Some results on energy shaping feedback

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13/08/2007

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The problem

• Given an open-loop mechanical control system $\Sigma_{ol} = (Q, G_{ol}, V_{ol})$:

$$\Sigma_{ol} = (Q, G_{ol}, V_{ol}, \mathcal{F}) \quad \text{feedback} \quad \Sigma_{cl} = (Q, G_{cl}, V_{cl}, F_{cl})$$

• The closed-loop system $\Sigma_{cl} = (Q, G_{cl}, V_{cl})$ should have some desired properties, e.g., an equilibrium point $q_0$ should be stable.
• In equations:

Given:

\[ \nabla_{\gamma'}(t)\gamma'(t) = -G^{\sharp}_{\text{ol}} \circ dV_{\text{ol}}(\gamma(t)) + \sum_{a=1}^{m} u^a(t)G^{\sharp}_{\text{ol}} \circ F^a(\gamma(t)). \]

Find: Feedback controls \( u_{\text{shp}} : \text{TQ} \rightarrow \mathbb{R}^m \) such that the closed-loop system has governing equations

\[ \nabla_{\gamma'}(t)\gamma'(t) = -G^{\sharp}_{\text{cl}} \circ dV_{\text{cl}}(\gamma(t)) + G^{\sharp}_{\text{cl}} \circ F_{\text{cl}}(\gamma'(t)). \]

• Form of \( F_{\text{cl}} \): \( F_{\text{cl}} = F_{\text{cl, diss}} + F_{\text{cl, gyr}} \) where
  1. \( F_{\text{cl, diss}} \) is a dissipative force and
  2. \( F_{\text{cl, gyr}} \) is a quadratic gyroscopic force.

• Recall: A \textbf{quadratic gyroscopic force} is of the form

\[ \langle F_{\text{gyr}}(v_q); w_q \rangle = B_{\text{gyr}}(w_q, v_q, v_q), \]

where \( B_{\text{gyr}} \) is a \((0,3)\)-tensor field satisfying

\[ B_{\text{gyr}}(u_q, v_q, w_q) = -B_{\text{gyr}}(v_q, u_q, w_q) \]

(denote \( F_{\text{gyr}}(v_q) = B^q_{\text{gyr}}(v_q) \)).

\textbf{Punchline}: Preserves energy.
• **Assumed procedure:**

1. Find closed-loop (kinetic energy)/(quadratic gyroscopic force):

\[
G^\sharp_{\text{ol}} \circ F_{\text{kin}}(\gamma'(t)) = G^\sharp_{\text{cl}} \circ \nabla_{\gamma'(t)}^\gamma \gamma'(t) + G^\sharp_{\text{cl,gyr}}(\gamma'(t)) - G^\sharp_{\text{ol}} \circ \nabla_{\gamma'(t)}^\gamma \gamma'(t)
\]

2. Find closed-loop potential energy:

\[
F_{\text{pot}}(\gamma(t)) = G^\flat_{\text{ol}} \circ G^\sharp_{\text{cl}} \circ dV_{\text{cl}}(\gamma(t)) - dV_{\text{ol}}(\gamma(t)).
\]

3. Find closed-loop control:

\[
\sum_{a=1}^{m} u_{\text{shp}}^a(v_q) G^\sharp_{\text{ol}} \circ F^a(q) = -F_{\text{kin}}(v_q) - F_{\text{pot}}(q).
\]

• **Today:** Ignore dissipative forces.

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**Objectives of approach**

• What are the possible closed loop energies,

\[
E_{\text{cl}}(v_q) = \frac{1}{2} G_{\text{cl}}(v_q) + V_{\text{cl}}(q)?
\]

• For stabilisation: Want Hess \( V_{\text{cl}}(q_0) > 0 \)?

• Main limitation for stabilisation: only works for systems that are linearly stabilizable, i.e., doesn’t work for “hard” systems (i.e., requiring discontinuous feedback) if there are benefits, they are global in nature.
Partial literature review and comments

• Potential shaping:

• Hamiltonian approach (IDA-PBC):

• Lagrangian approach with symmetry:
• Equivalence of Lagrangian and Hamiltonian setting:

• Geometric formulation and weak integrability results:

• Extension to general Lagrangians:

• Extension to nonholonomic systems:

• Linear systems:
A theorem on potential energy shaping

The setup

- We have an open-loop system \((Q, \mathcal{G}_{ol}, V_{ol}, \{F^1, \ldots, F^m\})\) and have applied a control to shape the kinetic energy to \(G_{cl}\).
- Define \(\Lambda_{cl} = G_{cl}^\flat \circ G_{cl}^\sharp\).
- Let \(F\) be the codistribution generated by \(\{F^1, \ldots, F^m\}\) (control forces are thus \(\mathcal{F}\)-valued). Assume constant rank.
- Let \(\mathcal{F}_{cl}\) be the codistribution \(\Lambda_{cl}^{-1}(\mathcal{F})\).

Lemma 1 Given: \(\mathcal{G}_{ol}, V_{ol}, \mathcal{G}_{cl},\) and \(\mathcal{F}\).
A force \(F\) taking values in \(\mathcal{F}\) gives a closed-loop potential \(V_{cl}\) if and only if
\[
F(q) = \Lambda_{cl} \circ dV_{cl}(q) - dV_{ol}(q), \quad q \in Q.
\]

The potential energy shaping partial differential equation

- Define \(Q_{\mathbb{R}} = Q \times \mathbb{R}\) and \(\pi: Q_{\mathbb{R}} \to Q\) by \(\pi(q, V) = q\).
- A potential function \(V\) defines a section of \(Q_{\mathbb{R}}\): \(q \mapsto (q, V(q))\).
- We have the map \(\Phi_d : \mathcal{J}^1Q_{\mathbb{R}} \to T^*Q\) satisfying \(\Phi_d(j_1V(q)) = dV(q)\).
- Abbreviate \(\alpha_{cl} = \Lambda_{cl}^{-1} \circ dV_{ol}\).
- Let \(\pi_{\mathcal{F}_{cl}} : T^*Q \to T^*Q/\mathcal{F}_{cl}\) be the canonical projection.
- Define
\[
R_{pot} = \{j_1V(q) \in \mathcal{J}^1Q_{\mathbb{R}} \mid \pi_{\mathcal{F}_{cl}} \circ \Phi_d(j_1V(q)) = \pi_{\mathcal{F}_{cl}} \circ \alpha_{cl}(q)\}.
\]
Proposition 1 A section $F$ of $\mathcal{F}$ is a potential energy shaping feedback if and only if $F = \Lambda_{\text{cl}} \circ dV - dV_{\text{ol}}$ for a solution $V$ to $R_{\text{pot}}$.

- What’s the point of all the fanciness?
- You get a partial differential equation in the framework for applying the Goldschmidt\(^1\) theory for integrability of partial differential equations.

\(^1\) *J. Differential Geom.*, 1, 269–307, 1967

The statement

- Let $l_2(\mathcal{F}_{\text{cl}})$ be the two-forms in the algebraic ideal generated by $\mathcal{F}_{\text{cl}}$.

Theorem 1 Let $(Q, G_{\text{ol}}, V_{\text{ol}}, \mathcal{F})$ be an analytic simple mechanical control system and let $G_{\text{cl}}$ be an analytic Riemannian metric. Let $j_1 V(q_0) \in R_{\text{pot}}$. Assume that $q_0$ is a regular point for $\mathcal{F}$ and that $\mathcal{F}_{\text{cl}}$ is integrable in a neighbourhood of $q_0$. Then the following statements are equivalent:

(i) there exists a neighbourhood $\mathcal{U}$ of $q_0$ and an analytic potential energy shaping feedback $F$ defined on $\mathcal{U}$ which satisfies

\[ \Phi_d(j_1 V(q_0)) = F_{\text{cl}}(q_0) + \alpha_{\text{cl}}(q_0); \]

(ii) there exists a neighbourhood $\mathcal{U}$ of $q_0$ such that $d\alpha_{\text{cl}}(q) \in l(\mathcal{F}_{\text{cl}, q})$ for each $q \in \mathcal{U}$.

Moreover, if $V_{\text{cl}, 1}$ and $V_{\text{cl}, 2}$ are two closed-loop potential functions, then

\[ d(V_{\text{cl}, 1} - V_{\text{cl}, 2})(q) \in \mathcal{F}_{\text{cl}, q} \text{ for each } q \in Q. \]
The working version

• If $\mathcal{F}_{\text{cl}}$ is not integrable, replace it with the largest integrable codistribution contained in it.

• Since $\mathcal{F}_{\text{cl}}$ is integrable choose coordinates $(q^1, \ldots, q^n)$ for $Q$ such that
  
  $\mathcal{F}_{\text{cl},q} = \text{span}_{\mathbb{R}}\{dq^1(q), \ldots, dq^r(q)\}$.

• Write $\alpha_{\text{cl}} = G_{\text{cl},ij} G_{\text{ol}}^{jk} \frac{\partial V_{\text{ol}}}{\partial q^k} dq^i$.

• The potential shaping partial differential equation has a solution if

  \[ \frac{\partial \alpha_{\text{cl},a}}{\partial q^b} = \frac{\partial \alpha_{\text{cl},b}}{\partial q^a}, \quad a, b \in \{r + 1, \ldots, n\}. \]

• If $\tilde{V}_{\text{cl}}$ is some solution to the potential shaping partial differential equation, then any other solution has the form

  $V_{\text{cl}}(q^1, \ldots, q^n) = \tilde{V}_{\text{cl}}(q^1, \ldots, q^n) + F(q^1, \ldots, q^r)$.

Discussion

• The proof of the existence part of the theorem is not constructive. It merely tells you that there are no obstructions to constructing a Taylor series solution order-by-order.

• Note that the integrability condition for potential shaping is a condition on $\alpha_{\text{cl}} = \Lambda_{\text{cl}}^{-1} \circ dV_{\text{ol}}$. This is dependent on $G_{\text{cl}}$.

  The point: A bad design for $G_{\text{cl}}$ can make it impossible to do any potential energy shaping.

• Note that we require integrability of $\mathcal{F}_{\text{cl}} = \Lambda_{\text{cl}}^{-1}(\mathcal{F})$. This is dependent on $G_{\text{cl}}$.

  The point: A bad design for $G_{\text{cl}}$ can make it impossible to achieve any flexibility in the character of the possible closed-loop potential functions.
An affine connection formulation of kinetic energy shaping

• For a Riemannian metric $\mathcal{G}$, define $\text{KE}_\mathcal{G} : \mathbb{T} \to \mathbb{R}$ by $\text{KE}_\mathcal{G}(v_q) = \frac{1}{2} \mathcal{G}(v_q, v_q)$.

• An affine connection $\nabla$ is $\mathcal{G}$-energy-preserving if $\mathcal{L}_{\gamma''(t)} \text{KE}_\mathcal{G}(\gamma'(t)) = 0$ for every geodesic $\gamma$ of $\nabla$.

Lemma 2 $\nabla$ is $\mathcal{G}$-energy preserving if and only if $\text{Sym}(\nabla \mathcal{G}) = 0$.

Theorem 2 Given: $\mathcal{G}_\text{ol}$ and $\mathcal{F}$.
The solutions to the following problems are in 1–1 correspondence:

(i) when there exist $\mathcal{G}_\text{cl}$ and a gyroscopic tensor $B_\text{cl}$ such that
$$\overline{\nabla}_{\gamma'(t)} \gamma'(t) + \mathcal{G}_\text{cl}^d B_\text{cl}(\gamma'(t)) - \overline{\nabla}_{\gamma'(t)} \gamma'(t) \in \mathcal{G}_\text{ol}^d(\mathcal{F});$$

(ii) when there exist $\mathcal{G}_\text{cl}$ and a $\mathcal{G}_\text{cl}$-energy preserving connection $\nabla$ such that
$$\nabla_{\gamma'(t)} \gamma'(t) - \overline{\nabla}_{\gamma'(t)} \gamma'(t) \in \mathcal{G}_\text{ol}^d(\mathcal{F}).$$
The kinetic energy shaping partial differential equation

Geometric formulation of partial differential equation

- Recall that the set of torsion-free affine connections on $Q$ is an affine subbundle
  \[ \text{Aff}_0(Q) = \{ \Gamma \in T^*Q \otimes J^1TQ \mid \Gamma \circ \pi^1_0 = \text{id}_{TQ}, \]
  \[ (j_1Y - \Gamma(Y))(X) - (j_1X - \Gamma(X))(Y) = [X, Y] \}
  modelled on $S^2(T^*Q) \otimes TQ$.

- Let $Y_{KE} = \{ (\Gamma, \mathcal{G}) \in \text{Aff}_0(Q) \times S^2_+(T^*Q) \mid \Gamma$ is $\mathcal{G}$-energy preserving $\}$.

- Define $\Phi_{\text{LC}} : J^1S^2_+(T^*Q) \to \text{Aff}_0(Q)$ by $\Phi_{\text{LC}}(j_1\mathcal{G}) = \mathcal{G}\nabla$.

- Define the quasilinear partial differential equation
  \[ R_{\text{kin}} = \{ (j_1\Gamma(q), j_1\mathcal{G}(q)) \in J^1Y \mid \pi_{\mathcal{F}}(\Gamma(q) - \Phi_{\text{KE}}(j_1\mathcal{G}(q))) = 0 \}, \]
  where $\pi_{\mathcal{F}} : S(T^*Q) \otimes TQ \to S(T^*Q) \otimes TQ/\mathcal{G}_\#(\mathcal{F})$ is the canonical projection.

An observation

- Define two subsets of $\text{Aff}_0(Q)$:
  \[ \text{Aff}_0(Q, \mathcal{F}, \nabla) = \nabla + S^2(T^*M) \otimes \text{coann}(\mathcal{F}), \]
  \[ \text{EP}(Q) = \{ \nabla \in \text{Aff}_0(Q) \mid \nabla$ is $\mathcal{G}$-energy preserving for some $\mathcal{G} \}$.

- The solutions $(\nabla, \mathcal{G}_{\text{cl}})$ to $R_{\text{kin}}$ are then described by asking that
  \[ \nabla \in \text{Aff}_0(Q, \mathcal{F}, \nabla) \cap \text{EP}(Q). \]

- $\text{Aff}_0(Q, \mathcal{F}, \nabla)$ is easy to understand.

- What about $\text{EP}(Q)$?

- And when $\nabla \in \text{EP}(Q)$ what does $\{ \mathcal{G} \mid \text{Sym}(\nabla \mathcal{G}) = 0 \}$ look like?
Relationship to an inverse problem in calculus of variations

- Consider the following subset of $\text{EP}(Q)$:

$$\text{LC}(Q) = \{ \nabla \in \text{Aff}_0(Q) \mid \nabla \text{ is the Levi-Civita connection for some } G \}.$$ 

- The problem was initially investigated by Eisenhart and Veblen\(^1\) who give necessary conditions and a sufficient condition with strong hypotheses.

- Comparison of problems:

<table>
<thead>
<tr>
<th>LC(Q)</th>
<th>EP(Q)</th>
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<tr>
<td>$\nabla G = 0$ has solution</td>
<td>Sym($\nabla G$) = 0 has solution</td>
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- The Eisenhart and Veblen problem is “nice:” it has an involutive symbol.

- The symbol for our generalisation is not involutive \(\rightarrow\) work to do here.

\(^1\)Proceedings of the National Academy of Sciences of the United States of America, 8, 19–23, 1922

Summary

- The method of energy shaping has been applied in certain cases, sometimes with some generality. However...

- The question, “If I give you a system, can you determine whether it can be stabilised using energy shaping” remains unresolved.