Symplectic Tubes for Cotangent-Lifted Symmetries

Tanya Schmah

Department of Computer Science, University of Toronto,
Department of Mathematics, Macquarie University, Australia

BIRS, August 2007
Outline

1. Introduction
2. Tubes for multiplication actions
3. The case of $SO(3)$
4. Actions on general $T^* Q$
5. Outlook
The problem: constructing symplectic tubes

Given a $G$ action on a manifold $M$, a *tube* around the orbit of a point $z \in M$ is a local change of coordinates that factorizes out the “group direction” as much as possible. More precisely . . .

**Definition**

A *tube* at $z$ is a $G$-equivariant diffeomorphism

$$
\Phi : G \times_{G_z} S \longrightarrow U \subseteq M,
$$

where $G_z$ is the isotropy group of $z$, $S$ is a manifold on which $G_z$ acts, and $U$ is a $G$-invariant neighbourhood of $z$.

If $G$ acts properly, then tubes always exist. $S$ may be chosen to be a neighbourhood of 0 in some complement $N$ to $T_z (G \cdot z)$, and $\Phi$ may be assumed to map $[e,0]_{G_z}$ to $z$. 
Suppose $M$ has a symplectic form.

**Symplectic Tube Theorem**

There are always tubes that are *symplectic* with respect to a certain “natural” symplectic form on $G \times_{G_z} S$, where $S$ is a neighbourhood of $(0, 0)$ in $(g_\mu / g_z)^* \oplus N_s$, where $N_s$ is the symplectic normal space at $z$.

The local model given by this theorem is a natural starting point for studies of singular reduction and bifurcations of relative equilibria. But the proofs are not constructive.

**Aim**

Explicitly construct symplectic tubes, particularly when $M = T^*Q$ and $G$ acts by cotangent lifts.

---

1Marle ‘85, Guillemin & Sternberg ‘84, Ortega & Ratiu ‘02, Benoist ‘02
A partial result

Let $M = T^*Q$, and $J$ be the standard momentum map for a cotangent-lifted action of $G$ on $M$. Let $\mu = J(z)$, with isotropy group $G_\mu$.

Cotangent Bundle Tube Theorem $^2$

If $G_\mu = G$, then a symplectic tube can be constructed explicitly (in terms of a Riemannian exponential on $Q$).

A partial result

Let $M = T^*Q$, and $J$ be the standard momentum map for a cotangent-lifted action of $G$ on $M$. Let $\mu = J(z)$, with isotropy group $G_\mu$.

**Cotangent Bundle Tube Theorem**

If $G_\mu = G$, then a symplectic tube can be constructed explicitly (in terms of a Riemannian exponential on $Q$).

**Special case**: $Q$ and $G$ arbitrary, $\mu = 0$.

---

A partial result

Let $M = T^*Q$, and $J$ be the standard momentum map for a cotangent-lifted action of $G$ on $M$. Let $\mu = J(z)$, with isotropy group $G_\mu$.

**Cotangent Bundle Tube Theorem**

If $G_\mu = G$, then a symplectic tube can be constructed explicitly (in terms of a Riemannian exponential on $Q$).

**Special case:** $Q$ and $G$ arbitrary, $\mu = 0$.

**Non-example (not covered by theorem):**
$Q$ arbitrary, $G = SO(3)$ and $\mu \neq 0$

---

A partial result

Let $M = T^*Q$, and $J$ be the standard momentum map for a cotangent-lifted action of $G$ on $M$. Let $\mu = J(z)$, with isotropy group $G_{\mu}$.

Cotangent Bundle Tube Theorem \(^2\)

If $G_{\mu} = G$, then a symplectic tube can be constructed explicitly (in terms of a Riemannian exponential on $Q$).

Special case: $Q$ and $G$ arbitrary, $\mu = 0$.

Non-example (not covered by theorem):
$Q$ arbitrary, $G = SO(3)$ and $\mu \neq 0$

... even if the action is free

---

A partial result

Let \( M = T^*Q \), and \( J \) be the standard momentum map for a cotangent-lifted action of \( G \) on \( M \). Let \( \mu = J(z) \), with isotropy group \( G_{\mu} \).

**Cotangent Bundle Tube Theorem** \(^2\)

If \( G_{\mu} = G \), then a symplectic tube can be constructed explicitly (in terms of a Riemannian exponential on \( Q \)).

**Special case:** \( Q \) and \( G \) arbitrary, \( \mu = 0 \).

**Non-example (not covered by theorem):**
\( Q \) arbitrary, \( G = SO(3) \) and \( \mu \neq 0 \)

...even if the action is free ... and even if \( Q = SO(3) \)!

---

The specific problem for today

Let $M = T^* G$, and consider the left multiplication action of $G$ on itself, cotangent-lifted to an action on $T^* G$.

**Problem**

Construct a symplectic tube at an arbitrary point $z \in T^* G$, of the same form as in the general Symplectic Tube Theorem.

Why focus on this case?

- relatively simple – the action is free and transitive
- applicable to mechanics on Lie groups, eg. (affine-)rigid body motion;
- can be used as a bootstrap for a more general solution.
The left multiplication action of $G$ on $T^*G$

- Left-trivialise: $T^* G \cong G \times g^*$, so $J(g, \nu) = \text{Ad}_{g^{-1}}^{*} \nu$
- Without loss of generality, let $z = (e, \mu)$
- $N_s := \ker DJ(z)/(g_{\mu} \cdot z) \cong T_{\mu}O_{\mu} \cong g_{\mu}^\perp$, where $g_{\mu}^\perp$ is an arbitrary complement to $g_{\mu}$ in $g$.

Wanted

A $G$-equivariant symplectic diffeomorphism

$$
\Phi : \left(G \times g_{\mu}^* \times g_{\mu}^\perp, \Omega_Y\right) \longrightarrow \left(G \times g^*, \Omega_c\right),
$$
$$
(e, 0, 0) \longmapsto (e, \mu),
$$

where $\Omega_c$ is canonical and

$$
\Omega_Y(g, \nu, \zeta)((\xi_1, \dot{\nu}_1, \eta_1), (\xi_2, \dot{\nu}_2, \eta_2))
=: \langle \mu + \nu, [\xi_1, \xi_2] \rangle + \langle \dot{\nu}_2, \xi_1 \rangle - \langle \dot{\nu}_1, \xi_2 \rangle - \langle \mu, [\eta_1, \eta_2] \rangle.
$$
Tubes for the left multiplication action of $G$ on $T^*G$

By $G$-equivariance, $\Phi$ must be of the form

$$\Phi(g, \nu, \zeta) = (g F_1(\nu, \zeta), F_2(\nu, \zeta)).$$

A brute-force pull-back calculation yields:

**Proposition**

$\Phi^* \Omega_c = \Omega_Y$ if and only if

$$\Phi(g, \nu, \zeta) = \left( g F(\nu, \zeta), \text{Ad}^*_{F(\nu, \zeta)} (\mu + \nu) \right), \quad \text{and}$$

$$\langle \mu + \nu, \left[ F(\nu, \zeta)^{-1} (DF(\nu, \zeta) \cdot (\dot{\nu}_1, \eta_1)), F(\nu, \zeta)^{-1} (DF(\nu, \zeta) \cdot (\dot{\nu}_2, \eta_2)) \right] \rangle$$

$$+ \langle \dot{\nu}_2, F(\nu, \zeta)^{-1} (DF(\nu, \zeta) \cdot (\dot{\nu}_1, \eta_1)) \rangle$$

$$- \langle \dot{\nu}_1, F(\nu, \zeta)^{-1} (DF(\nu, \zeta) \cdot (\dot{\nu}_2, \eta_2)) \rangle$$

$$= \langle \mu, [\eta_1, \eta_2] \rangle.$$

(1)
Tubes for the left multiplication action of $G$ on $T^*G$

The general construction of a symplectic tube is still an open problem.

One interesting observation is that the first line of the previous formula is similar to:

**Lemma**

Suppose $\psi : T_\mu \mathcal{O}_\mu \to \mathcal{O}_\mu$ is of the form $\psi (- \text{ad}_{\zeta}^* \mu) = \text{Ad}_{f(\zeta)}^{-1} \mu$, for some $f : g^\perp_\mu \to G$. Then $\psi$ preserves the $\pm$KKS forms if and only if

$$\left\langle \mu, \left[ f(\zeta)^{-1} (Df(\zeta) \cdot \eta_1), f(\zeta)^{-1} (Df(\zeta) \cdot \eta_2) \right] \right\rangle = \langle \mu, [\eta_1, \eta_2] \rangle.$$

(2)

Can this be modified slightly to give a symplectic tube?
If $G = SO(3)$, then a symplectic $\psi : T_\mu O_\mu \rightarrow O_\mu$ is just an area-preserving map from a plane to a sphere!

In polar coordinates $(r, \theta)$ on the plane and spherical coordinates $(\theta, \phi)$ on the sphere, we can take $\phi = 2 \arcsin \left( \frac{r}{2} \right)$. To write $\psi$ in the form of the lemma, $\psi (- \text{ad}_{\zeta}^* \mu) = \text{Ad}^*_{f(\zeta)^{-1}} \mu$, we define

$$f(\zeta) := \exp \left( 2 \arcsin \left( \frac{\|\zeta\|}{2} \right) \cdot \frac{\zeta}{\|\zeta\|} \right).$$

Now we guess: insert a factor of $\sqrt{\frac{\|\mu\|}{\|\mu + \nu\|}}$ into the formula, to transform Equation 2 into Equation 1 . . .
A symplectic tube for $SO(3)$

Theorem

The following is an $SO(3)$-equivariant symplectic diffeomorphism (with respect to the symplectic form given earlier), in a neighbourhood of $(e, 0, 0)$:

$$\Phi : G \times g^*_\mu \times g^\perp_\mu \longrightarrow G \times g^*,$$

$$(g, \nu, \zeta) \longmapsto \left( g F(\nu, \zeta), \text{Ad}^*_F(\nu, \zeta)(\mu + \nu) \right)$$

where

$$F(\nu, \zeta) := \exp \left( 2 \arcsin \left( \frac{\|\zeta\|}{2} \sqrt{\frac{\|\mu\|}{\|\mu + \nu\|}} \right) \cdot \frac{\zeta}{\|\zeta\|} \right).$$
To prove this theorem by calculation, we need to be able to differentiate \( \exp \). For \( SO(3) \) this can be done using Rodrigues’ formula. In particular, that formula implies that, if \( \mathbf{v} \perp \mathbf{w} \), then

\[
\exp(-\hat{\mathbf{v}}) \left( D\exp(\hat{\mathbf{v}}) \cdot \hat{\mathbf{w}} \right) = \frac{\sin \|\mathbf{v}\|}{\|\mathbf{v}\|} \hat{\mathbf{w}} - \frac{1}{2} \left( \frac{\sin \frac{\|\mathbf{v}\|}{2}}{\frac{\|\mathbf{v}\|}{2}} \right)^2 (\mathbf{v} \times \mathbf{w})^\wedge
\]
Equations of motion in slice coordinates

\[ G = \text{SO}(3), \ g \cong \mathbb{R}^3. \]

Let \( z = (e, \mu) \), with \( \mu \neq 0 \), e.g. \( z \) a relative equilibrium.

The coordinates \((g, \nu, \zeta) \in G \times g^*_\mu \times g^\perp_\mu\) are called “slice coordinates”. \((e, 0, 0)\) corresponds to the original \( z \).

\[
\begin{align*}
\dot{g} &= g \frac{\partial h}{\partial \nu} \quad \text{(reconstruction equation)} \\
\dot{\nu} &= 0 \left( = \text{Pr}_{g^*_\mu} \left( \text{ad}^*_{\frac{\partial h}{\partial \nu}} \nu \right) \right) \\
\dot{\zeta} &= \mathbb{J} \frac{\partial h}{\partial \zeta}
\end{align*}
\]
In slice coordinates, the momentum level set $J^{-1}(\mu)$ is defined by $\nu = 0$ and $g \in G_\mu$, so

$$J^{-1}(\mu) = G_\mu \times \{0\} \times g^\perp_\mu \cong SO(2) \times \mathbb{R}^2,$$

with the second factor being the symplectic reduced space.

Compare this with the usual left-trivialised coordinates $(g, \nu) \in G \times g^*$. The momentum level set is

$$J^{-1}(\mu) = \{(g, \nu) : \text{Ad}_{g^{-1}}^* \nu = \mu\}.$$ For $G = SO(3)$, this is a circle bundle over a sphere, with the sphere being the symplectic reduced space.
A symplectic tube for the multiplication action of $G$ on $T^*G$ can be combined with the earlier Cotangent Bundle Slice Theorem (CBST) to give symplectic tubes for more general actions of $G$. Recall that the earlier theorem applies wherever $G_\mu = G$, i.e. at fully isotropic momentum values.

Let $M = T^*Q$ and $z \in T_q^*Q$. Let $\mu = J(z)$ (with no constraint on $\mu$). Let $A$ be a $G_q$-invariant complement to $g \cdot q$ in $T_qQ$, and let $\alpha = z|_A$. Considering $\alpha \in T_q^*qQ$, we have $J(\alpha) = 0$, so the fully-isotropic CBST can be applied there.
Simplest case: \( G_q = \{ e \} \)

\( G_z = \{ e \} \) but \( G_\mu \) is arbitrary, e.g. a Lagrangian equilibrium of the 3-body problem.

The symplectic normal space at \( \alpha \) is \( N_s(\alpha) \cong T^*B \), where \( B \) is a \( G_q \)-invariant complement to \( g \cdot \alpha \) in \( A \).

The symplectic normal space at \( z \) is \( N_s(z) \cong T_\mu O_\mu \oplus T^*B \).

Combining our results gives at symplectic tube

\[
G \times g^*_\mu \times T_\mu O_\mu \times T^*B \longrightarrow G \times g^* \times T^*B \longrightarrow T^*Q,
\]

\[
(e, 0, 0, 0) \longmapsto (e, \mu, 0, 0) \longmapsto z.
\]
The “bootstrapping” idea works whenever $G_q \subseteq G_\mu$. 
Outlook

- Application of tubes for $SO(3)$ actions:
  - Bifurcations of relative equilibria for rigid bodies and $n$-body problems
  - Attitude control of rigid body at nonzero angular momentum?
- Actions of $SE(3)$ and other classical groups
- Generalise bootstrapping to actions on arbitrary $T^*Q$
- General Lie groups?