delta v = \delta v^+ + \delta v^-$ where $\delta v^+ \in \text{vert}_v^- \mathcal{V}$ and $\delta v^- \in \text{vert}_v^+ \mathcal{V}$. The signs may appear notationally reversed but they are mnemonic in the sense the one wants more often to apply $T\partial^+$ not to elements of $\text{vert}_v^- \mathcal{V}$, which would result in zero, but rather to elements of $\text{vert}_v^+ \mathcal{V}$. The convention $\delta v^+ \in \text{vert}_v^- \mathcal{V}$ means that, usually, ‘+’ goes with ‘+’ to make something nonzero, while zero results when ‘+’ goes with ‘-’.

Remark 5. Lagrangian discretizations are towards constructing discrete Lagrangian models which have a stature with continuous Lagrangian models. However, not every construct that is well defined in the context of continuous models is also well defined in the context of discrete models. In the diagram above, at left, any point of \mathcal{M} near the area straddling $\partial^-(v)$ and $\partial^+(v)$ could be considered the base point of the discrete tangent vector v . The finite — as opposed



to infinitesimal — nature of the discrete tangent vectors precludes an unambiguous association of configurations to elements of the discrete phase spaces. In the discrete context, the association of configurations to points of velocity phase space is artificial — like the invocation of a metric or connection where none is really natural. This, of course, is somewhat unintuitive after such concentration on the continuous systems. The reflex to associate a unique configuration to a velocity has to be unlearned.

Lemma 3. *For all $v \in \mathcal{V}$, $T_v \partial^-$ [resp. $T_v \partial^+$] is a linear isomorphism from $\text{vert}_v^- \mathcal{V}$ [resp. $\text{vert}_v^+ \mathcal{V}$] to $T_{\partial^-(v)} \mathcal{M}$ [resp. $T_{\partial^+(v)} \mathcal{M}$].*

Proof. If $\delta v \in \text{vert}_v^+ \mathcal{V}$ and $T_v \partial^-(\delta v) = 0$ then $\delta v \in \ker T \partial^+ \cap T \partial^-$, so $\delta v = 0$ by Definition 2. Thus $T_v \partial^-$ has trivial kernel on $\text{vert}_v^+ \mathcal{V}$ and the result follows because $\dim \text{vert}_v^+ \mathcal{V} = \dim \mathcal{M} = \dim T_m \mathcal{M}$. \square

In particular, if $\delta m \in T_m \mathcal{M}$ and $\partial^+(v) = m$ [resp. $\partial^-(\tilde{v}) = m$], then there is a unique $\delta v \in \text{vert}_v^+ \mathcal{V}$ [resp. $\delta \tilde{v} \in \text{vert}_{\tilde{v}}^+ \mathcal{V}$] such that $T \partial^+(\delta v) = \delta m$ [resp. $T \partial^-(\delta \tilde{v}) = \delta m$]. If v is understood, then we will write $\delta v = \delta m^+$ [resp. $\delta v = \delta m^-$] i.e. δm^+ and δm^- satisfy

$$T \partial^-(\delta m^+) = 0, \quad T \partial^+(\delta m^+) = \delta m, \quad T \partial^-(\delta m^-) = \delta m, \quad T \partial^+(\delta m^-) = 0.$$

Combining, if $v, \tilde{v} \in \mathcal{V}$ are such that $\partial^+(v) = m = \partial^-(\tilde{v})$, then there are unique vectors $\delta v \in \text{vert}_v^+ \mathcal{V}$ and $\delta \tilde{v} \in \text{vert}_{\tilde{v}}^+ \mathcal{V}$ such that

$$T \partial^+(\delta v) = \delta m = T \partial^-(\delta \tilde{v}).$$

This provides a linear isomorphism $\tau_{\tilde{v}, v}: \text{vert}_v^+ \mathcal{V} \rightarrow \text{vert}_{\tilde{v}}^+ \mathcal{V}$ as follows: if $\delta v \in \text{vert}_v^+ \mathcal{V}$, then $\tau_{\tilde{v}, v}(\delta v) \equiv \delta \tilde{v}$ is the unique vector in $\text{vert}_{\tilde{v}}^+ \mathcal{V}$ such that $T_{\tilde{v}} \partial^-(\delta \tilde{v}) = T_v \partial^+(\delta v)$.

Suppose that $(\mathcal{V}, \partial^+, \partial^-)$ is a discrete tangent bundle of \mathcal{M} and θ is a smooth one form on \mathcal{V} . Let $v \in \mathcal{V}$, and set $m \equiv \partial^-(v)$ and $\tilde{m} \equiv \partial^+(v)$. We will have use of two bilinear forms on $T_m \mathcal{M} \times T_{\tilde{m}} \mathcal{M}$, denoted $d^{\pm} \theta(v)$ and $d^{\pm} \theta(v)$, and defined as follows. Given $\delta m \in T_m \mathcal{M}$ and $\delta \tilde{m} \in T_{\tilde{m}} \mathcal{M}$, choose (local) vector fields X and \tilde{X} with values in $\text{vert}^+ \mathcal{V}$ and $\text{vert}^- \mathcal{V}$ respectively, such that

$$T \partial^-(X(m)) = \delta m, \quad T \partial^- X(x) = T \partial^- X(y) \quad \text{if } \partial^-(x) = \partial^-(y),$$

and also this with $\delta \tilde{m}$, \tilde{X} , and ‘+’ instead of δm , X , and ‘-’, respectively. Such vector fields X and \tilde{X} commute, because

$$T \partial^- [\tilde{X}, X](m) = \frac{d}{dt} \Big|_{t=0} T \partial^- (F_t^{\tilde{X}})^* X(m) = \frac{d}{dt} \Big|_{t=0} T \partial^- X(F_t^{\tilde{X}}(m)) = 0.$$

Define

$$\mathbf{d}^\mp\theta(v)(\delta m, \delta\tilde{m}) \equiv (X(\iota_{\tilde{X}}\theta))(v), \quad \mathbf{d}^\pm\theta(v)(\delta m, \delta\tilde{m}) \equiv (\tilde{X}(\iota_X\theta))(v). \quad (10)$$

This well defines $\mathbf{d}^\mp\theta(v)$: if X' and \tilde{X}' are other choices of such vector fields, then, using the identity $\iota_{[V,W]}\alpha = L_V\iota_W\alpha - \iota_W L_V\alpha$,

$$(X(\iota_{\tilde{X}}\theta))(v) = L_{X'}\iota_{\tilde{X}}\theta(v) = \iota_{\tilde{X}}L_{X'}\theta(v) = \iota_{\tilde{X}'}L_{X'}\theta(v) = X'(\iota_{\tilde{X}'}\theta)(v),$$

and similarly $\mathbf{d}^\pm\theta(v)$ is well defined. Also, note that

$$\mathbf{d}\theta(X(v), \tilde{X}(v)) = X(\iota_{\tilde{X}}\theta)(v) - \tilde{X}(\iota_X\theta)(v) - \theta([X, \tilde{X}])(v) = \mathbf{d}^\mp\theta(v)(\delta m, \delta\tilde{m}) - \mathbf{d}^\pm\theta(v)(\delta m, \delta\tilde{m}),$$

so $\mathbf{d}^\mp\theta = \mathbf{d}^\pm\theta$ if and only if θ is closed.

3 Lagrange–d’Alembert principle

3.1 Continuous Lagrange–d’Alembert principle

Let Q be a manifold of system configurations, and $\mathcal{D} \subseteq TQ$ be a submanifold such that $\tau_Q|_{\mathcal{D}}$ is a submersion. To a given Lagrangian $L: TQ \rightarrow \mathbb{R}$, the corresponding *action functional* assigns to curves $q(t) \in Q$ the number

$$S \equiv \int_a^b L(q'(t)) dt. \quad (11)$$

The *variational derivative* of S is

$$\mathbf{d}S(q(t))\delta q(t) \equiv \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} S(q_\epsilon(t)) = 0,$$

where $\delta q(t)$ is a curve in TQ , and $q_\epsilon(t)$ satisfies

$$q_\epsilon(t) \Big|_{\epsilon=0} = q(t), \quad \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} q_\epsilon(t) = \delta q(t).$$

Let \mathcal{E} be a subbundle of the pull-back bundle $(\tau_Q|_{\mathcal{D}})^*(TQ)$ i.e. a smooth assignment of subspaces of T_qQ to each $v_q \in \mathcal{D}$. By definition, the curve $q(t)$ is an *evolution* if it satisfies $\text{LdA}(L, \mathcal{D}, \mathcal{E})$, defined as:

$$\left\{ \begin{array}{l} \text{LdA-1 (given constraint): } q'(t) \in \mathcal{D}. \\ \text{LdA-2 (criticality): } \mathbf{d}S(q(t))\delta q(t) = 0 \text{ for all } \delta q \text{ which satisfy} \\ \quad \text{LdA-2a (given constraint forces): } \delta q(t) \in \mathcal{E}_{q'(t)}; \\ \quad \text{LdA-2b (fixed boundary): } \delta q(a) = 0 \text{ and } \delta q(b) = 0. \end{array} \right.$$

This is the general version, where the annihilator of \mathcal{E}_{v_q} is the vector space of the constraint forces at state v_q , and where \mathcal{D} is a nonlinear constraint on velocities.

Since τ_Q is assumed to be a submersion on \mathcal{D} , $\ker(T(\tau_Q|_{\mathcal{D}}))$ is a subbundle of $\text{vert } T\mathcal{D}$ with fiber dimension $\dim \mathcal{D} - \dim Q$, and

$$\dot{\mathcal{D}}_{v_q} \equiv \{w_q \in TQ : \text{vert}_{v_q} w_q \in T\mathcal{D}\}, \quad (12)$$

is a subbundle of $(\tau_Q \mathcal{D})^*(TQ)$ with the same fiber dimension. One possibility for \mathcal{E} is *Chetaev's rule* $\mathcal{E} \equiv \dot{\mathcal{D}}$, but other choices may be appropriate, as discussed for example in [14]. Chetaev's rule specializes to the usual case of *linear constraints* if \mathcal{D} is a distribution on TQ , because then $\dot{\mathcal{D}}_{v_q} = \mathcal{D}_q$.

The Lagrange–d'Alembert principle above is written for curves $q(t) \in Q$. We now transform it a variational principle for curves $v(t) \in TQ$, by placing $q(t)$ and $v(t)$ in one-to-one correspondence using $q(t) = \tau_Q v(t)$ and $v(t) = q'(t)$. The transformed variational principle has the additional constraint (the first order constraint) $v(t) = (\tau_Q v(t))'$ on curves $v(t) \in TQ$. Substituting $q'(t) = v(t)$ into Equation (11) transforms the action to

$$S \equiv \int_a^b L(v(t)) dt,$$

and we extend S to all curves $v(t)$ by this same formula. The *variational derivative* of S is

$$dS(v(t)) \delta v(t) \equiv \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} S(v_\epsilon(t)),$$

where $\delta v(t)$ is a curve in TTQ , and $v_\epsilon(t)$ satisfies

$$v_\epsilon(t) \Big|_{\epsilon=0} = v(t), \quad \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} v_\epsilon(t) = \delta v(t).$$

The variation $v_\epsilon(t)$ implies a variation $q_\epsilon(t) = \tau_Q v_\epsilon(t)$; differentiating this in ϵ gives $\delta q(t) \equiv T\tau_Q \delta v(t)$, and thus the constraint $\delta q(t) \in \mathcal{E}_{v(t)}$. The first order constraint gives $v_\epsilon(t) = (\tau_Q v_\epsilon(t))'$ and differentiating in ϵ gives the constraint on $\delta v(t)$ corresponding to the first order constraint:

$$\delta v(t) = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} v_\epsilon(t) = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \frac{\partial}{\partial t} \tau_Q v_\epsilon(t) = s_Q \frac{d}{dt} T\tau_Q \delta v(t) = s_Q \delta q(t)',$$

where s_Q is the canonical involution on TTQ . Thus, the Lagrange–d'Alembert principle transforms to $LdA'(L, \mathcal{D}, \mathcal{E})$:

$$\left\{ \begin{array}{l} \text{LdA}'\text{-1 (constraints):} \\ \quad \text{LdA}'\text{-1a (given constraint): } v(t) \in \mathcal{D}; \\ \quad \text{LdA}'\text{-1b (first order constraint): } v(t) = (\tau_Q v(t))'. \\ \text{LdA}'\text{-2 (criticality): } dS(v(t)) \delta v(t) = 0 \text{ for all } \delta v \text{ which satisfy} \\ \quad \text{LdA}'\text{-2a (given constraint forces): } \delta q(t) \in \mathcal{E}_{v(t)}, \text{ where } \delta q \equiv T\tau_Q \delta v(t); \\ \quad \text{LdA}'\text{-2b (fixed boundary): } \delta q(a) = 0 \text{ and } \delta q(b) = 0; \\ \quad \text{LdA}'\text{-2c (first order constraint): } \delta v = s_Q \delta q'. \end{array} \right.$$

This transformed principle LdA' has some technical advantages and is better suited to construct discretizations where the discrete states are elements of TQ . It has been used in [18], where the higher dimension of TQ as opposed to Q provides some required freedom in a desingularization of continuous Lagrangian systems at time interval zero.

Remark 6. Implementing the first order constraint as a Lagrange multiplier gives

$$S \equiv \int_a^b L(v(t)) + \langle p(t), q'(t) - v(t) \rangle dt \tag{13}$$

where naturally $p(t) \in T^*Q$. This is the *Hamilton-Pontryagin principle* [26, 27]. The HP principle variationally identifies the Legendre transform as the Lagrange multiplier of the first order constraint i.e. it implies the constraint $p = FL$. See Section 3.3 for a few more comments on discretizations of the HP principle.

3.2 Discrete Lagrange–d’Alembert principle

We will develop a variational principle for sequences of points in a discrete tangent bundle \mathcal{V} , analogously with the continuous LdA’ principle of Section 3.1, which is a variational principle for curves with values in TQ .

Let Q be a configuration manifold and $(\mathcal{V}, \partial^+, \partial^-)$ a discrete tangent bundle on Q . The underlying structure for the discrete variational principle, which we call the *Lagrange–d’Alembert principle* is as follows:

- Given a (discrete) Lagrangian $L_d: \mathcal{V} \rightarrow \mathbb{R}$, the corresponding *discrete action functional* assigns to sequences $v_d(k) \in \mathcal{V}, k = 1, \dots, N$, the number

$$S_{d,N}(v_d) \equiv \sum_{k=1}^N L_d(v_d(k)).$$

We have reserved the subscript d to distinguish the discrete and continuous contexts. The derivative of S_d is

$$dS_{d,N}(v_d) \delta v_d \equiv \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} S_{d,N}(v_{d,\epsilon})$$

where $\delta v_d \in T\mathcal{V}[1, N]$, and $v_{d,\epsilon} \in \mathcal{V}[1, N]$ satisfies

$$v_{d,0}(k) = v_d(k), \quad \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} v_{d,\epsilon}(k) = \delta v_d(k).$$

- The discrete velocity constraint is provided by a submanifold \mathcal{D}_d of \mathcal{V} , such that $\partial^+|_{\mathcal{D}_d}$ and $\partial^-|_{\mathcal{D}_d}$ are submersions.
- In the continuous context, the constraint forces are determined by an association \mathcal{E} of subspaces of TQ to velocities in the continuous constraint \mathcal{D} . For the discrete context we assume a subbundle \mathcal{E}_d of the pullback bundle $(\partial^+|_{\mathcal{D}_d})^*(TQ)$ i.e. \mathcal{E}_d is a smooth assignment of subspaces of $T_{\partial^+(v)}Q$ to $v \in \mathcal{D}_d$.
- A higher order of accuracy of discretizations of (continuous) nonholonomically constrained Lagrangian systems requires discrete analogues that do not fit the pattern just described. To accommodate this, we generalize and replace dL_d with a one form σ_d on $T\mathcal{V}$ and replace the derivative of the action dS_N with $\Sigma_{d,N}$ where

$$\Sigma_{d,N}(v_d) \delta v_d \equiv \sum_{k=1}^N \sigma_d(v_d(k)) \delta v_d(k).$$

The net effect is that the discrete analogue σ_d of the derivative of the Lagrangian is not necessarily closed, and it contributes to the discrete analogue of the derivative of the action, which also is not necessarily closed.

A *discrete constrained Lagrangian system (DCLS)* is a tuple $(\mathcal{V}, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ as above, where $\mathcal{V} \equiv (\mathcal{V}, \partial^+, \partial^-)$ is a discrete tangent bundle. If the constraint is absent, then the tuple (\mathcal{V}, σ_d) is simply a *discrete Lagrangian system (DLS)*. By definition, a sequence $v(k)$ is an *evolution* if it satisfies

$$\left\{ \begin{array}{l} \text{LdA}'_d-1 \text{ (constraints):} \\ \quad \text{LdA}'_d-1a \text{ (given constraint): } v(k) \in \mathcal{D}_d; \\ \quad \text{LdA}'_d-1b \text{ (first order constraint): } \partial^+(v_d(k)) = \partial^-(v_d(k+1)). \\ \text{LdA}'_d-2 \text{ (criticality): } \Sigma_{d,N}(v_d(k)) \delta v(k) = 0 \text{ for all } \delta v(k) \text{ which satisfy} \\ \quad \text{LdA}'_d-2a \text{ (given constraint forces): } T\partial^+(\delta v_d(k)) \in (\mathcal{E}_d)_{v_d(k)}; \\ \quad \text{LdA}'_d-2b \text{ (fixed boundary): } T\partial^-(\delta v(1)) = 0 \text{ and } T\partial^+(\delta v(N)) = 0; \\ \quad \text{LdA}'_d-2c \text{ (first order constraint): } T\partial^+(\delta v_d(k)) = T\partial^-(\delta v_d(k+1)). \end{array} \right.$$

This is the *discrete Lagrange–d’Alembert principle* $\text{LdA}'_d(\mathcal{V}, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$.

Definition 4.

$$\begin{aligned} C_{d,N} &\equiv \{w \in \mathcal{V}[1, N] : \partial^+(w(k)) = \partial^-(w(k+1))\}; \\ \mathcal{N}_{d,N} &\equiv \{w \in C_{d,N} : w(k) \in \mathcal{D}_d\}; \\ \mathcal{W}_{d,N} &\equiv \{\delta w \in \mathbf{T}_w C_{d,N} : \mathbf{T}\partial^-(\delta w(1)) = 0, \mathbf{T}\partial^+(\delta w(N)) = 0, \mathbf{T}\partial^+(\delta w(k)) \in (\mathcal{E}_d)_w\}. \end{aligned}$$

Altogether, a sequence v_d satisfies LdA'_d if and only if

1. v_d lies in the manifold $C_{d,N}$; and
2. v_d satisfies the constraint $v_d \in \mathcal{N}_{d,N}$; and
3. v_d is *critical*, meaning $\Sigma_{d,N}(v_d) \delta v_d = 0$ for all $\delta v_d \in \mathcal{W}_{d,N}$.

The restriction to the first order submanifold $C_{d,N}$ is implemented first. In part this is because the distribution $\mathcal{W}_{d,N}$ has no natural extension away from $C_{d,N}$, since it is the *common value* of the backward and forward projected variations that have to be in \mathcal{E} .

Remark 7. LdA'_d is potentially a skew critical problem, in that one seeks points in a constraint — $\mathcal{N}_{d,N}$ — where a one form — $\Sigma_{d,N}$ — annihilates a distribution — $\mathcal{W}_{d,N}$ — but that distribution is not necessarily the tangent bundle of the constraint. See [8] and Section 7.

As shown in Theorem 4 below, the discrete Lagrange–d’Alembert principle for sequences of arbitrary length is equivalent to the same principle for consecutive pairs of the sequence. This is critical to the construction of integrators, because it reduces an optimization on length N sequences to an iteration on length $N = 2$ sequences. Thus, the $N = 2$ case occurs often, and it is helpful to abbreviate its notations.

Definition 5.

$$\begin{aligned} \Sigma_d(v, \tilde{v}) &\equiv \sigma_d(v) + \sigma_d(\tilde{v}), \\ C_d &\equiv \{(v, \tilde{v}) \in \mathcal{V} \times \mathcal{V} : \partial^+(v) = \partial^-(\tilde{v})\}, \\ \mathcal{N}_d &\equiv \{(v, \tilde{v}) \in C_d : v, \tilde{v} \in \mathcal{D}_d\}, \\ \mathcal{W}_d &\equiv \{(\delta v, \delta \tilde{v}) \in \mathbf{T}C_d : \mathbf{T}\partial^-(\delta v) = 0, \mathbf{T}\partial^+(\delta \tilde{v}) = 0, \mathbf{T}\partial^+(\delta v) \in (\mathcal{E}_d)_v\}. \end{aligned}$$

i.e. the atomic $N = 2$ case is abbreviated by dropping the N subscript. (v, \tilde{v}) is a solution pair if it satisfies LdA' for $N = 2$.

Theorem 4. *Let $(\mathcal{V}, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ be a DCLS. Then a sequence $v_d(k)$ is a discrete evolution if and only if each pair $(v, \tilde{v}) = (v_d(k), v_d(k+1))$ is a discrete evolution, $1 \leq k \leq N-1$.*

Proof. Obviously, $v_d \in \mathcal{N}_{d,N}$ if and only if every pair $(v_d(k), v_d(k+1)) \in \mathcal{N}_d$. So it is only necessary to show that, for $v_d \in \mathcal{N}_{d,N}$, $\Sigma_{d,N}(v_d)$ annihilates every $\delta v_d \in (\mathcal{W}_{d,N})_{v_d}$ if and only if $\Sigma_d(v(k), v(k+1))$ annihilates every $(\delta v, \delta \tilde{v}) \in (\mathcal{W}_d)_{(v(k), v(k+1))}$, $1 \leq k \leq N-1$.

A sequence $\delta v_d \in \mathcal{W}_{d,N}$ has $\delta v_d(1)^- = 0$ and $\delta v_d(N)^+ = 0$ by definition, and so is the sum of the sequences which are the rows of the following array:

$$\begin{array}{ccccccc} \delta v_d(1)^+ & \delta v_d(2)^- & 0 & 0 & \cdots & 0 & 0 \\ 0 & \delta v_d(2)^+ & \delta v_d(3)^- & 0 & \cdots & 0 & 0 \\ 0 & 0 & \delta v_d(3)^+ & \delta v_d(4)^- & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \delta v_d(N-1)^+ & \delta v_d(N)^- \end{array}$$

As is easily verified, this corresponds to the direct sum decomposition

$$\mathcal{W}_{d,N} = \bigoplus_{k=1}^{N-1} (\mathcal{W}_d)_{(v(k),v(k+1))} \quad (14)$$

where elements of the subspaces $(\mathcal{W}_d)_{(v(k),v(k+1))}$ are understood to be, after appropriate padding with zeros, sequences of length N . Thus $\Sigma_{d,N}$ vanishes on $\mathcal{W}_{d,N}$ if and only if $\Sigma_{d,N}$ vanishes on each factor $(\mathcal{W}_d)_{(v(k),v(k+1))}$. But, for $\delta v_d \in \mathcal{W}_{d,N}$,

$$\begin{aligned} \Sigma_{d,N}(v_d) \delta v_d &= \sum_{k=1}^N \sigma_d(v_d(k)) \delta v_d(k) \\ &= \sum_{k=1}^N \sigma_d(v_d(k)) (\delta v_d(k)^- + \delta v_d(k)^+) \\ &= \sigma_d(v_d(1)) \delta v_d(1)^- + \sigma_d(v_d(N)) \delta v_d(N)^+ + \sum_{k=1}^{N-1} (\sigma_d(v_d(k+1)) \delta v_d(k+1)^- + \sigma_d(v_d(k)) \delta v_d(k)^+) \\ &= \sum_{k=1}^N \Sigma_d(v_d(k), v_d(k+1)) (\delta v_d(k)^+, \delta v_d(k+1)^-), \end{aligned}$$

i.e. with respect to the decomposition (14), $\Sigma_{d,N} = \bigoplus_{k=1}^{N-1} \Sigma_d$. \square

Remark 8. Many fundamental physical systems have continuous variational formulations with a fixed boundary constraint, and with action defined as an integral of a local Lagrangian. The solutions of such variational formulations have the essential property of *localization*: restrictions of solutions are solutions. This follows directly from the variational principle. Indeed, the action of a solution is a sum of the action over the restriction of a solution and the complement of that, and a fixed boundary variation of such a restriction is a variation of the whole. So the restriction is critical under such variations, because under them the whole is critical and the action is constant on the complement of the restriction. The proof of Theorem 4, which is also purely variational, shows that the discrete skew critical problem, for arbitrarily long sequences, is equivalent to successive skew critical problems, for sequences of length 2. This is because the discrete action is a sum over σ_d , and because of the fixed boundary constraint. Thus the discrete systems have localization to the discretization scales for the same reasons that the continuous systems have localization to arbitrary scales.

3.3 The discrete Hamilton-Pontryagin principle

One approach to discretizations of Lagrangian systems is through discretizations of the Hamilton-Pontryagin principle (13) as in [6, 13]. The Hamilton-Pontryagin principle does not immediately discretize in the formalism of this article, because

1. it requires the difference $q'(t) - v(t)$, but discrete tangent bundles do not support linear operations; and
2. it requires $q(t)$ from $v(t)$, whereas there is no unique projection to configurations from discrete tangent bundles.

To recover the HP principle in our context, one might posit additional constructs sufficient to intrinsically write the principle itself. For example, a discrete analogue of the difference $q'(t) - v(t)$ could be constructed using an appropriate

submersion $\Delta: \mathcal{V} \times \mathcal{V} \rightarrow TQ$. We choose not to pursue this here, but rather note that one can apply Lagrange multipliers to the variations, *after* differentiating the action and *after* imposing the second order constraint in phase space. That is, the discrete HP principle is obtained by removing LdA'_d-2c and replacing LdA'_d-2 with

$$\Sigma_{d,N}(v_d(k)) \delta v(k) + \langle p_k, T\partial^-(\delta v_d(k+1)) - T\partial^+(\delta v_d(k)) \rangle = 0.$$

The difference is valid because it occurs in the single tangent fiber of TQ at $\partial^+(v_d(k)) = \partial^-(v_d(k+1))$. Reverting to $N = 2$ gives

$$\sigma_d(v) \delta v + \sigma_d(\tilde{v}) \delta \tilde{v} + \langle p, T\partial^-(\delta \tilde{v}) \rangle - \langle p, T\partial^+(\delta v) \rangle = 0,$$

or, putting separately $\delta v = \delta q^+$, $\delta \tilde{v} = 0$ and then $\delta v = 0$, $\delta \tilde{v} = \delta q^-$,

$$\langle p, \delta q \rangle = \langle \sigma_d(v), \delta q^+ \rangle, \quad \langle p, \delta q \rangle = -\langle \sigma_d(\tilde{v}), \delta q^- \rangle.$$

This identifies the discrete Legendre transforms

$$\langle F^+L(v), \delta q \rangle \equiv \langle \sigma_d(v), \delta q^+ \rangle, \quad \langle F^-L(v), \delta q \rangle \equiv -\langle \sigma_d(\tilde{v}), \delta q^- \rangle,$$

and we have the commutative diagrams

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{F^+L} & T^*Q \\ & \searrow \partial^+ & \downarrow \tau_Q^* \\ & & Q \end{array} \quad \begin{array}{ccc} \mathcal{V} & \xrightarrow{F^-L} & T^*Q \\ & \searrow \partial^- & \downarrow \tau_Q^* \\ & & Q \end{array}$$

4 Equations of motion

4.1 Continuous equations of motion

In the continuous context, localization as explained in Remark 8 lends itself to the expectation of differential equations of motion [15]. Defining the *second order submanifold*

$$\ddot{Q} \equiv \{q''(0) : q(t) \text{ a } C^2 \text{ curve in } Q\},$$

there is a unique section ∂L of $\text{hom}(TQ, T^*Q)$ and a unique section $\delta L: \ddot{Q} \rightarrow T^*Q$ of the bundle $(\tau_{TQ}|_{\ddot{Q}})^*(T^*Q)$, such that

$$dS(q(t)) \delta q(t) = \int_a^b \delta L(q''(t)) \delta q(t) dt + \partial L(q'(t)) \delta q \Big|_a^b.$$

Defining the one form $\theta_L(v_q)w_{v_q} \equiv \partial L(v_q) T\tau_Q(w_{v_q})$,

$$dS(q(t)) \delta q(t) = \int_a^b \delta L(q''(t)) \delta q(t) dt + \theta_L(q'(t)) \delta v \Big|_a^b, \quad (15)$$

where

$$\delta q(t) \equiv \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} q_\epsilon(t), \quad \delta v(t) \equiv \frac{\partial}{\partial \epsilon} \Big|_{\epsilon=0} \frac{\partial}{\partial t} \Big|_{t=0} q_\epsilon(t).$$

So the variational principle identifies δL and θ_L directly, and a curve $q^i(t)$ is an evolution if and only it satisfies

$$q'(t) \in \mathcal{D}, \quad \delta L(q'(t)) = -\lambda(t) \in \text{ann } \mathcal{E}. \quad (16)$$

These are the (continuous) Lagrange–d’Alembert equations for curves in \mathcal{Q} . In coordinates, $\ddot{Q} = \{(q^i, \dot{q}^i, \ddot{q}^i, \ddot{q}^i)\}$, and

$$\delta L = \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) dq^i = \left(\frac{\partial L}{\partial q^i} - \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \dot{q}^j - \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \ddot{q}^j \right) dq^i,$$

so that a curve $q^i(t)$ is an evolution if and only it satisfies the familiar

$$\frac{dq^i}{dt} \in \mathcal{D}, \quad \frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = -\lambda_i(t) \in \text{ann } \mathcal{E}.$$

All this corresponds to the principle LdA i.e. for curves on \mathcal{Q} . To cast it to the form of LdA’ i.e. for curves on $T\mathcal{Q}$, one must assume that $v(t)$ is first order, or else the integration-by-parts inherent in Equation (15) will fail. Under that restriction the various formulae transform easily, and Equations (16) become

$$v(t) = (\tau_{\mathcal{Q}} v(t))', \quad v(t) \in \mathcal{D}, \quad \delta L(v(t)) = -\lambda(t) \in \text{ann } \mathcal{E}.$$

These are the (continuous) Lagrange–d’Alembert equations for curves in $T\mathcal{Q}$.

The first and second fiber derivatives [1] of L are the maps $FL: T\mathcal{Q} \rightarrow T^*\mathcal{Q}$ and $F^2L: T\mathcal{Q} \rightarrow T_0^2\mathcal{Q}$ defined by

$$FL(v_q)w_q \equiv \mathbf{D}(L|T_q\mathcal{Q})(v_q)w_q, \quad F^2L(v_q)(w_q, \tilde{w}_q) \equiv \mathbf{D}^2(L|T_q\mathcal{Q})(v_q)(w_q, \tilde{w}_q).$$

A Lagrangian L is called $(\mathcal{D}, \mathcal{E})$ -regular if, for all $v_q \in \mathcal{D}$, the condition: $w_q \in \dot{\mathcal{D}}_{v_q}$ (recall Equation (12)) and $F^2L(v_q)(w_q, \tilde{w}_q) = 0$ for all $\tilde{w}_q \in \mathcal{E}_{v_q}$ implies $w_q = 0$. If L is $(\mathcal{D}, \mathcal{E})$ -regular, then the fiber dimension of $\dot{\mathcal{D}}$ and the fiber dimension of \mathcal{E} are necessarily equal i.e. regularity implies

$$\dim \mathcal{D} - \dim \mathcal{Q} = \text{fdim } \mathcal{E}$$

or equivalently

$$2 \dim \mathcal{Q} - \dim \mathcal{D} = \dim \mathcal{Q} - \text{fdim } \mathcal{E}.$$

The number of constraints in LdA (or its equivalent LdA’) is the codimension of \mathcal{D} in $T\mathcal{Q}$, while the dimension of the space of constraint forces is the fiber dimension of the annihilator subbundle of \mathcal{E} in $(\tau_{\mathcal{Q}})^*(T\mathcal{Q})$ i.e. there are

$$2 \dim \mathcal{Q} - \dim \mathcal{D}, \quad \dim \mathcal{Q} - \text{fdim } \mathcal{E}$$

independent constraints and constraint forces, respectively. At the outset of LdA, \mathcal{D} and \mathcal{E} are hypothesized and independent, and the number of independent constraints is unrelated to the number of independent constraint forces. Given regularity, equality of these is assured, and we set

$$r \equiv \dim \mathcal{D} - \dim \mathcal{Q} = \text{fdim } \mathcal{E},$$

and also there is [19] a unique second order vector field $Y_{\delta L}$ on \mathcal{D} such that $\delta L \circ Y_{\delta L}$ annihilates \mathcal{E} . Existence and uniqueness for LdA on the phase space \mathcal{D} follows because its evolutions are the integral curves of $Y_{\delta L}$.

4.2 Discrete equations of motion

To develop discrete equations of motion, we make the following definitions:

1. $\check{Q}_d \equiv \{(v, \tilde{v}) : \partial^+(v) = \partial^-(\tilde{v})\}$, which is a submanifold of $\mathcal{V} \times \mathcal{V}$. This is set theoretically the same as C_d , however for \check{Q}_d we consider $\mathcal{V} \times \mathcal{V}$ to be a discrete tangent bundle of \mathcal{V} , whereas for C_d we consider $\mathcal{V} \times \mathcal{V}$ to be the atomic two-point evolutions in \mathcal{V} .
2. $\delta\sigma_d(v, \tilde{v}) \delta q \equiv \sigma_d(\tilde{v}) \delta q^- + \sigma_d(v) \delta q^+$, which is a section of the pullback bundle $\pi_d^* TQ$, where $\pi_d: \check{Q} \rightarrow Q$ by $\pi_d(v, \tilde{v}) = \partial^+(v) = \partial^-(\tilde{v})$.
3. $\theta_{\sigma_d}^-(v) \delta v \equiv -\sigma_d(v) \delta v^-$, $\theta_{\sigma_d}^+(v) \delta v \equiv \sigma_d(v) \delta v^+$, which are both one forms on \mathcal{V} .
4. $\omega_{\sigma_d}^- \equiv -d\theta_{\sigma_d}^-$ and $\omega_{\sigma_d}^+ \equiv d\theta_{\sigma_d}^+$.

Remark 9. $\theta_{\sigma_d}^+ - \theta_{\sigma_d}^- = \sigma_d$ so $\omega_{\sigma_d}^+ = \omega_{\sigma_d}^-$ if σ_d is closed, and in this case we write ω_{σ_d} for either.

From the proof of Theorem 4, the discrete analogue of Equation (15) is Equation (17) below. This is a critical equation for our development and so it is separated here as a theorem.

Theorem 5. *If $(\mathcal{V}, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ is a DCLS then*

$$\iota_{C_{d,N}}^* \Sigma_{d,N}(v_d) \delta v_d = \sum_{k=1}^{N-1} \delta\sigma_d(v_k, v_{k+1}) \delta q(k) + \theta_{\sigma_d}^+(v_N) \delta v(N) - \theta_{\sigma_d}^-(v_1) \delta v(1) \quad (17)$$

where $\iota_{C_{d,N}}: C_{d,N} \rightarrow \mathcal{V}[1, N]$ is the inclusion and $\delta q \equiv T\partial^+(\delta v_d(k)) = T\partial^-(\delta v_d(k+1))$.

Thus, a sequence $v(k)$ is an evolution if and only if it consists of pairs (v, \tilde{v}) which satisfy

$$\partial^+(v) = \partial^-(\tilde{v}), \quad v, \tilde{v} \in \mathcal{D}_d, \quad \delta\sigma_d(v, \tilde{v}) \in \text{ann } \mathcal{E}_d. \quad (18)$$

These are the *discrete Lagrange–d’Alembert equations*.

There is no general existence and uniqueness result for the nonlinear algebraic Equations (18). However, local existence and uniqueness can be analyzed at the level of linearizations, using the inverse function theorem: View (18) as equations for $\tilde{v} \in \mathcal{D}_d$ given fixed $v \in \mathcal{D}_d$, and denote $q \equiv \partial^+(v)$. Smoothly choose (local) vector fields $X_{\delta q}$ on Q , linearly parametrized by elements of $\delta q \in (\mathcal{E}_d)_v$, such that $X_{\delta q}(q) = \delta q$ i.e. the vector fields $X_{\delta q}$ are extensions of $\delta q \in (\mathcal{E}_d)_v$. Each $X_{\delta q}$ lifts to a vector field $X_{\delta q}^-$ taking values in $\text{vert}^+ \mathcal{V}$ and such that $T\partial^- X_{\delta q}^- = X_{\delta q}^- \partial^-$. Equations (18) may be written

$$\tilde{v} \in (\partial^- | \mathcal{D}_d)^{-1}(q), \quad \sigma_d(\tilde{v}) X_{\delta q}^-(\tilde{v}) = -\sigma_d(v) \delta q, \quad \delta q \in (\mathcal{E}_d)_v.$$

The linearization of these in \tilde{v} is the derivative with respect to \tilde{v} , in the direction $\delta\tilde{v} \in T_{\tilde{v}}((\partial^- | \mathcal{D}_d)^{-1}(q))$, of the left side of the second equation. Such $\delta\tilde{v}$ are obtained one-to-one as $\delta\tilde{q}^+$ from $\delta\tilde{q} \in T\partial^+(\text{vert}_{\tilde{v}}^- \mathcal{V} \cap T_{\tilde{v}} \mathcal{D}_d)$. So, from Equations (10), the required condition is that the bilinear form

$$(\delta q, \delta\tilde{q}) \mapsto d^\pm \sigma_d(\tilde{v})(\delta q, \delta\tilde{q}), \quad \delta q \in (\mathcal{E}_d)_v, \quad \delta\tilde{q} \in T\partial^+(\text{vert}_{\tilde{v}}^- \mathcal{V} \cap T_{\tilde{v}} \mathcal{D}_d) \quad (19)$$

is nonsingular.

In continuous Lagrangian mechanics, the term ‘regular’ refers to linear conditions that provide proper equations of motion (hyperregular is the global condition that the Legendre transform is a diffeomorphism), and there is a single notion of regular, which is equivalent to nondegeneracy of the Lagrange two-form. But linear conditions are of the

infinitesimal, and they do not migrate well to the discrete context, which is finite. The discrete tangent bundle does not intrinsically support linear operations. So it is not that surprising to find a variety of notions of regularity in the discrete context, and we collect some of these here. There are more possibilities than the below: for example, more can be generated by replacing \mathbf{d}^\pm with \mathbf{d}^\mp .

Definition 6. Let $(\mathcal{V}, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ be a DCLS.

1. σ_d is regular⁻ if, for all $(v, \tilde{v}) \in \tilde{Q}$ such that $v, \tilde{v} \in \mathcal{D}_d$, the conditions (1) $\mathbf{d}^\pm \sigma_d(\tilde{v})(\delta q, \delta \tilde{q}) = 0$ for all $\delta \tilde{q} \in (\mathcal{E}_d)_{\tilde{v}}$, and (2) $\delta q \in T\partial^-(\text{vert}_v^+ \mathcal{V} \cap T_v \mathcal{D}_d)$, imply $\delta q = 0$;
2. σ_d is regular⁺ if, for all $(v, \tilde{v}) \in \tilde{Q}$ such that $v, \tilde{v} \in \mathcal{D}_d$, the conditions (1) $\mathbf{d}^\pm \sigma_d(\tilde{v})(\delta q, \delta \tilde{q}) = 0$ for all $\delta q \in (\mathcal{E}_d)_v$, and (2) $\delta \tilde{q} \in T\partial^+(\text{vert}_{\tilde{v}}^- \mathcal{V} \cap T_{\tilde{v}} \mathcal{D}_d)$, imply $\delta \tilde{q} = 0$;
3. σ_d is regular if it is regular⁻ and regular⁺;
4. σ_d is (V, \tilde{V}) -regular, $V, \tilde{V} \subseteq \mathcal{D}_d$ open, if it is regular, and, for all $v \in V$ there is a unique $\tilde{v} \in \tilde{V}$ such that (v, \tilde{v}) satisfies the discrete Lagrange–d’Alembert Equations (18).

The fiber dimension of $T\partial^+(\text{vert}^- \mathcal{V} \cap T\mathcal{D}_d)$ is $\dim \mathcal{D}_d - \dim Q$ and so, if σ_d is regular⁺, then necessarily

$$\dim \mathcal{D}_d - \dim Q = \text{fdim } \mathcal{E}_d, \quad (20)$$

and, as in the continuous context, we denote the common value by r . Thus regularity implies dimensional equality of the constraints and constraint forces, just as in the continuous context.

A discrete Lagrangian vector field is a map $Y_{\sigma_d}: U \rightarrow \tilde{Q}$, where $U \subseteq \mathcal{V}$ is open, such that $\partial_{\mathcal{V} \times \mathcal{V}}^- Y_{\sigma_d}(v) = v$ and

$$\delta \sigma_d Y_{\sigma_d}(v) \delta q = 0 \quad \text{for all } \delta q \in (\mathcal{E}_d)_v.$$

By Theorem 4, v_d is a discrete evolution if

$$(v_d(k), v_d(k+1)) = Y_{\sigma_d}(v_d(k)),$$

which says that the discrete derivative of the sequence v_d at k is the discrete Lagrange vector field at $v_d(k)$. Discrete evolutions can be obtained from a discrete Lagrangian vector field by iterations of maps F defined by $Y_{\sigma_d}(v) = (v, F(v))$.

5 Structures of discrete Lagrangian systems

Beginning with [15], and continuing with [19], there is an effective procedure for the recognition of structure for variational theories, specifically symplecticity, momentum preservation, and the equations of motion. In summary, this procedure uses the decomposition of the action into boundary and nonboundary parts, such as Equations (15) and (17). This decomposition is pulled back by the inclusion ι which maps solutions into the domain of the action functional. In the context of a Lagrangian L and action S_t , the procedure consists of the following steps:

Momentum structure: write $\mathbf{i}_\xi(\iota^* \mathbf{d}S_t) = 0$;

Symplectic structure: write $\mathbf{d}(\iota^*(\mathbf{d}S_t)) = 0$;

Symplectic equations structure: note $L = \left. \frac{d}{dt} \right|_{t=0} \iota^* S_t$, write $\mathbf{d}L = \left. \frac{d}{dt} \right|_{t=0} \iota^* \mathbf{d}S_t$ and use $\mathbf{L}_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha$.

In this section we apply this procedure to extract the discrete structure preservation properties of a DCLS. From the remaining of Section 5 through Section 7, let $(\mathcal{V}, \partial^-, \partial^+, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ be a given DCLS with be an evolution map $F: U_F \rightarrow V_F$.

5.1 Decomposition

The evolution map F defines an insertion ι_F of U_F into solutions of the DCLS by $\iota_F(v) \equiv (v, F(v))$. Pulling back Σ_d by ι_F gives, from Equation (17),

$$\iota_F^* \Sigma_d(v) \delta v = \delta \sigma_d(v, F(v)) \mathbf{T} \partial^+(\delta v) + \theta_{\sigma_d}^+(F(v)) \mathbf{T} F(\delta v) - \theta_{\sigma_d}^-(v) \delta v$$

i.e.

$$\iota_F^* \Sigma_d = F^* \theta_{\sigma_d}^+ - \theta_{\sigma_d}^- + \alpha_F \tag{21}$$

where α_F is the one form defined by

$$\alpha_F(v) \delta v \equiv \delta \sigma_d(v, F(v)) \mathbf{T} \partial^+(\delta v). \tag{22}$$

An important fact is that

$$\alpha_F(v) \delta v = 0 \text{ for all } \delta v \in \mathbf{T}_v \mathcal{D}_d \text{ such that } \mathbf{T} \partial^+(\delta v) \in (\mathcal{E}_d)_v, \tag{23}$$

because $(v, F(v))$ is a solution pair, and because of Equations (18) and (22).

5.2 Momentum

We begin with an definition of a *symmetric DCLS*. Let \mathcal{G} be a Lie group with Lie algebra \mathfrak{g} .

Definition 7. An action of a group \mathcal{G} on the DCLS $(\mathcal{V}, \partial^-, \partial^+, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$, where $\partial^- : \mathcal{V} \rightarrow \mathcal{Q}$ and $\partial^+ : \mathcal{V} \rightarrow \mathcal{Q}$, means actions of \mathcal{G} on \mathcal{Q} and \mathcal{V} such that

1. ∂^- and ∂^+ are intertwining; and
2. $\sigma_d, \mathcal{D}_d, \mathcal{E}_d$ are invariant; and
3. $\sigma_d(v)(\xi v) = 0$ for all $\xi \in \mathfrak{g}$ and all $v \in \mathcal{D}_d$.

In a symmetric DCLS, the derived constructs $\mathcal{C}_{d,N}, \mathcal{N}_{d,N}, \mathcal{W}_{d,N}, \Sigma_d$ are all invariant under diagonal actions of \mathcal{G} , and the symmetry group preserves the solutions. For the remainder Section 5.2, we posit the action of a group as in Definition 7.

Equivariance of evolutions requires uniqueness and hence has to rely on regularity. However, infinitesimal equivariance can be recovered with only an infinitesimal flavor of regularity.

Theorem 6.

1. If σ_d is regular⁺ then $\mathbf{T}F(\xi v) = F(\xi v)$ for all v in the domain of F and all $\xi \in \mathfrak{g}$.
2. Suppose σ_d is (U_F, V_F) -regular⁺. Then $\mathbf{T}F(gv) = gF(v)$ for all $v \in U_F$ and $g \in \mathcal{G}$ such that $gv \in V_F$.

Proof. From discussion following Equation (18), σ_d regular⁺ implies local existence and uniqueness of the evolutions via the inverse function theorem. Thus, for small t , \mathcal{G} invariance of the evolutions implies $F(\exp(\xi t)v) = \exp(\xi t)\tilde{v}$ and the first statement is obtained by differentiation at $t = 0$. The second statement follows from the global existence and uniqueness of (U_F, V_F) -regularity. \square

Generally, momentum is defined by insertion of the infinitesimal generator into the analogue of the Lagrange one form. There are two momenta since the DCLS context includes two Lagrange one forms $\theta_{\sigma_d}^+$ and $\theta_{\sigma_d}^-$.

Definition 8. *The momentum maps are the two functions $J^+ : \mathcal{V} \rightarrow \mathfrak{g}^*$ and $J^- : \mathcal{V} \rightarrow \mathfrak{g}^*$ defined by*

$$J_{\xi}^-(v) \equiv \langle J^-(v), \xi \rangle \equiv \theta_{\sigma_d}^-(v)(\xi v) = -\sigma_d(v)(\xi v)^-, \quad J_{\xi}^+(v) \equiv \langle J^+(v), \xi \rangle \equiv \theta_{\sigma_d}^+(v)(\xi v).$$

Group invariance provides that both the momentum maps intertwine the action on \mathcal{V} and the coadjoint action of \mathfrak{g}^* i.e. the momenta are CoAd-equivariant. Furthermore, if $v \in \mathcal{D}_d$ then $J^-(v) = J^+(v)$ and the superscript on J may be dropped, because

$$0 = \langle \sigma_d(v), \xi v \rangle = \langle \sigma_d(v), (\xi v)^- \rangle + \langle \sigma_d(v), (\xi v)^+ \rangle = -J_{\xi}^-(v) + J_{\xi}^+(v).$$

The momentum conservation structure for a DCLS is as follows:

Theorem 7. *If σ_d is regular⁺, $v \in \mathcal{D}_d$, and F is an evolution map, then $J_{\xi}(F(v)) = J_{\xi}(v)$ for any $\xi \in \mathfrak{g}$ such that $\xi \partial^+(v) \in (\mathcal{E})_v$.*

Remark 10. The momenta J_{ξ} are of course not generally conserved in continuous nonholonomic mechanics. Theorem 7 does not imply conservation of arbitrary momenta for a DCLS, because if $v_d \in \mathcal{V}[1, N]$ is an evolution then it is not usually possible to arrange the condition $\xi \partial^+(v_d(k)) \in (\mathcal{E})_{v_d(k)}$ for constant ξ independent of k . Rather, one will have a map $\mathcal{V} \ni v \rightarrow \xi^{\mathcal{V}}(v)$ which satisfies $\xi^{\mathcal{V}}(v) \partial^+(v) \in (\mathcal{E})_v$, and then

$$J_{\xi^{\mathcal{V}}(F(v))}(F(v)) - J_{\xi^{\mathcal{V}}(v)}(v) = J_{\xi^{\mathcal{V}}(F(v))}(v) - J_{\xi^{\mathcal{V}}(v)}(v) = \langle J(v), \xi^{\mathcal{V}}(F(v)) - \xi^{\mathcal{V}}(v) \rangle,$$

which is called the discrete nonholonomic momentum equation [7].

Proof of Theorem 7. Insertion of the infinitesimal generator $\xi_{\mathcal{D}_d}$ into Equation (21) gives, because σ_d annihilates infinitesimal generators,

$$0 = \langle \theta_{\sigma_d}^+(F(v)), \xi F(v) \rangle - \langle \theta_{\sigma_d}^-(v), \xi v \rangle = J_{\xi}^+(F(v)) - J_{\xi}^-(v).$$

There is no contribution from α_F because of the Equation (23). □

5.3 Symplectic

In the continuous nonholonomic systems LdA with constraint \mathcal{D} and variations \mathcal{E} , the distribution

$$\mathcal{K}_{\mathcal{D}, \mathcal{E}} \equiv \{ \delta v \in \mathbf{T}_v \mathcal{D} : \mathbf{T}\tau_{\mathcal{Q}}(\delta v) \in \mathcal{E}_v \}$$

is an important object because it supports the associated nonholonomic semi-symplectic structure [3, 9, 19, 22]. For a DCLS we explore three possible discrete analogues of $\mathcal{K}_{\mathcal{D}, \mathcal{E}}$:

$$\begin{aligned} \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^- &\equiv \{ \delta v \in \mathbf{T}_v \mathcal{D}_d : \mathbf{T}\partial^-(\delta v) \in (\mathcal{E}_d)_{\partial^-(v)} \}, \\ \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^0 &\equiv \{ \delta v \in \mathbf{T}_v \mathcal{V} : \mathbf{T}\partial^-(\delta v) \in (\mathcal{E}_d)_{\partial^-(v)}, \mathbf{T}\partial^+(\delta v) \in (\mathcal{E}_d)_{\partial^+(v)} \}, \\ \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+ &\equiv \{ \delta v \in \mathbf{T}_v \mathcal{D}_d : \mathbf{T}\partial^+(\delta v) \in (\mathcal{E}_d)_{\partial^+(v)} \}. \end{aligned} \tag{24}$$

Under the dimension condition (20), which is implied by regularity, all three of these discrete analogues have the same fiber dimension as the continuous $\mathcal{K}_{\mathcal{D}, \mathcal{E}}$:

Lemma 8. *If $r \equiv \dim \mathcal{D}_d - \dim \mathcal{Q} = \text{fdim } \mathcal{E}_d$ then*

$$\text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^0 = \text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+ = \text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^- = 2r.$$

Proof. By definition, $\partial^+|\mathcal{D}_d$ is a submersion, so

$$\dim \mathcal{Q} + r - \text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+ = \dim \mathcal{Q} - \text{fdim } \mathcal{E},$$

so $\text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+ = 2r$ follows and $\text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^- = 2r$ is similar. The fibers of $(T\partial^+)^{-1}\mathcal{E}_d$ and $(T\partial^-)^{-1}\mathcal{E}_d$ are transversal subspaces of the fibers of $T\mathcal{V}$, since

$$\text{vert}^+ \mathcal{V} = \ker T\partial^+ \subseteq (T\partial^+)^{-1}\mathcal{E}_d, \quad \text{vert}^- \mathcal{V} = \ker T\partial^- \subseteq (T\partial^-)^{-1}\mathcal{E}_d,$$

and $\text{vert}^+ \mathcal{V} \oplus \text{vert}^- \mathcal{V} = T\mathcal{V}$. The codimension of the intersection of transversal subspaces is the sum of the codimensions, so

$$\text{codim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^0 = \text{codim}(T\partial^+)^{-1}\mathcal{E}_d + \text{codim}(T\partial^-)^{-1}\mathcal{E}_d = 2(\dim \mathcal{Q} - r)$$

and the result follows because

$$\text{codim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^0 = \dim \mathcal{V} - \text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^0 = 2 \dim \mathcal{Q} - \text{fdim } \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^0.$$

□

The distribution $\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+$ is the kernel of the canonical vector bundle mapping $\nu: T\mathcal{V} \rightarrow T\mathcal{V}/\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+$ and admits [19] a curvature two form $\Delta_{\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+}$ on $T\mathcal{V}$ such that, for all vector fields $X, Y \in \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+$,

$$\Delta_{\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+}(X, Y) \equiv -\nu[X, Y].$$

Clearly $\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+$ is involutive if and only if $\Delta_{\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+} = 0$.

Theorem 9. *Let \mathcal{K}_d^0 be a subbundle of $\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+$ over $\mathcal{D}_d^0 \subseteq \mathcal{D}_d$. Suppose that*

1. \mathcal{K}_d^0 is TF invariant;
2. $\Delta_{\mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+} = 0$ on \mathcal{K}_d^0 ;
3. $d\sigma_d = 0$ on \mathcal{K}_d^0 .

Then F preserves $\omega_{\sigma_d} = \omega_{\sigma_d}^+ = \omega_{\sigma_d}^-$ on \mathcal{K}_d^0 .

Proof. If $\delta v, \delta w \in (\mathcal{K}_d^0)_v$, then, remembering the inclusion $\iota_F(v) \equiv (v, F(v))$,

$$d(\iota_F^* \Sigma_d)(\delta v, \delta w) = \iota_F^* d\Sigma_d(\delta v, \delta w) = d\sigma_d(v)(\delta v, \delta w) + d\sigma_d(F(v))(TF(\delta v), TF(\delta w)) = 0,$$

because of Items 1 and 3. In the same way, $\sigma_d = \theta_{\sigma_d}^+ - \theta_{\sigma_d}^-$ and $d\sigma_d = 0$ on \mathcal{K}_d^0 imply $\omega_{\sigma_d}^+ = -d\theta_{\sigma_d}^+ = d\theta_{\sigma_d}^- = \omega_{\sigma_d}^-$ on \mathcal{K}_d^0 . Thus, from the exterior derivative of Equation (21),

$$F^* \omega_{\sigma_d}^+ = d\omega_{\sigma_d}^- + d\alpha_F,$$

on \mathcal{K}_d^0 , so it is sufficient to show $d\alpha_F = 0$ on \mathcal{K}_d^0 . Extending δv and δw to vector fields $V \in \mathcal{K}_d^0$ and $W \in \mathcal{K}_d^0$,

$$d\alpha_F(\delta v, \delta w) = V(\alpha_F(W))(v) - W(\alpha_F(V))(v) - \alpha_F([V, W])(v),$$

and the result follows from Equation (23) because Item 2 implies $[V, W](v) \in \mathcal{K}_{\mathcal{D}_d, \mathcal{E}_d}^+$. □

6 Discrete linear and holonomic constraints

The continuous constrained Lagrangian systems LdA commonly have linear constraints, where the constraint is a distribution on Q , and the same distribution also provides the variations i.e. the special case where \mathcal{D} is a distribution and $\mathcal{E} \equiv (\tau_Q \mathcal{D})^* \mathcal{D}$ is the usual one. In this section we construct discrete analogues of this special case.

Recall [23] that if $m, \tilde{m} \in \mathcal{M}$, and \mathcal{F} is a distribution on \mathcal{M} , then \tilde{m} is \mathcal{F} -reachable from m if there is a piecewise smooth curve $c: [a, b] \rightarrow \mathcal{M}$ such that $c'(t) \in \mathcal{F}$ and $c(a) = m$, $c(b) = \tilde{m}$. \mathcal{F} -reachability is an equivalence relation on \mathcal{M} and the equivalence classes are called the *orbits* of \mathcal{F} .

Definition 9. The DCLS $(\mathcal{V}, \partial^+, \partial^-, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ has linear constraints if there is a distribution \mathcal{D}_d^Q on Q such that

1. $(\mathcal{E}_d)_v = (\mathcal{D}_d^Q)_{\partial^+(v)}$ for all $v \in \mathcal{D}_d$; and
2. $\partial^-(v)$ and $\partial^+(v)$ are \mathcal{D}_d^Q -reachable for all $v \in \mathcal{D}_d$; and
3. $\dim \mathcal{D}_d = \dim \mathcal{D}_d^Q$.

$(\mathcal{V}, \partial^+, \partial^-, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ is holonomic if \mathcal{D}_d^Q is involutive and $d\sigma_d = 0$.

To compare this definition with the continuous context, the first condition corresponds to using the distribution \mathcal{D}^Q of Q for the variations \mathcal{E} . The second condition is fulfilled, for example, in the case where the tangent bundle \mathcal{V} arises from a discretization which curve segments are integral curves of \mathcal{D}_d^Q . Such curves can be regarded as discrete analogues of the elements of \mathcal{D}_d^Q , so the second condition corresponds to equating \mathcal{D}_d^Q with the velocity constraint. Thus a DCLS with linear constraints is a discrete analogue of a (continuous) LdA with linear constraints, in that a single distribution on configuration space generates both the velocity constraint and the variations.

Remark 11. In the context of Definition 9 we will set $r \equiv \text{fdim } \mathcal{E}_d = \text{fdim } \mathcal{D}_d^Q$. In particular, $\dim \mathcal{D}_d = \dim Q + r$.

For the remainder of Section 6, let the DCLS $(\mathcal{V}, \partial^+, \partial^-, L_d, \sigma_d, \mathcal{E}_d)$ have linear constraint distribution \mathcal{D}_d^Q . If \mathcal{D}_d^Q is integrable, then the condition that $\partial^-(v)$ and $\partial^+(v)$ are \mathcal{D}_d^Q -reachable is strong, because it confines these to be in the same r -dimensional leaf of \mathcal{D}_d^Q . Lemma 10 is a first step in this line of reasoning.

Lemma 10. Suppose \mathcal{D}_d^Q is involutive and let $q \in Q$. Then ∂^+ and ∂^- are local diffeomorphisms from (respectively) $\mathcal{V}_q^- \cap \mathcal{D}_d$ and $\mathcal{V}_q^+ \cap \mathcal{D}_d$ to the leaf of \mathcal{D}_d^Q through q .

Proof. Let \mathcal{L}_q be the leaf of \mathcal{D}_d^Q containing $q \in Q$. By Item 2 of Definition 9, ∂^+ immerses the r -dimensional submanifold $\mathcal{V}_q^- \cap \mathcal{D}_d$ into \mathcal{L}_q , which also has dimension r . Thus ∂^+ is a local diffeomorphism from $\mathcal{V}_q^- \cap \mathcal{D}_d$ into \mathcal{L}_q . Similarly, ∂^- a local diffeomorphism from $\mathcal{V}_q^+ \cap \mathcal{D}_d$ into \mathcal{L}_q . \square

Lemma 11. If \mathcal{D}_d^Q is involutive then

1. σ_d is regular⁻ if and only if for all $v \in \mathcal{D}_d$, the conditions (1) $d^\pm \sigma_d(v)(\delta q, \delta \tilde{q}) = 0$ for all $\delta \tilde{q} \in (\mathcal{D}_d^Q)_{\partial^+(v)}$, and (2) $\delta q \in (\mathcal{D}_d^Q)_{\partial^-(v)}$, imply $\delta q = 0$.
2. σ_d is regular⁺ if and only if for all $v \in \mathcal{D}_d$, the conditions (1) $d^\pm \sigma_d(v)(\delta q, \delta \tilde{q}) = 0$ for all $\delta q \in (\mathcal{D}_d^Q)_{\partial^-(v)}$, and (2) $\delta \tilde{q} \in (\mathcal{D}_d^Q)_{\partial^+(v)}$, imply $\delta \tilde{q} = 0$.

Proof. If $v, \tilde{v} \in \mathcal{D}_d$ and $\partial^+(v) = \partial^-(\tilde{v})$, then $(\mathcal{E}_d)_v = (\mathcal{D}_d^Q)_{\partial^-(\tilde{v})}$. Lemma 11 gives $T\partial^+(\text{vert}_{\tilde{v}}^- \mathcal{V} \cap T_{\tilde{v}} \mathcal{D}_d) = (\mathcal{D}_d^Q)_{\partial^-(\tilde{v})}$, and the result for regular⁺ is obtained by transcribing Definition 6; regular⁻ is similar. \square

A skew critical problem in the meaning of Remark 7 is *ordinary [resp. variational]* if the tangent bundle of the constraint is the equal to [resp. contains] the distribution used to differentiate the objective. Ordinary critical problems correspond to the standard constrained optimization problem that seeks critical points of an objective subject to a constraint. Continuous systems with linear constraints are variational exactly if the constraint distribution \mathcal{D} is integrable i.e. exactly if the system is holonomic in the usual meaning of the term [19]. In the discrete context there is Theorem 12 below.

Theorem 12. *The following are equivalent:*

1. \mathcal{D}_d^Q is involutive;
2. $\mathcal{W}_{d,N}$ is involutive;
3. $\mathcal{W}_{d,N} = \{\delta v_d \in \mathbf{TN}_{d,N} : \delta q^-(1) = 0, \delta q^+(N) = 0\}$.

Proof. Assume $N = 2$; the proof for arbitrary N is similar.

(1) \Rightarrow (3). Let $(v, \tilde{v}) \in \mathcal{N}_d$, and define

$$q^- \equiv \partial^-(v), \quad q \equiv \partial^+(v) = \partial^-(\tilde{v}), \quad \tilde{q}^+ \equiv \partial^+(\tilde{v}),$$

which are all in the same leaf of \mathcal{D}_d^Q . Temporarily define

$$(\dot{\mathcal{N}}_d)_{(v,\tilde{v})} \equiv \{(\delta v, \delta \tilde{v}) \in \mathbf{T}_{(v,\tilde{v})}(\mathcal{V} \times \mathcal{V}) : \mathbf{T}\partial^-(\delta v) = 0, \mathbf{T}\partial^+(\delta \tilde{v}) = 0, \mathbf{T}\partial^+(\delta v) = \mathbf{T}\partial^-(\delta \tilde{v}), \delta v \in \mathbf{T}\mathcal{D}_d, \delta \tilde{v} \in \mathbf{T}\mathcal{D}_d\}.$$

corresponding to the set on the right side of the equality in Item 3. Recall that

$$(\mathcal{W}_d)_{(v,\tilde{v})} \equiv \{(\delta v, \delta \tilde{v}) \in \mathbf{T}_{(v,\tilde{v})}\mathcal{C}_d : \mathbf{T}\partial^-(\delta v) = 0, \mathbf{T}\partial^+(\delta \tilde{v}) = 0, \mathbf{T}\partial^+(\delta v) = \mathbf{T}\partial^-(\delta \tilde{v}) \in \mathcal{D}_d^Q\}.$$

It is required to show that $\mathcal{W}_{(v,\tilde{v})} = \dot{\mathcal{N}}_{(v,\tilde{v})}$ i.e. the condition $\delta v, \delta \tilde{v} \in \mathbf{T}\mathcal{D}_d$ in the definition of $(\mathcal{W}_d)_{(v,\tilde{v})}$ amounts to the same thing as the condition $\mathbf{T}\partial^+(\delta v) = \mathbf{T}\partial^-(\delta \tilde{v}) \in \mathcal{D}_d^Q$ in the definition of $\dot{\mathcal{N}}_{(v,\tilde{v})}$. If $(\delta v, \delta \tilde{v}) \in \dot{\mathcal{N}}_{(v,\tilde{v})}$ then δv is tangent to $\mathcal{V}_q^- \cap \mathcal{D}_d$ and $\mathbf{T}_v\partial^+$ maps the tangent space of this at v (isomorphically) to $(\mathcal{D}_d^Q)_q$. Thus $\mathbf{T}\partial^+(\delta v) \in \mathcal{D}_d^Q$. On the other hand, if $(\delta v, \delta \tilde{v}) \in (\mathcal{W}_d)_{(v,\tilde{v})}$, find the unique $\delta v' \in \text{vert}_v^-\mathcal{V} \cap \mathcal{D}_d$ and $\delta \tilde{v}' \in \text{vert}_{\tilde{v}}^+\mathcal{V} \cap \mathcal{D}_d$ such that $\mathbf{T}\partial^+(\delta v') = \mathbf{T}\partial^-(\delta \tilde{v}')$. This implies $\delta v = \delta v'$ since $\mathbf{T}_v\partial^+$ is a local diffeomorphism from $\text{vert}_v^-\mathcal{V}$ to $\mathbf{T}_q\mathcal{Q}$. Thus $\delta v \in \mathbf{T}_v\mathcal{D}_d$ since $\delta v'$ is. Similarly, $\delta \tilde{v} \in \mathbf{T}_{\tilde{v}}\mathcal{D}_d$, so $(\delta v, \delta \tilde{v}) \in \dot{\mathcal{N}}_{(v,\tilde{v})}$.

(2) \Rightarrow (1). Let X and Y be vector fields with values in \mathcal{D}_d^Q and let $q \in \mathcal{Q}$. Arrange $v, \tilde{v} \in \mathcal{N}_d$ so that $\partial^+(v) = \partial^-(\tilde{v}) = q$. Define $\pi_d : \mathcal{N}_d \rightarrow \mathcal{Q}$ by $\pi_d(v, \tilde{v}) \equiv \partial^+(v) = \partial^-(\tilde{v})$. X and Y have unique lifts \tilde{X} and \tilde{Y} which are vector fields on \mathcal{N}_d with values in \mathcal{W}_d such that $\mathbf{T}\pi_d \tilde{X} = X \circ \pi_d$ and $\mathbf{T}\pi_d \tilde{Y} = Y \circ \pi_d$. Then $[X, Y] \circ \pi_d = \mathbf{T}\pi_d [\tilde{X}, \tilde{Y}]$ and by assumption $[\tilde{X}, \tilde{Y}]$ has values in \mathcal{W}_d , so

$$[X, Y](q) = \mathbf{T}\pi_d [\tilde{X}, \tilde{Y}](v, \tilde{v}) \in \mathcal{D}_d^Q,$$

as required.

(3) \Rightarrow (2). By assumption, \mathcal{W}_d is integrable because it is equal to the kernel of the derivative of the map from \mathcal{N}_d to $\mathcal{Q} \times \mathcal{Q}$ defined by $v_d \mapsto (\partial^-(v), \partial^+(\tilde{v}))$, and any such kernel is involutive. \square

There are the following specializations of the distributions (24) to the context with linear constraints:

$$\begin{aligned} \mathcal{K}_{\mathcal{D}_d}^- &\equiv \mathbf{T}\mathcal{D}_d \cap (\mathbf{T}\partial^-)^{-1}\mathcal{D}_d^Q, \\ \mathcal{K}_{\mathcal{D}_d}^0 &\equiv ((\mathbf{T}\partial^-)^{-1}\mathcal{D}_d^Q \cap (\mathbf{T}\partial^+)^{-1}\mathcal{D}_d^Q)|_{\mathcal{D}_d}, \\ \mathcal{K}_{\mathcal{D}_d}^+ &\equiv \mathbf{T}\mathcal{D}_d \cap (\mathbf{T}\partial^+)^{-1}\mathcal{D}_d^Q. \end{aligned}$$

The fiber dimensions of these are all $2r$ because the dimension condition of Lemma 8 is satisfied, as explained in Remark 11. Lemma 13 below shows simplifications if \mathcal{D}_d^Q is involutive, and it provides a discrete analogue of the continuous fact that \mathcal{K}_D is integrable if and only if \mathcal{D} is integrable.

Lemma 13.

1. The following are equivalent: (1A) \mathcal{D}_d^Q is involutive, (1B) $\mathcal{K}_{\mathcal{D}_d}^-$ is involutive, (1C) $\mathcal{K}_{\mathcal{D}_d}^0$ is involutive, (1D) $\mathcal{K}_{\mathcal{D}_d}^+$ is involutive.

2. The following are equivalent: (2A) $\mathcal{K}_{\mathcal{D}_d}^0 \subseteq T\mathcal{D}_d$, (2B) $\mathcal{K}_{\mathcal{D}_d}^0 = \mathcal{K}_{\mathcal{D}_d}^-$, (2C) $\mathcal{K}_{\mathcal{D}_d}^0 = \mathcal{K}_{\mathcal{D}_d}^+$.

Moreover, the statements in (1) imply the statements in (2).

Proof.

(1A) \Rightarrow (1B), (1A) \Rightarrow (1C), (1A) \Rightarrow (1D): pull backs and intersections of involutive distributions are involutive.

(1A) \Leftarrow (1B), (1A) \Leftarrow (1C), (1A) \Leftarrow (1D): If $\mathcal{K}_{\mathcal{D}_d}^0$ is involutive and X, Y are vector fields on Q with $X, Y \in \mathcal{D}_d^Q$, then $X^+, Y^+ \in \mathcal{K}_{\mathcal{D}_d}^0$, hence $[X^+, Y^+] \circ \partial^+ = T\partial^+[X^+, Y^+] \in \mathcal{D}_d^Q$. Thus \mathcal{D}_d^Q is involutive. (1A) \Leftarrow (1B) and (1A) \Leftarrow (1D) are similar after using Lemma 10 to lift X and Y .

(2A) \Leftrightarrow (2B) and (2A) \Leftrightarrow (2C): If $\mathcal{K}_{\mathcal{D}_d}^0 \subseteq T\mathcal{D}_d$ then $\mathcal{K}_{\mathcal{D}_d}^0 \subseteq T\mathcal{D}_d \cap (T\partial^-)^{-1}\mathcal{D}_d^Q = \mathcal{K}_{\mathcal{D}_d}^-$ and hence $\mathcal{K}_{\mathcal{D}_d}^0 = \mathcal{K}_{\mathcal{D}_d}^-$ by Lemma 8. If $\mathcal{K}_{\mathcal{D}_d}^0 = \mathcal{K}_{\mathcal{D}_d}^-$ then $\mathcal{K}_{\mathcal{D}_d}^0 \subseteq \mathcal{K}_{\mathcal{D}_d}^- \subseteq T\mathcal{D}_d$. Similarly (2A) \Leftrightarrow (2C).

(1A) \Rightarrow (2A): Suppose \mathcal{D}_d^Q is involutive and $\delta v \in \mathcal{K}_{\mathcal{D}_d}^0$. Then $T\partial^+(\delta v^+) = T\partial^+(\delta v) \in \mathcal{D}_d^Q$ and $\delta v^+ \in \text{vert}^- \mathcal{V}$. By Lemma 10, there is a $\delta v' \in T(\mathcal{V}_q^- \cap \mathcal{D}_d)$ such that $T\partial^+(\delta v') = T\partial^+(\delta v)$. Also $T\partial^-(\delta v) = T\partial^-(\delta v') = 0$, so $\delta v = \delta v' \in T\mathcal{D}_d$. \square

In a (continuous) nonholonomic system with linear constraints, the Lagrange two form is nonsingular on the distribution \mathcal{K}_D if and only if the Lagrangian is regular. For a DCLS, there is the following similar result in the holonomic case.

Theorem 14. *If $(\mathcal{V}, \partial^+, \partial^-, L_d, \sigma_d, \mathcal{E}_d)$ is holonomic then σ_d is regular $^\pm$ if and only if ω_{σ_d} is nondegenerate on $\mathcal{K}_{\mathcal{D}_d}^0$.*

Proof. Let $v \in \mathcal{D}_d$ and $\delta v, \delta w \in \mathcal{K}_{\mathcal{D}_d}^0$. Then $T\partial^-(\delta v) \in \mathcal{D}_d^Q$, so there is a vector field $V \in \mathcal{D}_d^Q$ such that $T\partial^-(\delta v) = V(\partial^-(v))$. Similarly choose vector fields \tilde{V}, W , and \tilde{W} such that

$$T\partial^+(\delta v) = \tilde{V}(\partial^+(v)), \quad T\partial^-(\delta w) = W(\partial^-(w)), \quad T\partial^+(\delta w) = \tilde{W}(\partial^+(w)).$$

Then $\delta v = \tilde{V}^+(v) + V^-(v)$, $\delta w = \tilde{W}^+(w) + W^-(w)$. Also,

$$\begin{aligned} \omega_{\sigma_d}(\tilde{V}^+ + V^-, \tilde{W}^+ + W^-) &= -(\tilde{V}^+ + V^-)(\theta_{\sigma_d}^+(\tilde{W}^+ + W^-)) + (\tilde{W}^+ + W^-)(\theta_{\sigma_d}^+(\tilde{V}^+ + V^-)) + \theta_{\sigma_d}^+([\tilde{V}^+ + V^-, \tilde{W}^+ + W^-]) \\ &= -(\tilde{V}^+ + V^-)(\sigma_d(\tilde{W}^+ + W^-)^+) + (\tilde{W}^+ + W^-)(\sigma_d(\tilde{V}^+ + V^-)^+) + \sigma_d([\tilde{V}^+ + V^-, \tilde{W}^+ + W^-]^+) \\ &= -V^-(\sigma_d(\tilde{W}^+)) + (W^-)(\sigma_d(\tilde{V}^+)) - d\sigma_d(\tilde{V}^+, \tilde{W}^+) \\ &= d^\mp \sigma_d(W, \tilde{V}) - d^\mp \sigma_d(V, \tilde{W}) \end{aligned}$$

so

$$\omega_{\sigma_d}(\delta v, \delta w) = d^\pm \sigma_d(v)(T\partial^-(\delta w), T\partial^+(\delta v)) - d^\pm \sigma_d(v)(T\partial^-(\delta v), T\partial^+(\delta w)).$$

Suppose σ_d is regular $^\pm$ and that $\omega_{\sigma_d}(\delta v, \delta w) = 0$ for all $\delta w \in (\mathcal{K}_d^0)_v$. Choosing $\delta w \in \text{vert}^- \mathcal{V}$ gives

$$d^\pm \sigma_d(v)(T\partial^-(\delta v), \delta \tilde{q}) = 0$$

for all $\delta \tilde{q} \in (\mathcal{D}_d^Q)_{\partial^+(v)}$. Thus $T\partial^-(\delta v) = 0$ as σ_d is regular $^-$. Similarly $T\partial^+(\delta v) = 0$ since σ_d is regular $^+$. Thus $\delta v = 0$, proving that ω_{σ_d} is nondegenerate. The converse — that σ_d is regular $^\pm$ if ω_d is nondegenerate on $\mathcal{K}_{\mathcal{D}_d}^0$, follows by reversing this argument. \square

In the holonomic case, symplecticity and preservation of momentum are expected and they are recovered in the following two corollaries.

Corollary 15. *If $F: U_F \subseteq \mathcal{V} \rightarrow V_F \subseteq \mathcal{V}$ is an evolution of the holonomic DCLS $(\mathcal{V}, \partial^+, \partial^-, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$, then F is symplectic on $\mathcal{K}_{\mathcal{D}_d}^0$ i.e. $\mathcal{K}_{\mathcal{D}_d}^0$ is TF invariant and $\omega_d(v)(TF(\delta v), TF(\delta w)) = \omega_d(v)(\delta v, \delta w)$ for all $\delta v, \delta w \in (\mathcal{K}_{\mathcal{D}_d}^0)_v$, $v \in \mathcal{D}_d$.*

Proof. It suffices to verify the hypotheses of Theorem 9 with $\mathcal{K}_d^0 = \mathcal{K}_{\mathcal{D}_d}^0$. If $v \in \mathcal{D}_d$ and $\delta v \in (\mathcal{K}_{\mathcal{D}_d}^0)_v$ then $\delta v \in T\mathcal{D}_d$ and $T\partial^+(\delta v) \in (\mathcal{E}_d)_v = (\mathcal{D}_d^Q)_{\partial^+(v)}$. Since \mathcal{D}_d is F invariant, $TF(\delta v) \in T\mathcal{D}_d$. Also

$$T\partial^-(TF(\delta v)) = T(\partial^- \circ F)(\delta v) = T\partial^+(\delta v) \in \mathcal{D}_d^Q$$

so $T\partial^-(TF(\delta v)) \in \mathcal{K}_{\mathcal{D}_d}^- = \mathcal{K}_{\mathcal{D}_d}^0$. Furthermore $\Delta_{\mathcal{K}_{\mathcal{D}_d}^+} = 0$ because Lemma 13 implies $\mathcal{K}_{\mathcal{D}_d}^+$ is involutive, and $d\sigma_d = 0$ by hypothesis. \square

Corollary 16. *Suppose $F: U_F \subseteq \mathcal{V} \rightarrow V_F \subseteq \mathcal{V}$ is an evolution of the holonomic DCLS $\mathcal{V} \equiv (\mathcal{V}, \partial^+, \partial^-, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$, and suppose that a Lie group \mathcal{G} acts on \mathcal{V} and \mathcal{Q} such that*

1. ∂^+ and ∂^- are equivariant; and
2. \mathcal{D}_d^Q is \mathcal{G} invariant; and
3. $\mathfrak{g}\mathcal{Q} \subseteq \mathcal{D}_d^Q$.

Then \mathcal{V} is a symmetric DCLS and F is momentum conserving.

Proof. Invariance of \mathcal{E}_d follows from $(\mathcal{E}_d)_v \equiv (\mathcal{D}_d^Q)_{\partial^+(v)}$ and Items 1 and 2. From Item 3, the definition of $\mathcal{K}_{\mathcal{D}_d}^0$, and Lemma 13, we have $\mathfrak{g}\mathcal{D}_d \subseteq \mathcal{K}_{\mathcal{D}_d}^0 \subseteq T\mathcal{D}_d$, so \mathcal{D}_d is invariant and hence \mathcal{V} is symmetric. By Theorem 7, conservation of momentum follows from $\xi \partial^+(v) \in (\mathcal{E})_v$ for all $v \in \mathcal{D}_d$ and all $\xi \in \mathfrak{g}$, but this is implied by Item 3. \square

7 Lagrange-d'Alembert as a skew critical problem

From [8] we recover some basic definitions:

Definition 10. *Let M and N be manifolds, α be a one-form on M , \mathcal{D} be a C^k distribution on M , and let $g: M \rightarrow N$ be a submersion. We call (α, \mathcal{D}, g) a C^k skew critical problem. A point $m_c \in M$ is a skew critical point of (α, \mathcal{D}, g) at $n \in N$ if*

$$\begin{cases} \alpha(m_c)(v) = 0 \text{ for all } v \in \mathcal{D}_{m_c}, \\ g(m_c) = n. \end{cases}$$

A skew critical problem is called variational if \mathcal{D} is involutive.

Definition 11. Let m_c be a skew critical point of (α, \mathcal{D}, g) . Define the bilinear form $d_{\mathcal{D}}\alpha(m_c): T_{m_c}M \times \mathcal{D}_{m_c} \rightarrow \mathbb{R}$ by

$$d_{\mathcal{D}}\alpha(m_c)(u, v) \equiv \langle d(i_V\alpha)(m_c), u \rangle,$$

where V is a (local) vector field with values in \mathcal{D} such that $V(m_c) = v$. The skew Hessian of α with respect to g and \mathcal{D} is the bilinear form

$$d_{\mathcal{D},g}\alpha(m_c): \ker T_{m_c}g \times \mathcal{D}_{m_c} \rightarrow \mathbb{R}$$

obtained by restriction of $d_{\mathcal{D}}\alpha(m_c)$. Define $d_{\mathcal{D},g}\alpha(m_c)^b: \ker T_{m_c}g \rightarrow \mathcal{D}_{m_c}^*$ by

$$d_{\mathcal{D},g}\alpha(m_c)^b(u) \equiv d_{\mathcal{D},g}\alpha(m_c)(u, \cdot).$$

A skew critical point m_c of (α, \mathcal{D}, g) is called nondegenerate if $d_{\mathcal{D},g}\alpha(m_c)^b$ is a linear isomorphism.

We are using the term ‘skew critical problem’ in two slightly ways. First, it is any constrained critical point problem where the derivative is taken in directions that may be other than the constraint directions. Second, it is a problem within the technical meaning of Definition 10. The two are not really the same. For example, in Definition 10, the purpose of the function g is to provide a parametrization of the critical points, rather than a constraint. It is not necessarily a natural constraint e.g. its level sets do not necessarily correspond to the leaves of \mathcal{D} , in the case that \mathcal{D} is integrable.

If \mathcal{D}_d is the level set of a submersive constraint function $g_d: \mathcal{V} \rightarrow \mathcal{P}$ i.e. $\mathcal{D}_d = g_d^{-1}(p_0)$, then LdA'_d may be cast as a skew critical problem within Definition 10, in a variety of ways. One of the ideas, though, of such skew critical problems, is that their critical points should be isolated on the level sets of the constraint function, and should be smoothly parametrized by values of the constraint function. So, the constraint function of the skew critical problem representing LdA'_d should provide exactly the freedom required to fit the skew critical points to the constraint levels and to the initial conditions of the evolutions of LdA'_d . Failure of this would rule out (local) existence and uniqueness for discrete evolutions as a consequence of the natural persistence of (nondegenerate) skew critical points. The following is a formulation of LdA'_d as a skew critical problem in the atomic $N = 2$ case; the case for arbitrary N is similar.

Definition 12. The associated skew critical problem to the DCLS $(\mathcal{V}, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ is $(\Sigma_d, \mathcal{W}_d, \hat{g}_d)$, defined on C_d , and where

$$\hat{g}_d: C_d \rightarrow \mathcal{V} \times \mathcal{P}, \quad \hat{g}_d(v, \tilde{v}) \equiv (v, g_d(\tilde{v})).$$

Clearly, $\hat{g}_d^{-1}(\hat{p}_0) \subseteq \mathcal{N}_d$ for any choice $\hat{p}_0 \equiv (v, p_0)$ such that $g_d(v) = p_0$, and the solutions to $(\Sigma_d, \mathcal{W}_d, \hat{g}_d)$ as v varies over \mathcal{D}_d are exactly the solutions to LdA'_d . The justification for the particular choice of \hat{g}_d comes from the following.

Theorem 17. All solution pairs of a regular⁺ DCLS $(\mathcal{V}, \sigma_d, \mathcal{D}_d, \mathcal{E}_d)$ are nondegenerate skew critical points of the skew critical problem $(\Sigma_d, \mathcal{W}_d, \hat{g}_d)$.

Proof. Let (v_c, \tilde{v}_c) be a solution pair. It is required to prove that $T_{(v_c, \tilde{v}_c)}(\mathcal{V} \times \mathcal{V}) \ni (\delta v, \delta \tilde{v}) = 0$ if

$$\begin{cases} (\delta v, \delta \tilde{v}) \in \ker d\hat{g}_d, \text{ and} \\ d_{\mathcal{W}_d, \hat{g}_d} \Sigma_d((\delta v, \delta \tilde{v}), (\delta w, \delta \tilde{w})) = 0 \text{ for all } (\delta w, \delta \tilde{w}) \in (\mathcal{W}_d)_{(v_c, \tilde{v}_c)}. \end{cases} \quad (25)$$

Assuming (25), choose $\delta q \in \mathcal{W}_{\partial^+(v_c)}\mathcal{Q}$, and extend δq to a (local) vector field $X_{\delta q}$ on \mathcal{Q} , as in the development just after Theorem 5. Then $X \equiv (X_{\delta q}^+, X_{\delta q}^-)$ is a vector field in \mathcal{W}_d , so $(\delta w, \delta \tilde{w}) \in (\mathcal{W}_d)_{(v_c, \tilde{v}_c)}$ if $\delta w \equiv X_{\delta q}^+(v_c)$ and $\delta \tilde{w} \equiv X_{\delta q}^-(\tilde{v}_c)$. Also,

$$i_X \Sigma_d(v, \tilde{v}) = i_{X^+} \sigma_d(v) + i_{X^-} \sigma_d(\tilde{v}). \quad (26)$$

$(\delta v, \delta \tilde{v}) \in \ker \mathbf{d}\hat{g}_d$ implies, from the definition of \hat{g}_d , that $\delta v = 0$ and $\delta \tilde{v} \in T_{\tilde{v}_c} \mathcal{D}_d$ where $\mathcal{D}_d = g_d^{-1}(p_0)$ and $p_0 \equiv g_d(v)$. Since the skew critical problem occurs in C_d , $T\partial^-(\delta \tilde{v}) = T\partial^+(\delta v) = 0$, so it is enough to show $T\partial^+(\delta \tilde{v}) = 0$. The first term of Equation (26) differentiates to zero in the direction $(\delta v, \delta \tilde{v})$, and hence

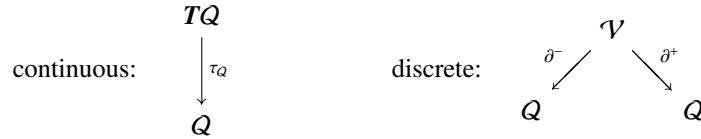
$$0 = \mathbf{d}_{\mathcal{W}_d \Sigma_d}((\delta v, \delta \tilde{v}), (\delta w, \delta \tilde{w})) = \mathbf{d}^\pm \sigma_d(\tilde{v}_c)(\delta q, \delta \tilde{q}),$$

where $\delta \tilde{q} = T\partial^+(\delta \tilde{v}) \in T\partial^+(\text{vert}_{\tilde{v}_c}^- \mathcal{V} \cap T_{\tilde{v}_c} \mathcal{D}_d)$. Then $\delta \tilde{q} = 0$, since δq is arbitrary and the DCLS is regular⁺. \square

8 Concluding remarks

Discretization is, in one view, the replacement of infinitesimal objects with finite, geometric ones, depending on what one wants to represent. In differential geometry this might mean the attachment of geometric objects to every point of a manifold. The ubiquitous linear bundles of differential geometry become bundles of geometric shapes; the linear fibers become sets on which most of the usual linear operations are absent. At $h = 0$ there is degeneration. Obtaining this limit requires desingularization, and, for coherent results along the whole of a continuous target, semiglobal analysis. For each h of a discretization, we have a *discrete analogue*, which is a simpler, abstract construct, because it has not the burden of supporting the limit $h \rightarrow 0$.

Using curve segments to discretize tangent bundles is geometrically vivid. The abstract notion of a discrete tangent bundle $(\mathcal{V}, \partial^+, \partial^-)$ (Definition 2) of a configuration space \mathcal{Q} , is an example of the clarity afforded by invariant differential geometry:



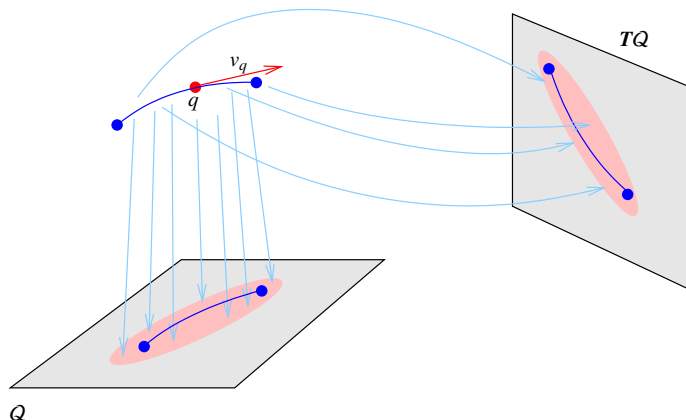
While they do not support linear operations, the discrete tangent bundles have other structures. The forward and backward discrete canonical projections result in decompositions because they split every fiber of $T\mathcal{V}$.

To work Hamilton's principle directly on velocity phase space is to view the equation $q(t)' = v$ as a constraint. The discrete analogue of this constraint for a sequence $v_i \in \mathcal{V}$ is that successive curve segments attached to the v_i join to make a continuous whole i.e. $\partial^+(v_i) = \partial^-(v_{i+1})$. That, together with the usual fixed endpoint constraints, and discrete action the sum of a discrete Lagrangian over v_i , altogether define the discrete variational principle that gives the discrete evolution. What follows that is more-or-less a straight application of the philosophy of [15, 19], which extracts structural properties directly from variational principles.

Practically, it is easier, and more direct, to generate curve segments rather than interpolate between the configurations $(q^+, q^-) \in \mathcal{Q}$. In [21] we derive, based on the discretizations of this article, numerical methods for explicitly constrained Lagrangian systems. The required curve segments may be generated using virtually any one step numerical integrator, and, automatically there follows a variational integrator of the same order. The current state of the art in geometric integrators for nonholonomic systems uses the MV discrete phase space [7, 10, 11, 17]. It is future work to address construction of and the error analysis of nonholonomic variational integration algorithms using the discretizations of this article.

The curve segments naturally shrink to points as $h \rightarrow 0$. This results in a well defined and precise approach to the limit $h \rightarrow 0$, of which Proposition 1 shows typical use. It is good to be respectful of this limit. In [20] we show that the error analysis of discrete holonomic variational integrators, which is also an issue of $h \rightarrow 0$, depends on a subtle symmetry, and has sometimes been oversimplified.

The discrete phase spaces and the continuous ones are conceptually separated; there is no innate association between discrete and continuous states, nor is there any unique configuration associated to a discrete velocity. This is an unnatural conceptual point and it has to be forcibly remembered, especially when, as is usual, the discrete phase space and the continuous one are the same, set theoretically. Since there is no innate association of continuous and discrete states, there is neither any association of discrete states with physical states. Such associations are inherently ambiguous if $h > 0$.



For example, when projecting to configuration space, there are not just the two possibilities ∂^+ and ∂^- , but also τ_Q , any other point on the curve segment, and, if $\mathcal{V} = TQ \subseteq \mathbb{R}^N$, any convex combination of ∂^+ and ∂^- . The continuous tangent vector associated to curve segments is similarly ambiguous. Without further motivation, any of these choices are as good as any others. With motivation, such choices reflect the motivation, not the presence of a preeminent choice. Suppose one has a variational integrator, and some association $TQ \rightarrow \mathcal{V}$. Conjugation by any structure preserving morphism, which is near to the identity to sufficiently high order in h , gives another association. The conjugated and original variational integrators are equivalent; the implied change in association of physical state to discrete states is not relevant. And, in any case, even though the discrete and continuous phase spaces may be the same set theoretically, the structures of the discretization do not usually have the same functional form as those of the continuous system. So what is the justification for identifying the phase spaces?

If a discretization of a structured model is not structure preserving, then it is subordinate and its states are slaved to continuous states. If such a discretization is structure preserving and has equal stature to a continuous model, then its states correspond to continuous ones only ambiguously. Any specific identification of continuous and discrete states, such as e.g. the identification of the Lagrange multiplier of the discrete first order constraint with the continuous momentum of the continuous Lagrangian, can only be admitted a status similar to a *possibly convenient special coordinate system*. Of course many such coordinates exist in geometry; but they are, at most, *important and useful intermediaries* which cannot properly be elevated to the stature of necessity or structural centrality.

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