ORDERINGS AND SIGNATURES OF HIGHER LEVEL ON MULTIRINGS AND HYPERFIELDS

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Abstract. Multirings are objects like rings but with multi-valued addition. They are a variant of other objects called hyperrings, defined by Krasner [12], [13]. In [16] the second author defines multirings, introduces a certain special class of multirings called real reduced multirings, defines a natural reflection $A \rightsquigarrow Q_{\text{red}}(A)$ from the category of multirings satisfying $-1 \notin \sum A^2$ to the full subcategory of real reduced multirings, provides an elementary first-order description of these objects, and proves that these objects are precisely the spaces of signs, also known as abstract real spectra, considered earlier in [1], [15]. In the present paper we extend results of E. Becker and others concerning orderings of higher level on fields and rings to orderings of higher level on hyperfields and multirings and, in the process of doing this, we establish higher level analogs of the results in [16]. In particular, we introduce a class of multirings called ℓ-real reduced multirings, define a natural reflection $A \rightsquigarrow Q_{\ell\text{-red}}(A)$ from the category of multirings satisfying $-1 \notin \sum A^{2^\ell}$ to the full subcategory of ℓ-real reduced multirings, and provide an elementary first-order description of these objects. The relationship between ℓ-real reduced hyperfields and the spaces of signatures defined by Mulcahy and Powers [20], [21], [22] is also examined.

1. Introduction

There has been considerable interest recently in hyperfields, hyperrings and multirings. This interest derives not so much from the actual objects themselves as from the success achieved in using these objects to understand and explain other objects and phenomena. Hyperfields and hyperrings arise in the study of the algebraic structure of the adèle class space of a global field and in exploring the deeper relationship between algebraic number fields and algebraic function fields [7], [8]. Hyperfields occur naturally in the context of quadratic form theory and spaces of orderings [16], Milnor K-theory [17], tropical geometry [23], commutative algebras over fields with semi-linear homomorphisms, abelian groups with injective homomorphisms, as well as non-desarguesian plane projective geometries [7]. Multirings are considered in [16], and spaces of signs, also known as abstract real spectra, objects which arise naturally in the study of constructible sets in real geometry [1], [15], are shown to be multirings of a particular sort.

Hyperrings and hyperfields were introduced first by Krasner [12], [13] in connection with his work on valued fields. Multirings and multifields were introduced later and independently in [16]. All of these objects are very natural and very useful,

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although they are not at all widely known. The reader may wish to look ahead to Section 2 for the precise definitions.

For hyperrings the strong distributive property

$$a(b+c) = ab + ac$$

is assumed, whereas, for multirings only the weaker distributive property

$$a(b+c) \subseteq ab + ac$$

is assumed. Every hyperring is a multiring, but the converse is false. Hyperfields and multifields, on the other hand, are exactly the same thing.

The object of the present paper is to develop higher level analogs of some of the level 1 results in [16]. In particular, we want to develop a reasonable theory of real reduced multirings and hyperfields of higher level. The paper extends work done by the first author in [9]. Some of the arguments are straightforward, some are non-trivial. For the higher level theory in the case of commutative rings and fields see [2], [3], [4], [5], [11], [18], [24].

It is important to note that by level ℓ we mean exponent $k = 2^\ell$, i.e., only orderings of 2-power exponent are considered here. This restriction seems to be necessary because the valuation-theoretic tools needed to deal with arbitrary even exponent are not available for general hyperfields.

As in the case of rings and fields [11], it is important to determine when

$$\sum A^k - \sum A^k = A$$

holds for a multiring or hyperfield A. We consider this question in Section 3. The arguments are more complicated than one might expect. In Section 4 we develop some of the basic Artin-Schreier theory for orderings of higher level on hyperfields. In Section 5 we prove a certain weak local-global principle for T-modules, extending to multirings a standard result for rings [3], [5], [24] which says that, for a maximal proper T-module M of A, T a preordering of higher level of A, the set $M \cap -M$ is a prime ideal of A. The argument is quite involved, and it also uses results from Section 3. In Section 6 we define orderings of higher level and the higher level real spectrum for multirings. In Section 7 we use the weak local-global principle to establish higher level versions of the Positivstellensatz for multirings, copying more or less directly the argument in [16]. In Section 8 we explain how results concerning real ideals extend to real ideals of higher level in multirings.

In Section 9 we construct a functor (a reflection)

$$A \rightsquigarrow Q_{\ell\text{-red}}(A)$$

from the category of multirings A satisfying $-1 \notin \sum A^{2^\ell}$ onto a certain (full) subcategory, called the category of ℓ-real reduced multirings, and we characterize ℓ-real reduced multirings as non-zero multirings satisfying the following simple axioms.

(1) $a^{2^\ell+1} = a$.
(2) $a + ab^{2^\ell} = \{a\}$.
(3) $a^{2^\ell} + b^{2^\ell}$ contains a unique element.

Actually, we do more. We construct an ℓ-real reduced multiring $Q_T(A)$ for each proper preordering T of level ℓ of A. $Q_{\ell\text{-red}}(A)$ is the multiring obtained from this construction when $T = \sum A^{2^\ell}$. Our results generalize level 1 results in [16]. Again, the argument is quite involved. It is necessary to modify substantially the argument in [16], using instead the weak local-global principle from Section 5. For a hyperfield satisfying axiom (1), axioms (2) and (3) reduce to the single axiom

(4) $1 + 1 = \{1\}$.

As is explained in [16], 1-real reduced hyperfields correspond to spaces of orderings, so it is natural to wonder if ℓ-real reduced hyperfields correspond to the spaces of signatures introduced in [20], [21], [22]. We consider this question in Section 10. We produce an example showing that, in fact, this is not the case, and we mention one additional axiom, a certain symmetry property

($*$) For all odd integers $1 \le k \le 2^\ell$, $a \in b + c \Rightarrow a^k \in b^k + c^k$,

which is satisfied by spaces of signatures but not by general ℓ-real reduced hyperfields.

2. Terminology

Because the terms multiring and hyperfield are not yet part of the everyday mathematical language, it is necessary to recall some definitions.

A *multiring* [16] is a system $(A, +, \cdot, -, 0, 1)$ where A is a set, $+$ is a multivalued binary operation on A, i.e., a function from $A \times A$ to the set of all subsets of A, $\cdot$ is a binary operation on A, $- : A \to A$ is a function, and $0, 1$ are elements of A such that

I. $(A, +, -, 0)$ is a canonical hypergroup, terminology as in [19], i.e.,

(1) $c \in a + b \Rightarrow a \in c + (-b)$,
(2) $a \in b + 0$ iff $a = b$,
(3) $(a + b) + c = a + (b + c)$, and
(4) $a + b = b + a$; and

II. $(A, \cdot, 1)$ is a commutative monoid, i.e., $(ab)c = a(bc)$, $ab = ba$, and $a1 = a$ for all $a, b, c \in A$; and

III. $a0 = 0$ for all $a \in A$; and

IV. $a(b + c) \subseteq ab + ac$.

A *multifield* is a multiring with $1 \neq 0$ such that every non-zero element has a multiplicative inverse. *Hyperrings* and *hyperfields* are defined by Krasner in [12] and [13]. A hyperring is a multiring which also satisfies the second half of the distributive property, i.e., $ab + ac \subseteq a(b + c)$.

For a multifield, the second half of the distributive property is automatic from the first half. If $a \neq 0$, $ab + ac = aa^{-1}(ab + ac) \subseteq a[a^{-1}(ab) + a^{-1}(ac)] = a(b + c)$. If $a = 0$, $ab + ac$ and $a(b + c)$ are both equal to zero. It follows that multifields and hyperfields are the same thing.

At the same time, there are many interesting examples of multirings which are not hyperrings. The real reduced multirings constructed in [16] are typically not hyperrings. The ℓ-real reduced multirings we construct in the present paper are typically not hyperrings.

Example 2.1. Let V be an algebraic set in R^n where R is a real closed field, and let A denote the coordinate ring of V, i.e., the ring of all polynomial functions $f : V \to R$. Define an equivalence relation $\sim$ on A by declaring $f \sim g$ to mean that f, g have the same sign ($+$,$-$, or 0) at each point of V. The set of equivalence classes is the real reduced multiring denoted by $Q_{\text{red}}(A)$ in [16]. It is made into a multiring as follows: Denote the equivalence class of f by $\overline{f}$. Define $\overline{f} \in \overline{g} + \overline{h}$ to mean $\exists\ f', g', h' \in A$ such that $f' = g' + h'$, $\overline{f'} = \overline{f}$, $\overline{g'} = \overline{g}$ and $\overline{h'} = \overline{h}$. Define $\overline{g}\overline{h} = \overline{gh}$, $-\overline{f} = \overline{-f}$, $0 = \overline{0}$, and $1 = \overline{1}$. The multiring $Q_{\text{red}}(A)$ is not a hyperring if $\dim(V) \geq 1$. For example, if $V = R$ (so $n = 1$ and A is the polynomial ring $R[x]$), and a, b, c and d are the classes of the polynomials $x, x, 1$ and $x^2 + x^3$, respectively, then $d \in ab + ac$ but $d \notin a(b + c)$. This is because d is positive for x close to zero, $x \neq 0$, but any element of $a(b + c)$ is negative for x close to zero, $x < 0$. We also use the fact that x^3 and x have the same sign.

See Example 9.12 below for a more thorough analysis of Example 2.1 and for an extension of Example 2.1 to higher level.

If S, T are subsets of a multiring A then $S + T :=$ the union of the sets $x + y$, $x \in S$, $y \in T$, and $ST := \{xy \mid x \in S, y \in T\}$. Also, $S - T := S + (-T)$, where $-T := \{-y \mid y \in T\}$. $\sum S$ denotes the union of all finite sums $x_1 + \cdots + x_n$, $x_1, \ldots, x_n \in S$, $n \geq 1$.

We refer the reader to [16] for basic terminology and basic facts concerning multirings and hyperfields. We recall parts of this. A *multiring homomorphism* from A to B, where A and B are multirings, is a function $f : A \to B$ satisfying $f(a + b) \subseteq f(a) + f(b)$, $f(ab) = f(a)f(b)$, $f(-a) = -f(a)$, $f(0) = 0$, and $f(1) = 1$. We say A is *strongly embedded* in B by a multiring homomorphism $i : A \to B$ if i is injective and, for all $a, b, c \in A$, $i(c) \in i(a) + i(b) \Rightarrow c \in a + b$. A multiring A is a submultiring of B if A is strongly embedded in B and, for $a, b \in A$, and $c \in B$, if $c \in i(a) + i(b)$, then $c \in i(A)$. The notions of strong embedding and submultiring coincide in the ring case, but are not the same in the multiring case.

Ideals and multiplicative sets are defined in an obvious way. If S is a multiplicative set in A and I is an ideal of A, then one can form the localization $S^{-1}A$ and the factor multiring A/I, and there are natural multiring homomorphisms $A \to S^{-1}A$ and $A \to A/I$. The principal ideal of A generated by $x \in A$ is the set $\sum Ax :=$ the union of all sets of the form $a_1x + \cdots + a_nx$, $a_i \in A$, $n \geq 1$. If A is a hyperring, this coincides with the set $Ax := \{ax \mid a \in A\}$.

We denote the hyperfield of fractions of a multidomain D by $\text{ff}(D)$, i.e., $\text{ff}(D) := (D \backslash \{0\})^{-1}D$. It is important to realize that the natural multiring homomorphism $D \to \text{ff}(D)$ is not injective in general, and even when it is injective, it need not be a strong embedding; see [16, Example 2.5(2)]. In particular, D need not be a submultiring of its hyperfield of fractions.

If S is a multiplicative subset in a multiring A, there is another construction one can perform, which we denote by $A/_mS$ and refer to as *the quotient construction* [7, Proposition 2.6], [13], [16, Example 2.6]. $A/_mS$ is the set of equivalence classes with respect to the equivalence relation $\sim$ on A defined by $a \sim b$ iff $as = bt$ for some $s \in S$. The operations on $A/_mS$ are the obvious ones induced by the corresponding operations on A. Denote by $\overline{a}$ the equivalence class of a. Then $\overline{a} \in \overline{b} + \overline{c}$ iff $as \in bt + cu$ for some $s, t, u \in S$, $\overline{a}\overline{b} = \overline{ab}$, $-\overline{a} = \overline{-a}$. Also, $0 = \overline{0}$, and $1 = \overline{1}$.

For a multiring A, we are interested in the set $\{x^k \mid x \in A\}$, which we denote by A^k for short, so $\sum A^k$ denotes the union of all finite sums $x_1^k + \cdots + x_n^k$, $x_1, \ldots, x_n \in A$, $n \geq 1$. We are especially interested in the case where $k = 2^\ell$.

Let A be a multiring, $\ell \geq 1$ an integer. A *preordering of level ℓ of A* is a subset T of A satisfying $T + T \subseteq T$, $TT \subseteq T$ and $a^{2^\ell} \in T$ for all $a \in A$. A preordering of level 1 is what is referred to as a preordering in [16]. We say the preordering T of A is *proper* if $-1 \notin T$. A *T-module of A* is a subset M of A satisfying $M + M \subseteq M$, $TM \subseteq M$, and $1 \in M$.

3. Sums and differences of 2^ℓ-th powers

Sums and differences of k-th powers in commutative rings and fields are considered in [11]. In this context one has the identity

$$k!x = \sum_{h=0}^{k-1} (-1)^{k-1-h} \binom{k-1}{h} [(x+h)^k - h^k],$$

see [11, Théorème 8.2.2], which shows that $\sum A^k - \sum A^k = A$ holds for any commutative ring A satisfying $k! \in A^*$. The corresponding result for hyperfields and multirings is much harder to prove.

We introduce some notation. For any hyperfield or multiring A, any $x \in A$, and any integer $n \geq 1$, we denote the subset

$$\underbrace{x + \cdots + x}_{n \text{ times}}$$

of A by $n \cdot x$.

3.1. **Hyperfield Case.** The *characteristic* of a hyperfield F is defined to be the least integer $n \geq 2$ such that $0 \in n \cdot 1$. We say the characteristic of the hyperfield F is *equal to zero* if no such n exists.

We aim to prove the following basic result:

Theorem 3.1. *Let F be a hyperfield,* $\text{char}(F) = 0$*, and let T be a preordering of F of finite level ℓ, i.e., finite exponent 2^ℓ. Then $-1 \in T \Rightarrow T = F$.*

Observe that Theorem 3.1 can be rephrased equivalently as follows:

Theorem 3.2. *Let F be a hyperfield,* $\text{char}(F) = 0$*. Then, for any integer $\ell \geq 0$, $F = \sum F^{2^\ell} - \sum F^{2^\ell}$.*

Note:

(1) In the field case the hypothesis $\text{char}(F) = 0$ can be weakened considerably. One might expect the same to hold in the hyperfield case.

(2) For level equal to 1, Theorem 3.1 is proved already in [16, Lemma 3.2]; in fact, it is proved under the weaker hypothesis that $\text{char}(F) \neq 2$, i.e., $-1 \neq 1$.

Proof. Fix a subset S of T containing 0 and 1 and closed under addition and multiplication which is maximal subject to the condition $S \cap -S = \{0\}$. Such a subset S exists by Zorn's Lemma, using our assumption that $\text{char}(F) = 0$. Replacing F by $F/_m S^*$, we can assume $S = \{0, 1\}$. Observe that this implies in particular that $1 + 1 = \{1\}$, so $a + a = a(1 + 1) = \{a\}$ for any $a \in F$.

Claim 1: If $\gamma \in T^*$ and $\gamma^2 = 1$ then $\gamma = \pm 1$. For the proof of this, consider $S' := S + \gamma S$. If $\gamma \in S$ then $\gamma = 1$, so we may suppose $\gamma \notin S$. It follows, by maximality of S, that $S' \cap -S' \neq \{0\}$. Say $p \in S' \cap -S'$, $p \neq 0$. Then $p \in s_1 + \gamma t_1$, $-p \in s_2 + \gamma t_2$, $s_i, t_i \in S$, so $0 \in p - p \subseteq (s_1 + s_2) + \gamma(t_1 + t_2)$. Then $0 \in s + \gamma t$ for some $s, t \in S$, not both zero. Then s, t are both $\neq 0$, i.e., $s = t = 1$, and $\gamma = -1$.

Claim 2: If $\gamma \in T^*$ and $\gamma \in \gamma^2 - 1$, then $\gamma = \pm 1$. The proof of this is completely similar to the proof of Claim 1.

The proof that $F = T$ is by contradiction. We suppose $F \neq T$. Fix $a \in F^*$, $a \notin T^*$. Replacing a by a^{2^k} for some suitable $k \geq 0$, we can suppose $a \notin T^*$ but $a^2 \in T^*$. Observe that $T + aT$ is a hyperfield which is a subhyperfield of F and which contains T properly. Replacing F by $T + aT$, we can assume that $F = T + aT$.

Claim 3: If $\gamma \in T^*$ and $b \in \gamma + a$ then $b^2 \notin T^*$. For suppose $b \in \gamma + a$, $b^2 \in T^*$. Then $b^2 \in \gamma^2 + \gamma a + \gamma a + a^2 = \gamma^2 + \gamma a + a^2$, so $\gamma a \in b^2 - \gamma^2 - a^2 \subseteq T$. Then $a = (\gamma a)(\frac{1}{\gamma}) \in T^*$, contradicting $a \notin T^*$.

Claim 4: F^*/T^* is cyclic. Suppose $b \in F^*$ has order 2 modulo T^*. We know that $b \in \gamma + \delta a$ for some $\gamma, \delta \in T$. Obviously $b \neq \gamma$, so $\delta \neq 0$. Then $\frac{b}{\delta} \in \frac{\gamma}{\delta} + a$ and $(\frac{b}{\delta})^2 \in T^*$ so, by Claim 3, $\frac{\gamma}{\delta} = 0$, i.e., $b = \delta a$. Thus aT^* is the unique element of order 2 in F^*/T^*. Since the exponent of F^*/T^* is a 2-power, this implies F^*/T^* is cyclic.

Observe that, since $\gamma + a \neq \emptyset$, Claim 3 proves, in particular, that the level cannot be 1, which is what was proved already in [16, Lemma 3.2]. Thus the level is 2 or more and there is an element b of F^* having order 4 modulo T^*. Since $F = T + aT$, $b \in \gamma + \delta a$ for some $\gamma, \delta \in T$. γ and δ must both be $\neq 0$ (since otherwise b would have order 1 or 2 modulo T^*). Then $\frac{b}{\delta} \in \frac{\gamma}{\delta} + a$, so, replacing b by $\frac{b}{\delta}$ and γ by $\frac{\gamma}{\delta}$, we may assume $\delta = 1$, i.e., $b \in \gamma + a$.

Claim 5: We can assume $a^2 = -1$. For, as explained above, we have $b \in \gamma + a$ where $\gamma \in T^*$ and b has order 4 modulo T^*. Then $b^2 \in \gamma^2 + \gamma a + a^2$, so $b^2 \in \delta + \gamma a$ for some $\delta \in \gamma^2 + a^2$. If $\delta \neq 0$, then dividing by γ we would have the element $\frac{b^2}{\gamma}$ of order 2 modulo T^* belonging to $\frac{\delta}{\gamma} + a$, and $\frac{\delta}{\gamma} \in T^*$, contradicting Claim 2. Thus $\delta = 0$, so $a^2 = -\gamma^2$. Then $(\frac{a}{\gamma})^2 = -1$, so we are done, replacing a by $\frac{a}{\gamma}$.

Claim 6: $T^* = \{1, -1\}$. For suppose $\exists \gamma \in T^*$, $\gamma \neq \pm 1$. Then $\gamma + a \neq \emptyset$, so $\exists b \in \gamma + a$. Take an element $b \in F^*$ with smallest order modulo T^* such that there exists $\gamma \in T^*$, $\gamma \neq \pm 1$, $b \in \gamma + a$. According to Claim 3, b has order 4 or more modulo T^*. Then $b^2 \in \gamma^2 + \gamma a + a^2 = \gamma^2 + \gamma a - 1$, so $b^2 \in \delta + \gamma a$, $\delta \in \gamma^2 - 1$. Dividing by γ, $\frac{b^2}{\gamma} \in \frac{\delta}{\gamma} + a$, and $\frac{b^2}{\gamma}$ has lower order than b, so $\delta = 0$ or $\delta = \pm\gamma$. If $\delta = 0$, then $\gamma^2 = 1$ and $\gamma = \pm 1$ by Claim 1, a contradiction. If $\delta = \pm\gamma$ then $\pm\gamma \in \gamma^2 - 1$ so, applying Claim 2 to $\pm\gamma$, we obtain $\pm\gamma = \pm 1$, i.e., $\gamma = \pm 1$, also a contradiction.

Combining Claims 4 and 6, we see that F^* is finite, so each set $1 + b$ is finite. Pick $b \in F$ with $1 + b$ smallest subject to the condition $b \notin T$. Then $1 + b \neq \emptyset$, so $\exists c \in 1 + b$. If $c \in T$, then $b \in -1 + T \subseteq T$, a contradiction to our choice of b. Thus $c \notin T$. Also, $1 + c \subseteq 1 + 1 + b = 1 + b$. This implies $1 + c = 1 + b$ so $c \in 1 + c$. But then $1 \in c - c = c(1 - 1) \subseteq cT$, which is impossible. □

Corollary 3.3. *Let F be a hyperfield and let T be a preordering of level ℓ of F. If F has characteristic zero then $T - T = F$. If $-1 \notin T$ then $T^* - T^* = F$.*

Proof. Since $T - T$ is a preordering containing -1, the first assertion is immediate from Theorem 3.1. Suppose now that $-1 \notin T$. Then F has characteristic zero, so $T - T = F$. Also, $T - T = (T^* - T^*) \cup T^* \cup -T^* \cup \{0\}$, so it suffices to show $T^* \cup -T^* \cup \{0\} \subseteq T^* - T^*$. To simplify notation, replace F by the quotient hyperfield $F/_m T^*$. Thus $T = \{0, 1\}$ and we wish to show $\{0, 1, -1\} \subseteq 1 - 1$. Obviously $0 \in 1 - 1$. Also $1 + 1 \subseteq T = \{0, 1\}$, and T is proper, so $0 \notin 1 + 1$, so $1 + 1 = 1$. It follows that $1, -1 \in 1 - 1$. □

3.2. **Local Case.** By a *local* multiring we mean a multiring A with a unique maximal ideal $\mathfrak{m}$ such that every $a \in A \backslash \mathfrak{m}$ is a unit. If A is a ring or, more generally, a hyperring [13], the latter condition is a consequence of the former. But, for a general multiring, it seems that a principal ideal (a) can be equal to (1) without a being a unit. If $\mathfrak{p}$ is a prime ideal of a multiring A then the localization $A_\mathfrak{p} := (A\backslash\mathfrak{p})^{-1}A$ of A at $\mathfrak{p}$ is local.

Theorem 3.4. *Let $(A,\mathfrak{m})$ be a local multiring such that $\left(\bigcup_{n\geq 2} n\cdot 1\right)\cap\mathfrak{m}=\emptyset$. Then, for any $\ell\geq 0$, $\sum A^{2^\ell}-\sum A^{2^\ell}=A$.*

Proof. Let $F=A/\mathfrak{m}$ and let $B=\sum A^{2^\ell}-\sum A^{2^\ell}$. F is a hyperfield of characteristic zero and B is a submultiring of A. Suppose $B\neq A$. Fix $x\in A$, $x\notin B$. We know $F=\sum F^{2^\ell}-\sum F^{2^\ell}$, by Theorem 3.2, so $\exists\ b_i,c_j\in A$ and $y\in\mathfrak{m}$ such that $x\in\sum b_i^{2^\ell}-\sum c_j^{2^\ell}+y$. This forces $y\notin B$. Thus $\exists\ y\in\mathfrak{m}$, $y\notin B$. Replacing y by a suitable 2-power, we can assume $y^2\in B$. Since $y\notin B$, $\exists\ a\in 1+\sum By$, $a\notin B$ (e.g., one can take any $a\in 1+y$). Observe that $a\notin\mathfrak{m}$, so a is a unit.

Claim 1: We may assume $a^2\in B$, i.e., that a has order 2 modulo B^*. For suppose a has order 2^k modulo B^*, $k\geq 2$. Then $a^2\in 1+\sum By+\sum By^2$ so $a^2\in b+c$, $b\in 1+\sum By^2$, $c\in\sum By$. Since $c\in\mathfrak{m}$ and $a\notin\mathfrak{m}$ we see that $b\notin\mathfrak{m}$, i.e., b is a unit. Since $b\in B$ this forces $b\in B^*$ and $\frac{a^2}{b}\in 1+\frac{c}{b}\subseteq 1+\sum By$. Thus $a_1:=\frac{a^2}{b}$ belongs to $1+\sum By$ and a_1 has order 2^{k-1} modulo B^*. The result follows now, by induction on the order of a.

So $\exists\ a\in 1+\sum By$ having order 2 modulo B^*. Say $a\in 1+z$, $z\in\sum By$. Then $z^2\in\sum By^2\subseteq B$ and $z\in -1+a$, so $z^2\in 1-a-a+a^2$, i.e., $z^2\in 1-\gamma a+a^2$ for some $\gamma\in 1+1$ (using the fact that a is a unit). Then $\gamma a\in 1+a^2-z^2\in B$ and $a=(\gamma a)(\frac{1}{\gamma})\in B$, a contradiction. □

3.3. **General Case.** We prove the following general result.

Theorem 3.5. *Let A be a multiring such that $\left(\bigcup_{n\geq 2} n\cdot s\right)\cap\mathfrak{m}=\emptyset$ for each maximal ideal $\mathfrak{m}$ of A and each $s\in A\backslash\mathfrak{m}$. Then, for each $\ell\geq 0$, $\sum A^{2^\ell}-\sum A^{2^\ell}=A$.*

Remark:

(1) The hypothesis of Theorem 3.5 is that, for each maximal ideal $\mathfrak{m}$ of A, the set $\bigcup_{n\geq 2} n\cdot 1$ in the multiring $A_\mathfrak{m}$ has empty intersection with the maximal ideal of $A_\mathfrak{m}$, or, equivalently, that the hyperfield of fractions of the multidomain $A/\mathfrak{m}$ has characteristic zero.

(2) If A is a hyperring then the hypothesis of Theorem 3.5 simplifies to $\bigcup_{n\geq 2} n\cdot 1\subseteq A^*$.

(3) We will not use Theorem 3.5 in what follows. For what we do, Theorems 3.2 and 3.4 will suffice.

In view of Theorem 3.4, to prove Theorem 3.5 it suffices to establish the following:

Theorem 3.6. *For any multiring A and any $\ell\geq 0$ the following are equivalent:*

(1) $A=\sum A^{2^\ell}-\sum A^{2^\ell}$.
(2) $A_\mathfrak{m}=\sum (A_\mathfrak{m})^{2^\ell}-\sum (A_\mathfrak{m})^{2^\ell}$ *for each maximal ideal $\mathfrak{m}$ of A.*

Note: See [11, Théorème 4.8] for the proof of Theorem 3.6 in the ring case. In [11] the result is proved for arbitrary exponents, not just 2-powers. The proof in [11] uses standard facts about extending primes ideals in integral extensions together with the well-known fact that if an A-module M satisfies $M_\mathfrak{m}=\{0\}$ for all maximal ideals $\mathfrak{m}$ of A, then $M=\{0\}$. The reader will find obvious traces of these original ingredients in the present proof.

Proof. The implication (1) $\Rightarrow$ (2) is clear. If $a \in A$, $s \in A\backslash\mathfrak{m}$. then $a/s = b/s^{2^\ell}$ where $b := as^{2^\ell-1}$. By (1) $\exists\ c_i, d_j \in A$ such that $b \in \sum c_i^{2^\ell} - \sum d_j^{2^\ell}$. Then $a/s \in \sum (c_i/s)^{2^\ell} - \sum (d_j/s)^{2^\ell}$.

For the implication (2) $\Rightarrow$ (1) we need the following:

Lemma 3.7. *Suppose B is a submultiring of a multiring A and $\forall\ a \in A\ \exists\ k \geq 0$ such that $a^{2^k} \in B$. Then for any proper ideal I of B, $\sum AI$ is a proper ideal of A.*

Proof. Suppose $1 \in \sum AI$, say $1 \in \sum_{i=1}^n c_i d_i$, $c_i \in A$, $d_i \in I$. Then $1 \in \sum CI$ where C is the submultiring of A generated by B and $c_1, \dots, c_n$. Since $c_i^{2^k} \in B$ for each i, there is a finite sequence of submultirings $B = C_0 \subseteq \cdots \subseteq C_m = C$ and elements $a_i \in C_i$ with $a_i^2 \in C_{i-1}$ and $C_i = C_{i-1} + \sum C_{i-1} a_i$, $i = 1, \dots, m$.

Claim 1: $1 \in \sum C_i I \Rightarrow 1 \in \sum C_{i-1} I$. Suppose $1 \in \sum C_i I$, so $1 \in p + q$, $p \in \sum C_{i-1} I$, $q \in \sum C_{i-1} I a_i$. Observe that $p^2 \in \sum C_{i-1} I$, $q^2 \in \sum C_{i-1} I$ (using the fact that $a_i^2 \in C_{i-1}$), and $q \in 1 - p \subseteq C_{i-1}$, so $pq \in \sum C_{i-1} I$. It follows that $1 = 1^2 \in p^2 + pq + pq + q^2 \subseteq \sum C_{i-1} I$.

Applying Claim 1 we see that $1 \in \sum CI$ implies $1 \in \sum BI = I$, contradicting $1 \notin I$. $\square$

(2) $\Rightarrow$ (1). Let $B = \sum A^{2^\ell} - \sum A^{2^\ell}$. Suppose $a \in A$, $a \notin B$. Let $I = \{x \in B \mid ax \in B\}$. I is an ideal of B which is proper, i.e., $1 \notin I$. By Lemma 3.7, $1 \notin \sum AI$. By Zorn's Lemma, $\exists$ a maximal ideal $\mathfrak{m}$ of A with $\sum AI \subseteq \mathfrak{m}$. We know $A_\mathfrak{m} = \sum (A_\mathfrak{m})^{2^\ell} - \sum (A_\mathfrak{m})^{2^\ell}$, so $a = \frac{b}{s^{2^\ell}}$, $b \in B$, $s \in A\backslash\mathfrak{m}$. Thus $ts^{2^\ell} a = tb$, so $(ts)^{2^\ell} a = t^{2^\ell} b \in B$, for some $t \in A\backslash\mathfrak{m}$. Then $(ts)^{2^\ell} \in I \subseteq \mathfrak{m}$, a contradiction. $\square$

4. Higher level Artin-Schreier theory

Fix a hyperfield F. An *ordering of level ℓ* of F is a proper preordering P of level ℓ of F such that F^*/P^* is cyclic. If $|F^*/P^*| = 2^\ell$ one says *P has exact level ℓ*.

Certain of the well-known results of the Artin-Schreier theory extend directly from the field case to the hyperfield case. See also [9, Theorems 1 and 2].

Theorem 4.1. *A preordering T of level ℓ of F is proper iff there exists an ordering P of level ℓ of F lying over T. In particular, F has an ordering of level ℓ iff $-1 \notin \sum F^{2^\ell}$.*

Proof. One implication is clear. For the other, suppose T is a proper preordering of level ℓ of A. Let P be a maximal proper preordering of A lying over T. Such a preordering P exists by Zorn's Lemma. Clearly P has level ℓ. We claim that P is an ordering. Since F^*/P^* has exponent dividing 2^ℓ, to prove this it suffices to show that F^*/P^* has a unique element of order 2. Suppose $a^2 \in P^*$, $a \notin P^*$. Then $P + Pa$ is a preordering of F which contains P properly, so $-1 \in P + Pa$, i.e., $-1 \in s + ta$, $s, t \in P$. Since $-1 \notin P$, we see that $t \neq 0$, so $\frac{1}{t} \in P$. Then $-ta \in 1 + s$ so $-a \in \frac{1}{t}(1 + s) \subseteq P^*$. $\square$

We say F is *ℓ-real* if $-1 \notin \sum F^{2^\ell}$. Note that every ℓ-real hyperfield has characteristic zero, so, by Theorem 3.2, $\sum F^{2^\ell} - \sum F^{2^\ell} = F$.

Theorem 4.2. *Suppose F is a hyperfield of characteristic zero and T is a preordering of level ℓ of F. Then T is equal to the intersection of all orderings of level ℓ of*

F lying over T. In particular, $\sum F^{2^\ell}$ is equal to the intersection of all orderings of level ℓ of F.

Proof. If T is improper then $T = F$, by Theorem 3.1, so the result is obvious in this case. Suppose now that T is proper. Suppose $a \in F$ lies in all orderings of F lying over T but $a \notin T$. Replacing a by a suitable 2-power of a, we may as well assume $a^2 \in T$. Let $T' = T - Ta$. If $-1 \in T'$ then $-1 \in s - ta$ for some $s, t \in T$. Since T is proper, $t \neq 0$, and $ta \in 1 + s$ so $a \in \frac{1}{t}(1 + s) \subseteq T$, a contradiction. This proves that $-1 \notin T'$, i.e., T' is a proper preordering of F, so, by Theorem 4.1 applied to T', there is an ordering P of F lying over T'. Then P lies over T and $-a \in P$, so $a \notin P$, a contradiction. □

Theorem 4.2 implies the existence of lots of orderings of higher level. On the other hand, the reader may want to refer to [2] or [4] for concrete examples of orderings of higher level coming from valuations on fields.

5. Weak local-global principle

Our goal in this section is to prove the following result:

Theorem 5.1. *Suppose A is a multiring, T is a proper preordering of A of level ℓ, and M is a T-module of A which is maximal subject to $-1 \notin M$. Then $M \cap -M$ is a prime ideal of A and $M \cup -M = A$.*

Note:

(1) The proof of Theorem 5.1 is complicated. Theorem 3.4 is needed at one point in the proof, to allow reduction to the case where $T - T = A$. There are other complications as well.

(2) See [9] for a slightly different proof of the special case $T - T = A$.

(3) The level 1 version of Theorem 5.1 is proved already in [16, Proposition 5.1]. Unfortunately, there is a mistake in the proof. It is not clear that the set M' defined in the proof of [16, Proposition 5.1] is closed under addition.

(4) See [24, Theorem 1.1.4] or [18, Theorem 1.6] or [5, Proposition 3] for the proof of Theorem 5.1 in the ring case, not just for 2-power exponent, but for arbitrary even exponent.

Proof. Assume first that $T - T = A$. Later we get rid of this assumption. We begin by showing that $\mathfrak{p} := M \cap -M$ is an ideal. Clearly $\mathfrak{p} + \mathfrak{p} \subseteq \mathfrak{p}$, $-\mathfrak{p} = \mathfrak{p}$, and $T\mathfrak{p} \subseteq \mathfrak{p}$. Since $T - T = A$ this implies $A\mathfrak{p} \subseteq \mathfrak{p}$.

Next we show that $\mathfrak{p}$ is prime. Suppose $ab \in \mathfrak{p}$, $a \notin \mathfrak{p}$. Replacing a by $-a$, if necessary, we can assume $a \notin M$. Thus -1 lies in the T-module $M + \sum Ta$. Then $-b^{2^\ell} \in Mb^{2^\ell} + \sum Tab^{2^\ell} \subseteq M$ (using the fact that $ab \in \mathfrak{p}$), so $b^{2^\ell} \in \mathfrak{p}$.

Thus we are reduced to showing $b^2 \in \mathfrak{p} \Rightarrow b \in \mathfrak{p}$. Suppose $b^2 \in \mathfrak{p}$, $b \notin \mathfrak{p}$. Without loss of generality, $b \notin M$. Write $-1 \in q + c$, $q \in M$, $c \in \sum Tb$. Note that $c^2 \in \sum Tb^2 \subseteq \mathfrak{p}$, so $c^2 \in \mathfrak{p}$. Replacing A by $A/\mathfrak{p}$, T by $(T + \mathfrak{p})/\mathfrak{p}$, and M by $M/\mathfrak{p}$, we may assume $\mathfrak{p} = \{0\}$. Thus $-1 \in q + c$, $q \in M$, $c^2 = 0$. Observe that $-c \in 1 + q$ so $0 = c^2 \in 1 + q + q + q^2$. If $q^2 \in T$, this yields $-1 \in q + q + q^2 \in M$, which is a contradiction. So we suppose $q^2 \notin T$ and we induct on the least k such that $q^{2^k} \in T$. Since $-c \in 1 + q$ we see that $-c \in M$. Also, multiplying by $-c$, $0 = c^2 \in -c - qc$, so $qc = -c$. Multiplying by q, $c = -cq \in q + q^2$, so $-q^2 \in q - c \in M$. Also $-q \in 1 + c$ so $q^2 \in 1 + c + c + c^2 = 1 + c + c$, so $q^2 \in 1 + d$, $d \in c + c$. Then $d^2 = 0$. Thus and $-1 \in q_1 + c_1$ where $c_1 := d$ and $q_1 := -q^2$. Moreover, $c_1^2 = 0$, $q_1 \in M$ and $q_1^{2^{k-1}} \in T$ if $q^{2^k} \in T$. This completes the proof that $\mathfrak{p}$ is prime.

Suppose now that $a \in A$, $a \notin M$, $a \notin -M$. Then $-1 \in M + \sum Ta$, $-1 \in M - \sum Ta$. Multiplying by a^{2^ℓ}, and noting that $a^{2^\ell-1}(\sum Ta) \subseteq T$, this yields $-a^{2^\ell} \in M + t_1 a$, $-a^{2^\ell} \in M - t_2 a$, $t_1, t_2 \in T$. Then $-t_1 a \in a^{2^\ell} + M \subseteq M$ and $t_1 a \in a^{2^\ell} + M \subseteq M$, so $t_1 t_2 a \in \mathfrak{p}$. If t_1 or t_2 is in $\mathfrak{p}$, then $-a^{2^\ell} \in M$, so $a \in \mathfrak{p}$. Thus $a \in \mathfrak{p}$ in any case, contradicting $a \notin M$ (and also $a \notin -M$). Thus proves $M \cup -M = A$.

Suppose now that $T - T \neq A$. Let $B := T - T$ and let $\tilde{M} = M \cap B$. If $x \in B$, $x \notin \tilde{M}$, then $x \notin M$ so $-1 \in M + \sum Tx$, i.e., $-1 \in \tilde{M} + \sum Tx$. This proves that the T-module $\tilde{M}$ of B is maximal subject to $-1 \notin \tilde{M}$ so, by what we have just proved, $\mathfrak{p} := \tilde{M} \cap -\tilde{M}$ is a prime ideal of B. The localization of B at $\mathfrak{p}$ is $S^{-1}B$ where $S := \{s^{2^\ell} \mid s \in B \backslash \mathfrak{p}\}$. Using Lemma 3.7 we have a maximal ideal $\mathfrak{m}$ of $S^{-1}A$ containing the prime ideal $S^{-1}\mathfrak{p}$ of $S^{-1}B$. If $x \in S^{-1}A \backslash \mathfrak{m}$, then $x^{2^\ell} \in S^{-1}B \backslash S^{-1}\mathfrak{p}$, so x is a unit of $S^{-1}A$, i.e., $S^{-1}A$ is local with maximal ideal $\mathfrak{m}$, and $\mathfrak{m} \cap S^{-1}B = S^{-1}\mathfrak{p}$. If $-1 \in S^{-1}M$ then $-s^{2^\ell} \in M$ for some $s \in B \backslash \mathfrak{p}$. Then $s^{2^\ell} \in \tilde{M} \cap -\tilde{M} = \mathfrak{p}$, a contradiction. If the subset $n \cdot 1$ of $S^{-1}A$ intersects $\mathfrak{m}$, then the subset $n \cdot s^{2^\ell}$ of B intersects $\mathfrak{p}$ for some $s \in B \backslash \mathfrak{p}$. Say $c \in n \cdot s^{2^\ell} \cap \mathfrak{p}$. Then $-s^{2^\ell} \in \tilde{M} - c \subseteq \tilde{M} + \mathfrak{p} = \tilde{M}$ so $s^{2^\ell} \in \tilde{M} \cap -\tilde{M} = \mathfrak{p}$, a contradiction. Thus, by Theorem 3.4, $S^{-1}A = S^{-1}B = S^{-1}T - S^{-1}T$. Let N be a $S^{-1}T$-module of $S^{-1}A$ containing $S^{-1}M$ and maximal subject to $-1 \notin N$. By what we have already proved we know that $N \cap -N$ is a prime ideal of $S^{-1}A$ and $N \cup -N = S^{-1}A$. By the maximality of M we have $M = \rho^{-1}(N)$ where $\rho : A \to S^{-1}A$ is the natural multiring homomorphism. It follows that $M \cap -M = \rho^{-1}(N \cap -N)$ is a prime ideal of A and $M \cup -M = \rho^{-1}(N \cup -N) = \rho^{-1}(S^{-1}A) = A$. □

The most useful part of Theorem 5.1 is not so much the assertion that $M \cup -M = A$, although this is certainly important, but rather it is the assertion that $M \cap -M$ is a prime ideal. It has the following consequence. For a T-module M of A and an ideal I of A, I is said to be M-*convex* if $(M + I) \cap -(M + I) = I$.

Corollary 5.2. *For any T-module M of A which is proper, there exists an M-convex prime ideal $\mathfrak{p}$ of A.*

Proof. Take $\mathfrak{p} = N \cap -N$ where N is any maximal proper T-module lying over M. □

We also note the following consequence of Corollary 5.2.

Corollary 5.3. *For any T-module M of A which is proper, any minimal prime ideal of A lying over $M \cap -M$ is M-convex.*

Note: $M \cap -M$ may not be an ideal, but it doesn't matter. One can still talk about the minimal prime ideals lying over $M \cap -M$.

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of A lying over $M \cap -M$, and let T' and M' denote the extensions of T and M to $A_\mathfrak{p}$. One checks easily that the T'-module M' is proper. By Corollary 5.2, $\exists$ a M'-convex prime $\mathfrak{q}'$ of $A_\mathfrak{p}$. Let $\mathfrak{q}$ be the inverse image of $\mathfrak{q}'$ under the natural multiring homomorphism $A \to A_\mathfrak{p}$. Then $\mathfrak{q}$ is an M-convex prime ideal of A, and $M \cap -M \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ so, by the minimality of $\mathfrak{p}$, $\mathfrak{q} = \mathfrak{p}$. □

6. Orderings of higher level on multirings

We define orderings of higher level on multirings. We also define a higher level real spectrum for multirings.

For a prime ideal $\mathfrak{p}$ of A, the *residue hyperfield* of A at $\mathfrak{p}$ is defined to be $\text{ff}(A/\mathfrak{p})$, the hyperfield of fractions of the multidomain $A/\mathfrak{p}$. For a preordering T of level ℓ of A, we denote by $T_\mathfrak{p}$ the extension of T to $\text{ff}(A/\mathfrak{p})$. The preordering $T_\mathfrak{p}$ is proper iff the prime ideal $\mathfrak{p}$ is T-convex.

By an *ordering of level* ℓ of a multiring A we mean a pair $(\mathfrak{p}, P)$ where $\mathfrak{p}$ is a prime ideal of A and P is an ordering of level ℓ on $\text{ff}(A/\mathfrak{p})$. The prime ideal $\mathfrak{p}$ is called the *support* of $(\mathfrak{p}, P)$. We denote by $\text{Sper}_\ell(A)$ the set of all orderings of level ℓ of A and by X_T the set of all orderings $(\mathfrak{p}, P)$ of level ℓ of A with P lying over $T_\mathfrak{p}$. For $a \in A$ define

$$Z(a) := \{(\mathfrak{p}, P) \in \text{Sper}_\ell(A) \mid a \in \mathfrak{p}\},$$

and

$$U(a) := \{(\mathfrak{p}, P) \in \text{Sper}_\ell(A) \mid a \notin \mathfrak{p},\ a + \mathfrak{p} \in P\}.$$

It is important to note that, for any $a \in A$, the sets

$$Z(a),\ U(a),\ U(-a),\ \text{and } U(-a^{2^k}),\ k = 1, \dots, \ell - 1$$

are pairwise disjoint and their union is all of $\text{Sper}_\ell(A)$. The *spectral topology* is the topology on $\text{Sper}_\ell(A)$ having the sets $U(a)$, $a \in A$ as subbasis.

Proposition 6.1. $\text{Sper}_\ell(A)$ *with the spectral topology is a spectral space* [10].

Proof. The proof follows a standard pattern, e.g., see [15] or [16]. Consider the mapping $\Phi : \text{Sper}_\ell(A) \to \{0, 1\}^A$ defined by $\Phi(\mathfrak{p}, P) = f$ where

$$f(a) := \begin{cases} 0 \text{ if } (\mathfrak{p}, P) \notin U(a) \\ 1 \text{ if } (\mathfrak{p}, P) \in U(a) \end{cases}.$$

Observe that if $\Phi(\mathfrak{p}, P) = f$ then

$$\mathfrak{p} = \{a \in A \mid f(a) = 0,\ f(-a) = 0,\ \text{and } f(-a^{2^k}) = 0 \text{ for } k = 1, \dots, \ell - 1\}$$

and

$$P = \{\frac{a + \mathfrak{p}}{b + \mathfrak{p}} \mid a, b \notin \mathfrak{p} \text{ and } f(ab^{2^\ell - 1}) = 1\} \cup \{0\}.$$

It follows that the map Φ is injective. The topology on $\text{Sper}_\ell(A)$ induced by Φ (giving $\{0, 1\}^A$ the product topology, where $\{0, 1\}$ is given the discrete topology) is the so-called patch topology, i.e., the topology with subbasis consisting of the sets $U(a)$ together with the complementary sets $\text{Sper}_\ell(A) \backslash U(a)$, $a \in A$. It suffices to show that $\text{Sper}_\ell(A)$ with the patch topology is a Boolean space or, equivalently, that the image of Φ is closed in $\{0, 1\}^A$. This is easy to check. □

If $f : A \to B$ is a multiring homomorphism and $\mathfrak{q}$ is a prime ideal of B, then $f^{-1}(\mathfrak{q})$ is a prime ideal of A and f induces a multiring homomorphism

$$f_\mathfrak{q} : \frac{a + f^{-1}(\mathfrak{q})}{b + f^{-1}(\mathfrak{q})} \mapsto \frac{f(a) + \mathfrak{q}}{f(b) + \mathfrak{q}},\ a, b \in A,\ f(b) \notin \mathfrak{q}$$

from the residue hyperfield A at $f^{-1}(\mathfrak{q})$ to the residue hyperfield of B at $\mathfrak{q}$. Moreover, if Q is any ordering of level ℓ of the residue hyperfield of B at $\mathfrak{q}$, then $f_\mathfrak{q}^{-1}(Q)$ is an ordering of level ℓ of the residue hyperfield of A at $f^{-1}(\mathfrak{q})$. In this way, Sper_ℓ

defines a contravariant functor from the category of multirings to the category of spectral spaces.

7. Higher level Positivstellensatz

In this section we establish a higher level version of the Positivstellensatz for multirings. The level 1 case was already proven in [16]. The special case of the higher level Positivstellensatz when $A = T - T$ was given in [9, Theorem 5]. Also, see [3], [5], and [24] for corresponding results in the ring case.

For a preordering T of level ℓ of A we see that

$$X_T = \bigcap_{a \in T} (Z(a) \cup U(a)).$$

Since each $Z(a) \cup U(a)$ is closed, it follows that X_T is closed.

For a preordering T of level ℓ of A, we define an equivalence relation $\sim$ on A, called *T-equivalence,* by $a \sim b$ iff $\forall$ orderings $(\mathfrak{p}, P) \in X_T$, either a, b are both in $\mathfrak{p}$ or $a, b \notin \mathfrak{p}$ and $\frac{a+\mathfrak{p}}{b+\mathfrak{p}} \in P$. According to the Artin-Schreier Theorem 4.2, this is equivalent to the assertion that $\forall$ T-convex primes $\mathfrak{p}$ either a and b are both in $\mathfrak{p}$, or $a, b \notin \mathfrak{p}$ and $\frac{a+\mathfrak{p}}{b+\mathfrak{p}} \in T_{\mathfrak{p}}$.

We denote the equivalence class of a by $\overline{a}$, so $\overline{a} = \overline{b}$ iff $a \sim b$. We refer to $\overline{a}$ as the *sign* of a on X_T. Write $\overline{a} = 0$ (resp., $\overline{a} \geq 0$, resp., $\overline{a} > 0$) at $(\mathfrak{p}, P)$ to mean that the image of a in $\text{ff}(A/\mathfrak{p})$ is zero, resp., in P, resp., in P but not zero.

Theorem 7.1. *Suppose $c, d \in A$. Then $\overline{d} \neq 0 \Rightarrow \overline{c} > 0$ holds on X_T iff $-d^{2^\ell k} \in T - \sum A^{2^\ell} c$ for some integer $k \geq 0$.*

Proof. Let $B = S^{-1}A$, $T' = S^{-1}T$, where $S := \{d^{2^\ell k} \mid k \geq 0\}$, and consider the $\sum A^{2^\ell}$-module $T - \sum A^{2^\ell} c$ and the $\sum B^{2^\ell}$-module $T' - \sum B^{2^\ell} c$. If $-S \cap (T - \sum A^{2^\ell} c) = \emptyset$, then $-1 \notin T' - \sum B^{2^\ell} c$, so there is a $\sum B^{2^\ell}$-module M in B containing $T' - \sum B^{2^\ell} c$ and maximal subject to $-1 \notin M$. By Theorem 5.1, $\mathfrak{q} := M \cap -M$ is a prime ideal of B. Also, $T' \subseteq M$, so $(T' + \mathfrak{q}) \cap -(T' + \mathfrak{q}) = \mathfrak{q}$. It follows that the preordering $T'' := \{(a + \mathfrak{q})/(b + \mathfrak{q}) \mid a, b \in T', b \notin \mathfrak{q}\}$ is a proper preordering of the residue hyperfield of B at $\mathfrak{q}$, and this residue hyperfield, denote it by F, obviously coincides with the residue hyperfield of A at $\mathfrak{p}$ where $\mathfrak{p} :=$ the inverse image of $\mathfrak{q}$ under the natural multiring homomorphism $A \to S^{-1}A = B$. Also, $\mathfrak{p}$ is T-convex and T'' is the extension of T to F. Since $d \notin \mathfrak{q}$ (d is invertible in B), it follows from our assumption that $c + \mathfrak{q} > 0$ at P for all orderings P of F containing T''. According to Theorem 4.2, this implies $c + \mathfrak{q} \in T''$. This yields elements $s, t \in T' + \mathfrak{q}$ with $s, t \notin \mathfrak{q}$ such that $sc = t$. Then $s^{2^\ell - 1} t \in T' + \mathfrak{q} \subseteq M$ and $s^{2^\ell - 1} t = s^{2^\ell} c \subseteq \sum B^{2^\ell} c \subseteq -M$, so $s^{2^\ell - 1} t \in M \cap -M = \mathfrak{q}$, a contradiction. □

Corollary 7.2. *$-1 \notin T$ iff $X_T \neq \emptyset$.*

Proof. Apply Theorem 7.1 with $c = 0$, $d = 1$ to deduce that $X_T = \emptyset$ iff $-1 \in T$. □

Corollary 7.3.

(1) *$\overline{a} = 0$ on X_T iff $-a^{2^\ell k} \in T$ for some $k \geq 0$.*
(2) *$\overline{a} > 0$ on X_T iff $-1 \in T - \sum A^{2^\ell} a$.*
(3) *$\overline{a} \geq 0$ on X_T iff $-a^{2^\ell k} \in T - \sum A^{2^\ell} a$ for some $k \geq 0$.*

(4) *Fix* $a \in b^{2^\ell} + c^{2^\ell}$. *Then* $\overline{b} = \overline{c}$ *on* X_T *iff* $-a^{2^\ell k} \in T - \sum A^{2^\ell} bc^{2^\ell - 1}$ *for some* $k \geq 0$.

Proof. Apply Theorem 7.1 as follows: (1) Take $c = 0$, $d = a$. (2) Take $c = a$, $d = 1$. (3) Take $c = a$, $d = a$. (4) Take $c = bc^{2^\ell - 1}$, $d = a$. □

8. ℓ-Real ideals

The results on real ideals in [16, Section 6] carry over to higher level. An ideal I of a multiring A is said to be *ℓ-real* if $(\sum a_i^{2^\ell}) \cap I \neq \emptyset \Rightarrow a_i \in I$ for each i. Every ℓ-real ideal is *radical* in the sense that $a^2 \in I \Rightarrow a \in I$, i.e., I is the intersection of prime ideals of A, by [16, Proposition 2.4]. The converse is not true in general.

Proposition 8.1. *For a prime ideal* $\mathfrak{p}$ *of a multiring* A*, the following are equivalent:*

(1) $\mathfrak{p}$ *is* ℓ*-real.*
(2) $\mathfrak{p}$ *is* $\sum A^{2^\ell}$*-convex.*
(3) *The residue hyperfield of* A *at* $\mathfrak{p}$ *is* ℓ*-real.*
(4) $\mathfrak{p}$ *is the support of some ordering of level* ℓ *of* A*.*

Proof. This is clear. See Theorem 4.1. □

The *ℓ-real radical* of an ideal I of A is

$$\sqrt[\ell,R]{I} := \{a \in A \mid \exists b_i \in A \text{ and } k \geq 0 \text{ such that } (a^{2^\ell k} + \sum b_i^{2^\ell}) \cap I \neq \emptyset\}.$$

Proposition 8.2. $\sqrt[\ell,R]{I}$ *is equal to the intersection of all* ℓ*-real prime ideals of* A *containing* I*.*

Proof. One inclusion is clear. For the other inclusion, use Corollary 7.3(1). Suppose $a \in \mathfrak{p}$ for each ℓ-real prime $\mathfrak{p}$ with $I \subseteq \mathfrak{p}$. consider $T := \sum A^{2^\ell} + I$ (the preordering of level ℓ of A generated by I). Then $\overline{a} = 0$ on X_T so, by Corollary 7.3(1), $-a^{2^\ell k} \in T$ for some $k \geq 0$. Then $(a^{2^\ell k} + \sum b_j^{2^\ell}) \cap I \neq \emptyset$ for some b_j, so $a \in \sqrt[\ell,R]{I}$ □

Proposition 8.3. *For an ideal* I *of a multiring* A*, the following are equivalent:*

(1) I *is* ℓ*-real.*
(2) $\sqrt[\ell,R]{I} = I$.
(3) I *is an intersection of* ℓ*-real prime ideals.*
(4) I *is radical and every minimal prime ideal over* I *is* ℓ*-real.*

Proof. Clearly (1) ⇔ (2). (2) ⇔ (3) by Proposition 8.2. If I is radical, then I is the intersection of the minimal prime ideals over I, so (4) ⇒ (3). It remains to show (3) ⇒ (4). Suppose $\mathfrak{q}$ is a minimal prime ideal over I which is not ℓ-real. Thus for each ℓ-real prime ideal $\mathfrak{p}$ lying over I there exists $a_\mathfrak{p} \in \mathfrak{p}$ with $a_\mathfrak{p} \notin \mathfrak{q}$. By the compactness of $\text{Sper}_\ell(A)$ in the patch topology (see Proposition 6.1), there exist finitely many elements $a_1, \dots, a_n$ of A such that $a_i \notin \mathfrak{q}$ for all i, and for each ℓ-real prime $\mathfrak{p}$ lying over I, $a_i \in \mathfrak{p}$, for some i. Let $a = a_1 \dots a_n$. Then $a \in \mathfrak{p}$ for each ℓ-real prime $\mathfrak{p}$ lying over I, so, by (3), $a \in I$. This contradicts $a \notin \mathfrak{q}$. □

A multiring A with $1 \neq 0$ is said to be *ℓ-real* if the ideal $\{0\}$ of A is ℓ-real. If I is an ℓ-real proper ideal of A, then A/I is ℓ-real. In particular, if $-1 \notin \sum A^{2^\ell}$, then $A/\sqrt[\ell,R]{\{0\}}$ is ℓ-real.

The notion of the ℓ-real radical of an ideal can be extended as follows: Let T be a preordering of level ℓ of a multiring A. The T-*radical* of an ideal I of A is defined to be

$$\sqrt[T]{I} := \{a \in A \mid \exists\, k \geq 0 \text{ such that } (a^{2^\ell k} + T) \cap I \neq \emptyset\}.$$

Proposition 8.4. *$\sqrt[T]{I}$ is the intersection of all T-convex prime ideals of A containing I.*

Proposition 8.5. *For a preordering T of level ℓ of a multiring A and an ideal I of A, the following are equivalent:*

(1) *I is radical and T-convex.*
(2) *$\sqrt[T]{I} = I$.*
(3) *I is an intersection of T-convex prime ideals.*
(4) *I is radical and every minimal prime ideal over I is T-convex.*

The proofs of Propositions 8.4 and 8.5 are simple modification of the proofs of Propositions 8.2 and 8.3.

9. ℓ-Real reduced multirings and hyperfields

The aim of this section is to extend to higher level the theory of real reduced multirings and real reduced hyperfields developed in [16]

9.1. **Hyperfield Case.** Recall that a hyperfield F is said to be ℓ-real if $-1 \notin \sum F^{2^\ell}$. Suppose F is an ℓ-real hyperfield, $\ell \geq 1$. For any proper preordering T of F of level ℓ, we can build the quotient hyperfield $Q_T(F) := F/_m T^*$ defined in Section 2. In particular, we can build $Q_{\sum F^{2^\ell}}(F)$, which we denote simply by $Q_{\ell\text{-red}}(F)$. If T_1, T_2 are level ℓ preorderings of F with $T_1 \subseteq T_2$ then the multiring homomorphism $F \to Q_{T_2}(F)$ factors through $Q_{T_1}(F)$.

Proposition 9.1. *For an ℓ-real hyperfield F the following are equivalent:*

(1) *The multiring homomorphism $F \to Q_{\ell\text{-red}}(F)$ is an isomorphism.*
(2) *$\sum F^{2^\ell} = \{0, 1\}$.*
(3) *$1 + 1 = \{1\}$ and $a^{2^\ell+1} = a$ for all $a \in F$.*

Proof. Assume (3). Then $a^{2^\ell} = 1$ if $a \neq 0$ and, by induction on n, 1 is the only element in the set $1 + \cdots + 1$ (n times) for any $n \geq 1$. It follows that $\sum F^{2^\ell} = F^{2^\ell} = \{0, 1\}$. Everything else is clear. □

A *ℓ-real reduced* hyperfield is defined to be an ℓ-real hyperfield satisfying the equivalent conditions of Proposition 9.1.

Proposition 9.2. *For a hyperfield F the following are equivalent:*

(1) *F is ℓ-real reduced.*
(2) *$1 + 1 = \{1\}$ and $a^{2^\ell+1} = a$ for all $a \in F$.*

Proof. Assume (2). As explained in the proof of Proposition 9.1, this implies $\sum F^{2^\ell} = \{0, 1\}$. If $-1 \in \{0, 1\}$, then $-1 = 0$, so $1 = 0$, or $-1 = 1$ so $0 \in 1+1$ which, by (2), implies $1 = 0$. This contradicts $1 \neq 0$. This proves that $-1 \notin \sum F^{2^\ell}$, so F is ℓ-real, and F is an ℓ-real reduced hyperfield by Proposition 9.1. The converse is clear. □

For any proper preordering T of level ℓ of an ℓ-real hyperfield F, $Q_T(F)$ is an ℓ-real reduced hyperfield. In particular, $Q_{\ell\text{-red}}(F)$ is an ℓ-real reduced hyperfield. If $p : F_1 \to F_2$ is a multiring homomorphism of ℓ-real hyperfields, then $p(\sum F_1^{2^\ell}) \subseteq \sum F_2^{2^\ell}$, so p induces a multiring homomorphism $Q_{\ell\text{-red}}(F_1) \to Q_{\ell\text{-red}}(F_2)$. In this way, $Q_{\ell\text{-red}}$ defines a functor (a reflection) from the category of ℓ-real hyperfields onto the subcategory of ℓ-real reduced hyperfields.

9.2. **Multiring Case.** Let A be a multiring such that $-1 \notin \sum A^{2^\ell}$. Fix a proper preordering T of A of level ℓ. For $a \in A$, denote by $\overline{a}$ the sign of a on X_T. For a prime ideal $\mathfrak{p}$ of A, denote by $a(\mathfrak{p})$ the image of $a \in A$ under the natural multiring homomorphism $A \to \operatorname{ff}(A/\mathfrak{p})$.

Theorem 9.3. *If $b \in \sum Ta_1 + \cdots + \sum Ta_n$ then $b(\mathfrak{p}) \in T_\mathfrak{p} a_1(\mathfrak{p}) + \cdots + T_\mathfrak{p} a_n(\mathfrak{p})$ for each T-convex prime $\mathfrak{p}$. Conversely, if $b(\mathfrak{p}) \in T_\mathfrak{p} a_1(\mathfrak{p}) + \cdots + T_\mathfrak{p} a_n(\mathfrak{p})$ for each T-convex prime $\mathfrak{p}$, then there exists $k, m \geq 0$ and $c \in b^{2^\ell m} + T$ such that $b^{2^\ell k+1} c \in \sum Ta_1 + \cdots + \sum Ta_n$.*

Proof. Assume $b(\mathfrak{p}) \in T_\mathfrak{p} a_1(\mathfrak{p}) + \cdots + T_\mathfrak{p} a_n(\mathfrak{p})$ for each T-convex prime $\mathfrak{p}$ of A. Let $B = S^{-1}A$ and $T' = S^{-1}T$ where $S := \{b^{2^\ell k} \mid k \geq 0\}$.

Claim 1. $-1 \in T' - \sum T' \frac{a_1}{b} - \cdots - \sum T \frac{a_n}{b}$. Suppose this is false. Choose a T'-module M of B containing $T' - \sum T' \frac{1_1}{b} - \cdots - \sum T \frac{a_n}{b}$ and maximal subject to $-1 \notin M$. According to Theorem 5.1, $\mathfrak{q} := M \cap -M$ is a prime ideal of B. Obviously $\mathfrak{q}$ is T' convex. The residue field of B at $\mathfrak{q}$ coincides with the residue field of A at $\mathfrak{p}$ where $\mathfrak{p}$ is the inverse image of $\mathfrak{q}$ under the natural multiring homomorphism $A \to S^{-1}A = B$. Also, $\mathfrak{p}$ is T-convex. This is clear. By the hypothesis, $t^{2^\ell} b \in \sum Ta_1 + \cdots + \sum Ta_n + x$ for some $t \in A \backslash \mathfrak{p}$, $x \in \mathfrak{p}$. Dividing by $-b$, this implies $-t^{2^\ell} \in M$, so $t^{2^\ell} \in M \cap -M = \mathfrak{q}$, i.e., $t \in \mathfrak{q}$, so $t \in \mathfrak{p}$, a contradiction.

From Claim 1 it follows that $bc' \in \sum T'a_1 + \cdots + \sum T'a_n$ for some $c' \in 1 + T'$. Clearing fractions, $\exists\ c \in b^{2^\ell m} + T$ such that $b^{2^\ell k+1} c \in \sum Ta_1 + \cdots + \sum Ta_n$, for some $k, m \geq 0$. This proves the second assertion. The first assertion is obvious. □

It is important to note that the element $b' = b^{2^\ell k+1} c$ defined in the statement of Theorem 9.3 satisfies $\overline{b'} = \overline{b}$. Also, we have the following:

Corollary 9.4. *If $\overline{a_i'} = \overline{a_i}$, $i = 1, \ldots, n$ and $b \in \sum Ta_1 + \cdots + \sum Ta_n$, then $\exists$ $b' \in \sum Ta_1' + \cdots + \sum Ta_n'$ such that $\overline{b'} = \overline{b}$.*

Proof. Compare to [16, Lemma 7.1.1(4)]. If $\overline{a'} = \overline{a}$ then, for each T-convex prime $\mathfrak{p}$, a' and a have the same sign at each support $\mathfrak{p}$ ordering of X_T so, by Theorem 4.2, $T_\mathfrak{p} a'(\mathfrak{p}) = T_\mathfrak{p} a(\mathfrak{p})$. The result is now immediate, using Theorem 9.3 in conjunction with this observation. □

As in [16] we denote the set $\{\overline{b} \mid b \in \sum Ta_1 + \cdots + \sum Ta_n\}$ by $D\langle \overline{a_1}, \ldots, \overline{a_n} \rangle$. Corollary 9.4 implies that $D\langle \overline{a_1}, \ldots, \overline{a_n} \rangle$ is well-defined.

Lemma 9.5. *For $1 \leq k < n$,*

$$D\langle \overline{a_1}, \ldots, \overline{a_n} \rangle = \bigcup \{ D\langle \overline{y}, \overline{z} \rangle \mid \overline{y} \in D\langle \overline{a_1}, \ldots, \overline{a_k} \rangle,\ \overline{z} \in D\langle \overline{a_{k+1}}, \ldots, \overline{a_n} \rangle \}.$$

Proof. If $x \in \sum Ta_1 + \cdots + \sum Ta_n$, then $x \in y + z$, $y \in \sum Ta_1 + \cdots + \sum Ta_k$, $z \in \sum Ta_{k+1} + \cdots + \sum Ta_n$. Since $y \in \sum Ty$ and $z \in \sum Tz$, this proves the inclusion ($\subseteq$). Suppose $\overline{x} \in D\langle \overline{y}, \overline{z} \rangle$, $\overline{y} \in D\langle \overline{a_1}, \ldots, \overline{a_k} \rangle$, $\overline{z} \in D\langle \overline{a_{k+1}}, \ldots, \overline{a_n} \rangle$. Thus

$\exists\, y', z'$ with $\overline{y'} = \overline{y}$, $\overline{z'} = \overline{z}$, $y' \in \sum Ta_1 + \cdots + \sum Ta_k$, $z' \in \sum Ta_{k+1} + \cdots + \sum Ta_n$. By Corollary 9.4, $\exists\, x'$ with $\overline{x'} = \overline{x}$ and $x' \in \sum Ty' + \sum Tz'$. Thus $x' \in \sum Ta_1 + \cdots + \sum Ta_n$. so $\overline{x} \in D\langle \overline{a_1}, \dots, \overline{a_n}\rangle$. This proves the inclusion ($\supseteq$). □

We will be applying Lemma 9.5 in the case $n = 3$. We also need the following basic result:

Lemma 9.6. *The following are equivalent:*

(1) $\exists\, a_i' \in A$ *such that* $0 \in a_0' + \cdots + a_n'$ *and* $\overline{a_i'} = \overline{a_i}$ *for* $i = 0, \dots, n$.
(2) $-\overline{a_i} \in D\langle \overline{a_1}, \dots, \overline{a_{i-1}}, \overline{a_{i+1}}, \dots, \overline{a_n}\rangle$ *for* $i = 0, \dots, n$.

Proof. See [16, Lemma 7.2]. (1) $\Rightarrow$ (2). By symmetry it suffices to show $\overline{-a_0} \in D\langle \overline{a_1}, \dots, \overline{a_n}\rangle$. Since $0 \in a_0' + \cdots + a_n'$, $-a_0' \in a_1' + \cdots + a_n'$, so $\overline{-a_0} = \overline{-a_0'} \in D\langle \overline{a_1'}, \dots, \overline{a_n'}\rangle = D\langle \overline{a_1}, \dots, \overline{a_n}\rangle$, using Corollary 9.4. (2) $\Rightarrow$ (1). We have a_i' with $\overline{a_i'} = \overline{a_i}$ such that $0 \in a_i' + \sum_{j\neq i} \sum Ta_j$. Then $0 \in 0 + \cdots + 0 \subseteq \sum_{i=0}^n (a_i' + \sum_{j\neq i}\sum Ta_j) = \sum_{i=0}^n (a_i' + \sum Ta_i)$, so there exists $a_i'' \in a_i' + \sum Ta_i$ such that $0 \in a_0'' + \cdots + a_n''$. Clearly $\overline{a_i''} = \overline{a_i}$. □

We define $Q_T(A) := \{\overline{a} \mid a \in A\}$. Here, $\overline{a}$ denotes the T-sign of a. Addition on $Q_T(A)$ is defined by $\overline{a} \in \overline{b} + \overline{c}$ iff $\exists\, a', b', c' \in A$ such that $\overline{a'} = \overline{a}$, $\overline{b'} = \overline{b}$, $\overline{c'} = \overline{c}$, and $a' \in b' + c'$. Multiplication on $Q_T(A)$ is defined by $\overline{a}\overline{b} = \overline{ab}$. Further, $0 := \overline{0}$, $1 := \overline{1}$, and $-\overline{a} := \overline{-a}$.

Theorem 9.7. *Suppose T is a proper preordering of A of level ℓ. Then*

(1) *$Q_T(A)$ is a multiring.*
(2) *$Q_T(A)$ is strongly embedded in the product multiring $\prod_{\mathfrak{p}} Q_{T_{\mathfrak{p}}}(\mathrm{ff}(A/\mathfrak{p}))$, product taken over all T-convex prime ideals of A.*

Note:
(1) Theorem 9.7 extends what is proved in level 1 in [16, Proposition 7.3].
(2) See Section 2 for the definition of strongly embedded. In concrete terms, part (2) of Theorem 9.7 is just saying that if $a, b, c \in A$ are such that $\overline{c(\mathfrak{p})} \in \overline{a(\mathfrak{p})} + \overline{b(\mathfrak{p})}$ for each T-convex prime $\mathfrak{p}$ of A, then $\overline{c} \in \overline{a} + \overline{b}$.

Proof. (1). See [16, Proposition 7.3(1)]. Everything is clear except the associativity of $+$. Suppose $x, u, v, w, p \in A$ are such that $\overline{p} \in \overline{u} + \overline{v}$ and $\overline{x} \in \overline{p} + \overline{w}$. Then $\overline{x} \in D\langle \overline{p}, \overline{w}\rangle$ and $\overline{p} \in D\langle \overline{u}, \overline{v}\rangle$, so $\overline{x} \in D\langle \overline{u}, \overline{v}, \overline{w}\rangle$, by Lemma 9.5. Also, $-\overline{u} \in -\overline{p} + \overline{v}$ and $-\overline{p} \in -\overline{x} + \overline{w}$, so $-\overline{u} \in D\langle -\overline{p}, \overline{v}\rangle$ and $-\overline{p} \in D\langle -\overline{x}, \overline{w}\rangle$, so $-\overline{u} \in D\langle -\overline{x}, \overline{v}, \overline{w}\rangle$, again by Lemma 9.5. Similarly, $-\overline{v} \in \overline{u} - \overline{p}$ and $-\overline{p} \in -\overline{x} + \overline{w}$, so $-\overline{v} \in D\langle -\overline{x}, \overline{u}, \overline{w}\rangle$, and $-\overline{w} \in \overline{p} - \overline{x}$ and $\overline{p} \in \overline{u} + \overline{v}$, so $-\overline{w} \in D\langle -\overline{x}, \overline{u}, \overline{v}\rangle$. According to Lemma 9.6 this implies there exist $x', u', v', w' \in A$ such that $\overline{x'} = \overline{x}$, $\overline{u'} = \overline{u}$, $\overline{v'} = \overline{v}$, $\overline{w'} = \overline{w}$, and $x' \in u' + v' + w'$. Pick $q \in v' + w'$ such that $x' \in u' + q$. Then $\overline{q} \in \overline{v} + \overline{w}$ and $\overline{x} \in \overline{u} + \overline{q}$. This completes the proof of (1).

(2). Let $a, b, c \in A$ and suppose $\overline{c(\mathfrak{p})} \in \overline{a(\mathfrak{p})} + \overline{b(\mathfrak{p})}$ holds for each T-convex prime $\mathfrak{p}$ of A. We want to show $\overline{c} \in \overline{a} + \overline{b}$. By hypothesis we have $s_{\mathfrak{p}}, t_{\mathfrak{p}}, u_{\mathfrak{p}} \in T_{\mathfrak{p}}^*$ such that $c(\mathfrak{p})s_{\mathfrak{p}} \in a(\mathfrak{p})t_{\mathfrak{p}} + b(\mathfrak{p})u_{\mathfrak{p}}$. This implies, in particular, that $c(\mathfrak{p}) \in a(\mathfrak{p})T_{\mathfrak{p}} + b(\mathfrak{p})T_{\mathfrak{p}}$, $-a(\mathfrak{p}) \in -c(\mathfrak{p})T_{\mathfrak{p}} + b(\mathfrak{p})T_{\mathfrak{p}}$, and $-b(\mathfrak{p}) \in -c(\mathfrak{p})T_{\mathfrak{p}} + a(\mathfrak{p})T_{\mathfrak{p}}$, for each T-convex prime $\mathfrak{p}$ of A, so $\overline{c} \in D\langle \overline{a}, \overline{b}\rangle$, $-\overline{a} \in D\langle -\overline{c}, \overline{b}\rangle$, and $-\overline{b} \in D\langle -\overline{c}, \overline{a}\rangle$, by Theorem 9.3. According to Lemma 9.6 this implies $\overline{c} \in \overline{a} + \overline{b}$. □

Remark 9.8. Theorem 9.7(2) can be strengthened a bit. For any $a_0,\dots,a_n\in A$, if $0\in\overline{a_0(\mathfrak{p})}+\cdots+\overline{a_n(\mathfrak{p})}$ for each T-convex prime $\mathfrak{p}$, then $0\in\overline{a_0}+\cdots+\overline{a_n}$. As in the proof of Theorem 9.7(2), the ingredients of the proof are Theorem 9.3 and Lemma 9.6.

We denote the multiring $Q_{\sum A^{2^\ell}}(A)$ by $Q_{\ell\text{-red}}(A)$ for short.

Theorem 9.9. *For a multiring A with $-1\notin\sum A^{2^\ell}$, the map $a\mapsto\overline{a}$ from A onto $Q_{\ell\text{-red}}(A)$ is an isomorphism iff A satisfies the following three properties (for all $a,b\in A$):*

(1) $a^{2^\ell+1}=a$.
(2) $a+ab^{2^\ell}=\{a\}$.
(3) $a^{2^\ell}+b^{2^\ell}$ *contains a unique element.*

Proof. One direction is more or less trivial. We know that $\overline{a}^{2^\ell+1}=\overline{a}$, $\overline{a}+\overline{a}\overline{b}^{2^\ell}=\{\overline{a}\}$, and $\overline{a}^{2^\ell}+\overline{b}^{2^\ell}$ contains a unique element, e.g., by applying Theorem 9.7(2) to reduce to the case where A is a hyperfield, so if $a\mapsto\overline{a}$ is an isomorphism, then (1), (2) and (3) obviously hold.

We focus now on the non-trivial direction. If $c\in a^{2^\ell}+b^{2^\ell}$, then $c^2\in(a^{2^\ell}+b^{2^\ell})(a^{2^\ell}+b^{2^\ell})\subseteq a^{2^\ell}+a^{2^\ell}b^{2^\ell}+a^{2^\ell}b^{2^\ell}+b^{2^\ell}=(a^{2^\ell}+a^{2^\ell}b^{2^\ell})+(b^{2^\ell}+a^{2^\ell}b^{2^\ell})$ Since a^{2^ℓ} is the unique element of $a^{2^\ell}+a^{2^\ell}b^{2^\ell}$ and $b^{2\ell}$ is the unique element of $b^{2^\ell}+a^{2^\ell}b^{2^\ell}$, the implies $c^2\in a^{2^\ell}+b^{2^\ell}$. Consequently, $c^2=c$, i.e., $c=c^2=c^3=\dots$, i.e., the unique element of $a^{2^\ell}+b^{2^\ell}$ is in A^{2^ℓ}. It follows by induction that, for any $a_1,\dots,a_n\in A$, $a_1^{2^\ell}+\cdots+a_n^{2^\ell}$ contains a unique element, which is an element of A^{2^ℓ}. In particular, $\sum A^{2^\ell}=A^{2^\ell}$.

Take $T=\sum A^{2^\ell}=A^{2^\ell}$. Suppose $\overline{a}=\overline{b}$. Let $c\in a^{2^\ell}+b^{2^\ell}$. Thus, by Corollary 7.3, $-c^{2^\ell k}\in A^{2^\ell}-\sum A^{2^\ell}ab^{2^\ell-1}$. Since $c^2=c$, $c^{2^\ell k}=c^{2^\ell}$. Thus, there exists $d\in\sum A^{2^\ell}ab^{2^\ell-1}$ with $d\in c^{2^\ell}+A^{2^\ell}$. Then $ac\in a(a^{2^\ell}+b^{2^\ell})\subseteq a+ab^{2^\ell}=\{a\}$, so $ac=a$. Similarly, $bc=b$ and $cd=c$. Thus $ad=(ac)d=a(cd)=ac=a$ and, similarly, $bd=b$. Say $d\in\sum e_i^{2^\ell}ab^{2^\ell-1}$. Then $a^{2^\ell-1}b=a^{2^\ell-1}bd\in\sum e_i^{2^\ell}a^{2^\ell}b^{2^\ell}\subseteq A^{2^\ell}$. This implies $a^{2^\ell-1}b\in A^{2^\ell}$, so $a^{2^\ell-1}b=(a^{2^\ell-1}b)^{2^\ell}=a^{2^\ell}b^{2^\ell}$. By symmettry, we also have $ab^{2^\ell-1}=a^{2^\ell}b^{2^\ell}$. Thus $a^{2^\ell}=a^{2^\ell}d\in a^{2^\ell}\sum e_i^{2^\ell}ab^{2^\ell-1}=a^{2\ell}\sum e_i^{2^\ell}a^{2^\ell}b^{2^\ell}\subseteq\sum e_i^{2^\ell}a^{2^\ell}b^{2^\ell}$. Since $\sum e_i^{2^\ell}a^{2^\ell}b^{2^\ell}$ is a singleton set, this implies $a^{2^\ell}=a^{2^\ell-1}b$. By symmetry, we also have $b^{2^\ell}=ab^{2^\ell-1}$. Then $a^{2^\ell}=b^{2^\ell}$ and $a=a^{2^\ell+1}=aa^{2^\ell}=ab^{2^\ell}=ab^{2^\ell-1}b=b^{2^\ell}b=b^{2^\ell+1}=b$. □

A multiring A satisfying $-1\notin\sum A^{2^\ell}$ and the equivalent conditions of Proposition 9.9 will be called an *ℓ-real reduced multiring.*

Corollary 9.10. *A non-zero multiring A is an ℓ-real reduced multiring iff the following properties hold (for all $a,b\in A$):*

(1) $a^{2^\ell+1}=a$.
(2) $a+ab^{2^\ell}=\{a\}$.
(3) $a^{2^\ell}+b^{2^\ell}$ *contains a unique element.*

Proof. As noted above, (1), (2), (3) imply $\sum A^{2^\ell}=A^{2^\ell}$. If $-1\in\sum A^{2^\ell}$, then $-1=a^{2^\ell}$ for some a, so $0\in 1+a^{2^\ell}$. By (2), $0=1$. This contradicts $A\neq\{0\}$.

Thus $-1 \notin \sum A^{2^\ell}$. Now apply Theorem 9.9 to conclude that A is an ℓ-real reduced multiring. The converse is obvious. □

Our earlier remarks for hyperfields carry over to multirings. For any multiring A and any proper preordering T of level ℓ of A, $Q_T(A)$ is an ℓ-real reduced multiring. This is clear. By Theorem 9.7(2) one is reduced to the case where A is a hyperfield. In particular, for any multiring A, if $-1 \notin \sum A^{2^\ell}$ then $Q_{\ell\text{-red}}(A)$ is an ℓ-real reduced multiring. If $p : A_1 \to A_2$ is a multiring homomorphism, where A_1, A_2 are multirings satisfying $-1 \notin \sum A_i^{2^\ell}$, $i = 1, 2$, then p induces a multiring homomorphism $Q_{\ell\text{-red}}(A_1) \to Q_{\ell\text{-red}}(A_2)$. In this way, $Q_{\ell\text{-red}}$ defines a functor (a reflection) from the category of multirings satisfying $-1 \notin \sum A^{2^\ell}$ onto the subcategory of ℓ-real reduced multirings.

Proposition 9.11. *If A is an ℓ-real reduced multiring then*

1. *Every ideal of A is ℓ-real.*
2. *Every factor multiring of A by an ideal I is ℓ-real reduced.*
3. *Every localization of A at a multiplicative set S is ℓ-real reduced.*
4. *Every residue hyperfield of A at a prime ideal is ℓ-real reduced.*

Actually, for (2) and (3) to be true, it is necessary to enlarge the class of ℓ-real reduced multirings to include the zero multiring. Or, if we don't want to do this, then it is necessary to require the ideal I considered in (2) to be proper, and the multiplicative set S considered in (3) to not contain 0.

Proof. (1) Suppose $\sum_{i=1}^n a_i^{2^\ell} \cap I \neq \emptyset$, where I is an ideal of A. We want to show $a_1 \in I$. Arguing as in the proof of Theorem 9.9, we know that $\sum_{i=2}^n a_i^{2^\ell}$ is a singleton element, say $\sum_{i=2}^n a_i^{2^\ell} = b^{2^\ell}$. Suppose $c \in I$ is the unique element of $\sum_{i=1}^n a_i^{2^\ell} = a_1^{2^\ell} + b^{2^\ell}$. Then $a_1^{2^\ell} c \in a_1^{2^\ell}(a_1^{2^\ell} + b^{2^\ell}) \subseteq a_1^{2^\ell} + a_1^{2^\ell} b^{2^\ell}$. But we know by property (2) of ℓ-real reduced multirings that $a_1^{2^\ell}$ is the unique element of $a_1^{2^\ell} + a_1^{2^\ell} b^{2^\ell}$, so this implies $a_1^{2^\ell} = a_1^{2^\ell} c \in I$, so $a_1 = a_1 a_1^{2^\ell} \in I$.

The proofs of (2) and (3) are straightforward and will be omitted. (4) follows by combining (2) and (3). □

For a multiring A with $-1 \notin \sum A^{2^\ell}$, and a proper preordering T of level ℓ of A, the primes of $Q_T(A)$ are the images under $A \to Q_T(A)$ of the supports of orderings of X_T, equivalently, the images under $A \to Q_T(A)$ of the T-convex primes of A, and $\mathrm{Sper}_\ell(Q_T(A))$ is identified naturally with X_T. In particular, the primes in $Q_{\ell\text{-red}}(A)$ are the images under $A \to Q_{\ell\text{-red}}(A)$ of the real primes of A, and $\mathrm{Sper}_\ell(Q_{\ell\text{-red}}(A))$ is identified naturally with $\mathrm{Sper}_\ell(A)$. If $\mathfrak{p}$ is a T-convex prime of A and $\overline{\mathfrak{p}}$ denotes the image of $\mathfrak{p}$ in $Q_T(A)$, and T' denotes the preordering of level ℓ of $\mathrm{ff}(A/\mathfrak{p})$ induced by T, then $\mathrm{ff}(Q_T(A)/\overline{\mathfrak{p}})$ is identified with the ℓ-real reduced hyperfield $Q_{T'}(\mathrm{ff}(A/\mathfrak{p}))$. In particular, if $\mathfrak{p}$ is a real prime of A and $\overline{\mathfrak{p}}$ is the image of $\mathfrak{p}$ in $Q_{\ell\text{-red}}(A)$, then $\mathrm{ff}(Q_{\ell\text{-red}}(A)/\overline{\mathfrak{p}})$ is identified with the the ℓ-real reduced hyperfield $Q_{\ell\text{-red}}(\mathrm{ff}(A/\mathfrak{p}))$.

Example 2.1 extends as follows:

Example 9.12. (1) Suppose A is an ℓ-real reduced multiring and there exist $\alpha, \beta \in \mathrm{Sper}_\ell(A)$ such that β is a proper specialization of α. This just means that

$$\beta \neq \alpha \text{ and } \forall\ a \in A,\ \beta \in U(a) \ \Rightarrow\ \alpha \in U(a).$$

Fix $x \in A$ such that $x < 0$ at α and $x = 0$ at β. We claim that $x(1+x) \neq x + x^2$. Any element of $1 + x$ is positive at β and consequently it is also positive at α, so any element of $x(1+x)$ is negative at α. At the same time, $x^2(x^{2^\ell-1}+1) \subseteq x^{2^\ell+1} + x^2 = x + x^2$ and any element of $x^2(x^{2^\ell-1}+1)$ is positive at α.

(2) It follows from (1) that a necessary condition for an ℓ-real reduced multiring to be a hyperring is that every point in $\mathrm{Sper}_\ell(A)$ is a closed point, i.e., the spectral topology and the patch topology coincide. It is not clear if this necessary condition is sufficient. But, anyway, it is clear that it doesn't happen very often. For example, if A is the coordinate ring of a real algebraic set $V \subseteq R^n$, R real closed, the spectral topology and the patch topology on $\mathrm{Sper}_\ell(Q_{\ell\text{-red}}(A)) = \mathrm{Sper}_\ell(A)$ coincide iff $\dim(V) = 0$, i.e., V is just a finite set of points, i.e., A is a product of finitely many copies of R [6, Proposition 7.5.6].

10. Relationship between orderings and signatures

Define $Q_{2^\ell} := \mu_{2^\ell} \cup \{0\}$ where μ_{2^ℓ} denotes the group of complex 2^ℓ-th roots of unity. Q_{2^ℓ} has natural hyperfield structure. For $s, t \in Q_{2^\ell}$, $s + t$ is defined as follows: If $t = 0$ then $s + t = s$. If $s = 0$ then $s + t = t$, If s, t are non-zero and $s \neq -t$ then $s + t = \{s, t\}$. If s, t are non-zero and $s = -t$, then $s + t = Q_{2^\ell}$. The rest of the hyperfield structure on Q_{2^ℓ} is clear. Observe that if $k \leq \ell$ then Q_{2^k} is strongly embedded in Q_{2^ℓ}, i.e., for $s, t, u \in Q_{2^k}$, $s \in t + u$ in Q_{s^k} iff $s \in t + u$ in Q_{2^ℓ}.

A *signature of level* ℓ on a hyperfield F is defined to be a multiring homomorphism $\sigma : F \to Q_{2^\ell}$. We say σ has *exact level* ℓ if σ is surjective.

If σ is a signature of level ℓ on a hyperfield F, then $P := \sigma^{-1}(\{0,1\})$ is an ordering of level ℓ on F. If σ has exact level ℓ then P has exact level ℓ. All of this is clear. If σ is a signature on F and k is an odd integer then σ^k, defined by $\sigma^k(a) := \sigma(a^k) = \sigma(a)^k$, is also a signature of F, and σ and σ^k have the same associated ordering. If σ has exact level ℓ, this yields $2^{\ell-1}$ signatures of F (corresponding to the $2^{\ell-1}$ automorphisms $x \mapsto x^k$, $1 \leq k \leq 2^\ell$, k odd, of Q_{2^ℓ}), all with the same associated ordering.

If P is an ordering on a hyperfield F then P is a proper preordering of F, so $P^* - P^* = F$, by Corollary 3.3. Scaling, we see that $aP^* + bP^* = F$ for any $a, b \in F^*$ such that $-\frac{a}{b} \in P^*$. In particular, $bP^* \subseteq aP^* - aP^*$, i.e., $aP^* \subseteq aP^* + bP^*$, and similarly $bP^* \subseteq aP^* + bP^*$, for any $a, b \in F^*$.

Proposition 10.1. *An ordering P on a hyperfield F arises from a signature on F iff the rigidity condition*

$$(1) \qquad \forall\, a, b \in F^* \text{ if } -\frac{a}{b} \notin P^* \text{ then } aP^* + bP^* = aP^* \cup bP^*$$

holds.

Proof. Suppose that $P = \sigma^{-1}(\{0,1\})$ for a signature $\sigma : F \to Q_{2^\ell}$. Then P is an ordering so, as explained already, $aP^* \cup bP^* \subseteq aP^* + bP^*$ holds for all $a, b \in F^*$. Using the fact that σ is a multiring homomorphism together with the definition of addition on the Q_{2^ℓ}, we see that the inclusion $aP^* + bP^* \subseteq aP^* \cup bP^*$ also holds for all $a, b \in F^*$ such that $-\frac{a}{b} \notin P^*$. Conversely, if P is any ordering of F satisfying (1), then $F/_m P^* \cong Q_{2^\ell}$ where ℓ is the exact level of P, so the composite map σ from F to Q_{2^ℓ} is a signature of exact level ℓ inducing P. $\square$

Proposition 10.2. *For an ordering P of a hyperfield F, the rigidity condition (1) holds if the level of P is 1 or 2, or if F is a field, or a hyperfield obtained by factoring a field by a preordering.*

Proof. The case of orderings of level 1 is covered by [16, Lemma 4.3], and the case of fields as well as hyperfields arising from factoring a field by a preordering is covered in [20, Theorem 5.5], so that it remains to show that the condition holds for an ordering P of level 2 in the hyperfield F.

Fix $a, b \in F^*$ with $-\frac{a}{b} \notin P^*$. Replacing F by $F/_m P^*$, we may assume $F = \{0\} \cup \{\pm 1, \pm x\}$, $x^2 = -1$, $P = \{0, 1\}$ and $1+1 = 1$. We want to show $a+b = \{a, b\}$. Since $P - P = F$, by Theorem 3.1, it follows that $1-1 = F$. This implies $a-a = F$ so $b \in a-a$, i.e., $a \in a+b$, and a similar argument shows that $b \in a+b$. Suppose now that $c \in a+b$, $c \neq a, b$. Scaling, we may suppose $a = 1$. Since $b \neq -a$ it follows that b is either 1, x or $-x$. If $b = 1$ then $c \in 1+1$, contradicting $1+1 = 1$, so $b = x$ or $b = -x$. Replacing x by $-x$ if necessary, we can assume $b = x$. Since $c \neq a, b$ it follows c is either -1 or $-x$. If $c = -1$ then $-1 \in 1+x$, i.e., $-x \in 1+1$, contradicting $1+1 = 1$. If $c = -x$, then $-x \in 1+x$, i.e., $-1 \in x+x$, i.e., $x \in 1+1$, contradicting $1+1 = 1$. □

The rigidity condition (1) fails in general for orderings of level ≥ 3.

Example 10.3. (1) Let $F = \{0\} \cup \{\pm 1, \pm a, \pm a^2, \pm a^3\}$, where $a^4 = -1$, with addition defined by

$1+1 = \{1\}$,
$1-1 = F$,
$1+a = \{1, a, -a^2, a^3\}$,
$1-a = \{1, -a\}$,
$1+a^2 = \{1, -a, a^2\}$,
$1-a^2 = \{1, -a^2, a^3\}$,
$1+a^3 = \{1, a^3\}$,
$1-a^3 = \{1, -a, a^2, -a^3\}$.

This addition is extended to all of F in the obvious way, i.e., $r+s := r(1+\frac{s}{r})$ if $r, s \neq 0$. One checks that, with this addition and with the obvious multiplication, F is a hyperfield, in fact it is a 3-real reduced hyperfield. The proof of this, especially the proof that the addition on F is associative, is quite tedious and will not be given here. The preordering $\{0, 1\}$ of F is obviously an ordering of F but it does not come from a signature, since the rigidity condition (1) fails.

(2) For $\ell > 3$, one can extend the 3-real reduced hyperfield F constructed in part (1) to an ℓ-real reduced hyperfield F' as follows: Take $F' = \{0\} \cup F'^*$ where F'^* is a cyclic group of order 2^ℓ extending F^*. For $x \in F'^*$ define $1+x = F'$ if $x = -1$, $1+x = \{1, x\}$ if $x \in F'^* \backslash F^*$ and define $1+x$ as in part (1) if $x \in F^*$, $x \neq -1$. For $r, s \in F'$, $r, s \neq 0$, define $r+s = r(1+\frac{s}{r})$. With this addition and with the obvious multiplication F' is an ℓ-real reduced hyperfield (this is a variant of the group extension construction in [20], [21], [22]), and $P = \{0, 1\}$ is an ordering of level ℓ of F' which does not come from a signature on F'.

Spaces of signatures of higher level are considered in sequence of papers by Mulcahy and Powers [20], [21], [22]. To every proper preordering T of level ℓ of a field F, one has an associated space of signatures (Sig_T, G_T) of level ℓ. (Actually, this works even if the exponent of F^*/T^* is not a 2-power.) The space of signatures

(Sig_T, G_T) carries exactly the same information as the ℓ-real reduced hyperfield $F/_mT^*$. The relationship goes as follows: $G_T = F^*/T^*$ and $\mathrm{Sig}_T =$ the set of signatures of $F/_mT^*$ (equivalently, the set of signatures of F lying over T). Addition on $F/_mT^*$ is recovered from the space of signatures (Sig_T, G_T) by the formula $a + b = D_T\langle a, b\rangle$, for all $a, b \in F/_mT^*$, $a, b \neq 0$.

More generally, to any (abstract) space of signatures (X, G) of level ℓ, one has the associated ℓ-real reduced hyperfield $F := G \cup \{0\}$ with addition defined by $a+b := D\langle a, b\rangle$, for all $a, b \in F$, $a, b \neq 0$, and, again, the space of signatures (X, G) can be recovered from F. Namely, $G = F^*$ and $X =$ the set of all signatures of F.

One can compute in a straightforward way all isomorphism classes of ℓ-real reduced hyperfields F with $|F^*| \leq 8$. There are ten of them altogether: four with $\ell = 1$, four with $\ell = 2$, and two with $\ell = 3$. One of the two with $\ell = 3$, namely, the one described in Example 10.3, does not correspond to a space of signatures. The remaining nine correspond to spaces of signatures.

The classification of finite spaces of signatures of higher level is carried out in [22], extending the classification of finite spaces of signatures of level 1 which was carried out earlier in [14], also see [15].

As explained already in [16], 1-real reduced hyperfields and spaces of signatures of level 1 are exactly the same thing. Example 10.3 provides us with an example of a 3-real reduced hyperfield which is not a space of signatures. It is natural to wonder what additional (first order) axioms are necessary on a ℓ-real reduced hyperfield to ensure that it is a space of signatures.

For example, if F is an ℓ-real reduced hyperfield coming from a space of signatures, then

$(*)$ For all odd integers $1 \leq k \leq 2^\ell$, $a \in b + c \Rightarrow a^k \in b^k + c^k$,

i.e., $a \mapsto a^k$ is a multiring automorphism of F, for each odd integer $1 \leq k \leq 2^\ell$. It is not known if this extra property $(*)$ is sufficient to ensure that F comes from a space of signatures. Probably not. However, we do at least have the following:

Proposition 10.4. *Suppose F is an ℓ-real reduced hyperfield satisfying $(*)$. Then every ordering of level ℓ of F comes from a signature.*

Proof. Let P be an ordering of F. Replacing F by $F/_mP^*$ we may assume $P = \{0, 1\}$. It suffices to show $a \neq -1 \Rightarrow 1 + a = \{1, a\}$. Let $b \in 1 + a$. We may assume $a \neq 1$. Suppose first that $\mathrm{ord}(a) > \mathrm{ord}(b)$. Say $\mathrm{ord}(a) = 2^n$. Then $a^{2^{n-1}} = -1$, $b^{2^{n-1}} = 1$. Let $k = 2^{n-1} + 1$. Then k is odd, so $b^k \in 1 + a^k$, i.e., $b \in 1 - a$. Then $a, -a \in 1 - b$, so $0 \in 1 - b$, i.e., $b = 1$. Suppose next that $\mathrm{ord}(a) = \mathrm{ord}(b)$. Taking $k = 2^\ell - 1$, we see that $\frac{1}{b} \in 1 + \frac{1}{a}$, i.e., $\frac{a}{b} \in 1 + a$. Noting that $\mathrm{ord}(\frac{a}{b}) < \mathrm{ord}(a)$, this yields $\frac{a}{b} = 1$, i.e., $b = a$, by our first case. Finally, if $\mathrm{ord}(a) < \mathrm{ord}(b)$ they we note that $-a \in 1 - b$, and reduce to the first case, but with a, b replaced with $-b, -a$. □

If A is a multiring we define a *signature of level ℓ on A* to be a multiring homomorphism $\sigma : A \to Q_{2^\ell}$. Every signature of level ℓ on A factors through $\mathrm{ff}(A/\mathfrak{p})$ where $\mathfrak{p}$ is the prime ideal of A defined by $\mathfrak{p} := \sigma^{-1}(\{0\})$. If A is ℓ-real reduced and $(*)$ holds for each residue hyperfield $\mathrm{ff}(A/\mathfrak{p})$ of A then $(*)$ also holds for A. This is clear.

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