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POLYNOMIALS NON-NEGATIVE ON A STRIP

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ABSTRACT. We prove that if $f(x,y)$ is a polynomial with real coefficients which is non-negative on the strip $[0,1]\times\mathbb{R}$, then $f(x,y)$ has a presentation of the form

$$f(x,y)=\sum_{i=1}^{k} g_i(x,y)^2+\sum_{j=1}^{\ell} h_j(x,y)^2 x(1-x),$$

where the $g_i(x,y)$ and $h_j(x,y)$ are polynomials with real coefficients.

1. Introduction

In [2] Hilbert showed that there are polynomials $f(x,y)\in\mathbb{R}[x,y]$ (necessarily of degree ≥ 6) which are non-negative on all of $\mathbb{R}^2$, but are not expressible as a sum of squares in $\mathbb{R}[x,y]$. The best-known example is the polynomial $f(x,y) = 1-3x^2y^2+x^4y^2+x^2y^4$ [6]. In contrast to this result, we prove:

Theorem 1.1. *Suppose $f(x,y)\in\mathbb{R}[x,y]$ is non-negative on the strip $[0,1]\times\mathbb{R}$. Then $f(x,y)$ is expressible as*

$$f(x,y)=\sigma(x,y)+\tau(x,y)x(1-x),$$

where $\sigma(x,y),\tau(x,y)$ are sums of squares in $\mathbb{R}[x,y]$.

This answers questions in [3] and [10] arising from the solution of the moment problem for cylinders with compact cross-section; see [3] and [16]. In [10] the authors claimed to know a proof of the result, but this claim was later withdrawn. Certain weak versions of the result were known already; see [3], [8] and [9].

A *preordering* of a ring A (commutative with 1) is a subset T of A satisfying $T+T\subseteq T$, $TT\subseteq T$, and $f^2\in T$ for all $f\in A$. The unique smallest preordering of A is $\sum A^2 :=$ the set of all (finite) sums of squares of elements of A. The preordering of A generated by finitely many elements $g_1,\cdots,g_s$ of A consists of all elements of the form $\sum_i \sigma_i g^i$, $\sigma_i\in\sum A^2$, $g^i := g_1^{i_1}\cdots g_s^{i_s}$, $i:=(i_1,\cdots,i_s)$ running through the set $\{0,1\}^s$.

A finitely generated preordering T of the polynomial ring $\mathbb{R}[x_1,\cdots,x_n]$ is said to be *saturated* if, for all $f\in\mathbb{R}[x_1,\cdots,x_n]$, $f\geq 0$ on $K_T \Rightarrow f\in T$. Here, $K_T := \{a\in\mathbb{R}^n \mid \forall\ g\in T\ g(a)\geq 0\}$. If $g_1,\cdots,g_s$ are generators of T, then K_T is the subset of $\mathbb{R}^n$ defined by the polynomial inequalities $g_i\geq 0$, $i=1,\cdots,s$.

By [12, Prop. 6.1], a finitely generated preordering T of $\mathbb{R}[x_1,\cdots,x_n]$ cannot be saturated if $\dim(K_T)\geq 3$. By [14, Th. 5.4], the same is true for $\dim(K_T)=2$, if

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T is stable. The preordering of $\mathbb{R}[x,y]$ consisting of sums of squares is stable, so the result of Hilbert referred to earlier can be seen as a special case of this latter result. Theorem 1.1 asserts that the preordering of $\mathbb{R}[x,y]$ generated by $x(1-x)$ is saturated. The identities $x = x^2+(1-x)^2$ and $1-x = x(1-x)+(1-x)^2$ imply that the preordering generated by $x(1-x)$ coincides with the preordering generated by x and $1-x$. Before the present paper was written, the only example of a finitely generated saturated preordering in the 2-dimensional non-compact case was the rather artificial example given in [15, Rem. 3.14] (the preordering of $\mathbb{R}[x,y]$ generated by $x, 1-x, y$ and $1-xy$). It is hoped that the techniques employed in the present paper will yield additional examples of this sort in the future. See [1] and [15] for examples of finitely generated saturated preorderings in the 2-dimensional compact case. See [3], [7], [12] and [13] for 1-dimensional examples.

2. Preliminary reductions

We assume $f \in \mathbb{R}[x,y]$, $f \geq 0$ on the strip $[0,1]\times\mathbb{R}$. We want to show f has a presentation $f = \sigma + \tau x(1-x)$, σ, τ sums of squares in $\mathbb{R}[x,y]$. By considering the behavior of f as $|y| \to \infty$, we see that f has even degree $2d$, as a polynomial in y, and that the leading coefficient is ≥ 0 on $[0,1]$, i.e., f has the form $f(x,y) = \sum_{i=0}^{2d} a_i(x)y^i$, $a_i(x) \in \mathbb{R}[x]$, $a_{2d}(x) \geq 0$ on $[0,1]$. If $d = 0$ the result is well-known, e.g., by [3, Th. 2.2] or [4, Prop. 2.7.3], so we assume always that $d \geq 1$.

Lemma 2.1. *We may assume $a_{2d}(x) > 0$ on $[0,1]$.*

Proof. Factor a_{2d} as $a_{2d} = \overline{a}\tilde{a}$ where $\overline{a}, \tilde{a} \in \mathbb{R}[x]$, $\overline{a}$ is > 0 on $[0,1]$ and $\tilde{a}$ is ± 1 times a product of linear factors $x-r$, $r \in [0,1]$. Then

$$\tilde{a}^{2d-1}f = \overline{a}(\tilde{a}y)^{2d} + a_{2d-1}(\tilde{a}y)^{2d-1} + \cdots + a_0\tilde{a}^{2d-1}.$$

Let $g := \overline{a}y^{2d} + a_{2d-1}y^{2d-1} + \cdots + a_0\tilde{a}^{2d-1}$. Using the fact that $\tilde{a} \geq 0$ on $[0,1]$, and the set of points (r,s) in the strip $[0,1]\times\mathbb{R}$ satisfying $\tilde{a}(r) \neq 0$ is dense in the strip, one sees that $g \geq 0$ on the strip. If we are able to show that $g = \sigma + \tau x(1-x)$, σ, τ sums of squares in $\mathbb{R}[x,y]$, then $\tilde{a}(x)^{2d-1}f(x,y) = \sigma(x, \tilde{a}(x)y) + \tau(x, \tilde{a}(x)y)x(1-x)$. Thus we are reduced to showing that if $b(x)f(x,y)$ has a presentation $b(x)f(x,y) = \sigma(x,y) + \tau(x,y)x(1-x)$ for some sums of squares $\sigma(x,y), \tau(x,y)$, where $b(x) \geq 0$ on the interval $[0,1]$, and $b(x)$ is ± 1 times a product of linear factors $x-r$, $r \in [0,1]$, then $f(x,y)$ also has such a presentation. The proof is by induction on the degree of $b(x)$. Suppose $x - r$ is a factor of $b(x)$, $0 \leq r \leq 1$. First suppose $0 < r < 1$. Then $b(x) = \overline{b}(x)(x-r)^2$. Also, $\sigma(x,y)$ and $\tau(x,y)$ vanish at $x = r$, so $\sigma(x,y) = \overline{\sigma}(x,y)(x-r)^2$, $\tau(x,y) = \overline{\tau}(x,y)(x-r)^2$, $\overline{\sigma}(x,y)$, $\overline{\tau}(x,y)$ sums of squares in $\mathbb{R}[x,y]$, and $\overline{b}(x)f(x,y) = \overline{\sigma}(x,y) + \overline{\tau}(x,y)x(1-x)$. If $r = 0$, then $b(x) = \overline{b}(x)x$ and $\sigma(x,y) = \overline{\sigma}(x,y)x^2$, $\overline{\sigma}(x,y)$ a sum of squares in $\mathbb{R}[x,y]$, and $\overline{b}(x)f(x,y) = \overline{\sigma}(x,y)x + \tau(x,y)(1-x)$. Using $x = x^2 + x(1-x)$ and $1 - x = (1-x)^2 + x(1-x)$ this yields $\overline{b}(x)f(x,y) = \sigma'(x,y) + \tau'(x,y)x(1-x)$, where $\sigma'(x,y)$ and $\tau'(x,y)$ are sums of squares in $\mathbb{R}[x,y]$. The argument for $r = 1$ is basically the same as that for $r = 0$. □

Lemma 2.2. *We may assume f has only finitely many zeros in the strip $[0,1]\times\mathbb{R}$.*

Proof. If $f = g^2h$ then, using the fact that the set of points (a,b) in the strip satisfying $g(a,b) \neq 0$ is dense in the strip [4, Prop. 1.1.2], we see that $h \geq 0$ on the strip. So we are reduced to showing the result for h. Thus we are reduced to the case where f is square-free. Since x and $1-x$ do not divide f (because they do not divide a_{2d}), f has only finitely many zeros on the boundary of the strip. If some irreducible factor p of f has infinitely many zeros in the interior of the strip then p has a nonsingular zero (a, b) in the interior of the strip which is not a zero of any other irreducible factor of f [4, Lem. 9.4.1]. Then p changes sign at (a, b) but each other irreducible factor of f has constant sign in a neighborhood of (a, b), contradicting the fact that f is ≥ 0 on the strip. It follows that f has only finitely many zeros in the strip. □

If f is square-free, then no irreducible factor of f can change sign in the interior of the strip, so each irreducible factor has constant sign on the strip. Replacing p by $-p$ if necessary, for each irreducible factor p, we may assume each irreducible factor of f is ≥ 0 on the strip. In this way we are reduced further, to the case where f itself is irreducible. But this does not seem to help us much in the proof.

3. The idea of the proof

Consider the case where the polynomial $f(x, y) := \sum_{i=0}^{2d} a_i(x)y^i$ is positive on $[0, 1] \times \mathbb{R}$, and $a_{2d}(x) > 0$ on $[0, 1]$. The form $F(x, y, z) := \sum_{i=0}^{2d} a_i(x)y^i z^{2d-i}$ is positive for $0 \leq x \leq 1$, $(y, z) \neq (0, 0)$ (since $F(x, y, z) = z^{2d} f(x, \frac{y}{z})$ if $z \neq 0$), so it achieves a positive minimum ϵ on the compact set

$$[0, 1] \times \mathbb{S}^1 := \{(x, y, z) \mid 0 \leq x \leq 1 \text{ and } y^2 + z^2 = 1\}.$$

Then, on the strip $[0, 1] \times \mathbb{R}$,

$$f(x, y) = F(x, y, 1) = F(x, \frac{y}{\sqrt{1+y^2}}, \frac{1}{\sqrt{1+y^2}})(1 + y^2)^d \geq \epsilon(1 + y^2)^d.$$

Using this, the argument in [3, Th. 5.1] shows that $f(x, y)$ has the required presentation. See [5, Th. 2.2] for additional explanation.

In the general case, one cannot possibly have such an ϵ. The idea is to replace ϵ by a polynomial $\epsilon(x)$. Specifically, we look for a polynomial $\epsilon(x) \in \mathbb{R}[x]$ such that

$$f(x, y) \geq \epsilon(x)(1 + y^2)^d$$

holds on the strip, $\epsilon(x) \geq 0$ on $[0, 1]$, and, $\forall\, x \in [0, 1]$, $\epsilon(x) = 0$ iff $f(x, y) = 0$ for some $y \in \mathbb{R}$. It is always possible to find such a polynomial $\epsilon(x)$, assuming that $a_{2d}(x) > 0$ on $[0, 1]$ and $f(x, y)$ has only finitely many zeros in the strip. Once this is established, one can show that a modified version of the argument in [3, Th. 5.1] carries through, with ϵ replaced by $\epsilon(x)$. This latter part of the argument is a bit technical: it is necessary to take pains to ensure that the continuous functions considered are analytic at the points where $\epsilon(x) = 0$, and to use a refined version of the Weierstrass Approximation Theorem.

4. The additional ingredients

We establish the additional results that we need in the proof of Theorem 1.1.

Lemma 4.1. *Suppose $f \in \mathbb{R}[x, y]$ is non-negative on a strip of the form $[0, \epsilon) \times \mathbb{R}$, $\epsilon > 0$, f has only finitely many zeros in this strip, and the leading coefficient of f is positive on the interval $[0, \epsilon)$. Then there exists a real constant $C > 0$, an even integer $m \geq 0$, and a real number δ, $0 < \delta \leq \epsilon$, such that $f(x, y) \geq Cx^m$ holds on the strip $[0, \delta) \times \mathbb{R}$. If f has no real zeros on the y-axis, we may take $m = 0$.*

We denote by $k((x))$ the formal power series field over a field k, i.e., the field of fractions of the formal power series ring $k[[x]]$.

Proof. The leading coefficient of f is positive at zero; in particular, it is a unit in $\mathbb{R}[[x]]$. By Puiseux's Theorem, f factors into linear factors in $\mathbb{C}((x^{\frac{1}{n}}))[y]$, for some $n \geq 1$. Each root has the form $z = \sum_{i=0}^{\infty} a_i x^{\frac{i}{n}}$, $a_i \in \mathbb{C}$, and is a complex analytic function of $x^{\frac{1}{n}}$ in a neighborhood of zero [11, Sect. 12.3]. $\overline{z} := \sum_{i=0}^{\infty} \overline{a_i} x^{\frac{i}{n}}$ is another root of f, and $\overline{z} \neq z$. (If all the a_i were real then the equation $y = \sum_{i=0}^{\infty} a_i x^{\frac{i}{n}}$ would define a real half-branch in the zero set of f, for x close to zero, $x > 0$, contradicting our assumption that f has only finitely many zeros in the strip $[0, \epsilon) \times \mathbb{R}$.) Write $a_i = b_i + c_i\sqrt{-1}$, $b_i, c_i \in \mathbb{R}$, so $z = z_1 + z_2\sqrt{-1}$, $\overline{z} = z_1 - z_2\sqrt{-1}$, where $z_1 := \sum b_i x^{\frac{i}{n}}$, $z_2 = \sum c_i x^{\frac{i}{n}}$, so $(y-z)(y-\overline{z}) = (y-z_1)^2 + z_2^2$. Let $k = \frac{i_0}{n}$ be the order of z_2 at zero, i.e., the least $\frac{i}{n}$ such that $c_i \neq 0$. For any real $x > 0$ close to zero, and any $y \in \mathbb{R}$,

$$\frac{(y-z)(y-\overline{z})}{x^{2k}} = (\frac{y-z_1}{x^k})^2 + (\frac{z_2}{x^k})^2,$$

and $\frac{z_2}{x^k}$ is a real number close to the non-zero real constant c_{i_0}. This implies there exists a real constant $C > 0$ such that $\frac{(y-z)(y-\overline{z})}{x^{2k}} > C$ for all real $x > 0$ sufficiently close to zero and all real y. Note: If f has no real zeros on the y-axis then a_0 is not real, i.e., $c_0 \neq 0$, so $k = 0$. Factoring f as $f(x,y) = a(x)\prod_{i=1}^{d}(y-z_i)(y-\overline{z_i})$, where $a(x)$ is the leading coefficient, this yields rationals $k_i \geq 0$ and real constants $C_i > 0$ and

$$\frac{f(x,y)}{\prod_{i=1}^{d} x^{2k_i}} = a(x)\prod_{i=1}^{d}\frac{(y-z_i)(y-\overline{z_i})}{x^{2k_i}} > a(x)\prod_{i=1}^{d} C_i$$

for all real y and all real $x > 0$ sufficiently close to zero. Finally, $f(x,y) \geq Cx^m$ for any real $x \geq 0$ sufficiently close to zero and any $y \in \mathbb{R}$, where C is $\prod_{i=1}^{d} C_i$ times the minimum value of $a(x)$ on $[0, \frac{\epsilon}{2}]$ and m is the least even integer $\geq 2\sum_{i=1}^{d} k_i$. $\square$

Lemma 4.2. *Suppose $f(x,y) = \sum_{i=0}^{2d} a_i(x)y^i$ is non-negative on the strip $[0,1]\times\mathbb{R}$, $f(x,y)$ has only finitely many zeros in the strip, and $a_{2d}(x)$ is positive on the interval $[0,1]$. Then there exists a polynomial $\epsilon(x) \in \mathbb{R}[x]$, $\epsilon(x) \geq 0$ on $[0,1]$, such that $f(x,y) \geq \epsilon(x)(1+y^2)^d$ holds on the strip and, for each $x \in [0,1]$, $\epsilon(x) = 0$ iff there exists $y \in \mathbb{R}$ such that $f(x,y) = 0$.*

Proof. For each $r \in [0,1]$, by Lemma 4.1 (applied to the new variables $t = x - r$ and $t = r - x$, or just to $t = x - r$, resp., just to $t = r - x$, if $r = 0$, resp., if $r = 1$), we have a real constant $C > 0$ and an even integer $m \geq 0$ (with $m = 0$ if f has no real zeros on the line $x = r$), such that $f(x,y) \geq C(x-r)^m$ holds for all (x,y) in the strip, x sufficiently close to r. By compactness of the interval $[0,1]$, there are finitely many $0 \leq r_1 < \cdots < r_k \leq 1$ and finitely many positive constants C_i and even integers $m_i \geq 0$ (with $m_i = 0$ if f has no real zeros on the line $x = r_i$) such that, for each (x,y) in the strip, $f(x,y) \geq C_i(x-r_i)^{m_i}$, for some i. We may assume each C_i is ≤ 1. Then $C_i(x-r_i)^{m_i} \geq \prod_{j=1}^{k} C_j(x-r_j)^{m_j}$ for each $x \in [0,1]$ and each i. Thus $f(x,y) \geq \epsilon_1(x)$ holds on the strip, where $\epsilon_1(x) := \prod_{j=1}^{k} C_j(x-r_j)^{m_j}$.

Since $f(x,y)$ has only finitely many zeros in the strip, $\exists$ a real constant $M > 0$ such that $f(x,y) > 0$ if $|y| \geq M$, $0 \leq x \leq 1$. Arguing with the form $F(x,y,z) := \sum_{i=0}^{2d} a_i(x)y^i z^{2d-i}$ as in Section 3, but with $\mathbb{S}^1 = \{(y,z) \mid y^2 + z^2 = 1\}$ replaced by the set $\{(y,z) \mid y^2+z^2=1, |y| \geq M|z|\}$, we see there exists a positive constant C such that $f(x,y) \geq C(1+y^2)^d$ for all (x,y) in the strip satisfying $|y| \geq M$. If $|y| \leq M$, then $1+y^2 \leq 1+M^2$, and $\frac{f(x,y)}{(1+y^2)^d} \geq \frac{f(x,y)}{(1+M^2)^d} \geq \frac{\epsilon_1(x)}{(1+M^2)^d}$. If $|y| \geq M$, $\frac{f(x,y)}{(1+y^2)^d} \geq C \geq \frac{C}{D}\epsilon_1(x)$, where $D := \max\{\epsilon_1(x) \mid x \in [0,1]\}$. So, in any case, $\frac{f(x,y)}{(1+y^2)^d} \geq \epsilon(x)$ holds on the strip, where

$$\epsilon(x) := \min\{\frac{1}{(1+M^2)^d}, \frac{C}{D}\}\epsilon_1(x).$$

□

Lemma 4.3. *Suppose $f \in \mathbb{R}[x,y]$ and there exists $\epsilon > 0$ such that f is non-negative on the strip $(-\epsilon,\epsilon) \times \mathbb{R}$ and the leading coefficient of f is positive on the interval $(-\epsilon,\epsilon)$. Then there exist g_1, g_2 polynomials in y whose coefficients are analytic functions of x defined in a neighborhood of zero, such that $f = g_1^2 + g_2^2$, for x sufficiently close to zero.*

Proof. Let p be an irreducible factor of f in $\mathbb{C}((x))[y]$ which is monic. By Puiseux's Theorem, p factors in $\mathbb{C}((x^{\frac{1}{n}}))[y]$, where n is the degree of p, as $p = \prod_{\omega \in \mu_n}(y - z_\omega)$, where μ_n denotes the group of complex n-th roots of 1, and $z_\omega = \sum_{i=0}^{\infty} a_i \omega^i x^{\frac{i}{n}}$ for each $\omega \in \mu_n$, where the a_i are complex numbers. The z_ω are complex analytic functions of $x^{\frac{1}{n}}$ in a neighborhood of zero [11, Sect. 12.3]. The coefficients of p are elementary symmetric function of the roots, so are complex analytic functions of x in some neighborhood of zero. Denote by $\overline{p}$ the polynomial in $\mathbb{C}((x))[y]$ obtained from p by conjugating coefficients in the obvious way. $\overline{p}$ is an irreducible factor of f. If $\overline{p} = p$, then z_1 coincides with one of the $\overline{z_\omega} := \sum_{i=0}^{\infty} \overline{a_i \omega^i} x^{\frac{i}{n}}$. This implies, in turn, that there are (two) real half-branches of f coming from p.[1] Since p changes sign at any such half-branch, p must appear in f with even multiplicity in this case. Thus f has a factorization of the form $f = a(x)\prod_{i=1}^{k} p_i \overline{p_i}$, where each p_i is irreducible and $a(x)$ is the leading coefficient. Then $f = g\overline{g}$ where $g = \sqrt{a(x)}p_1 \cdots p_k$. Decomposing g as $g = g_1 + g_2\sqrt{-1}$, $g_1, g_2 \in \mathbb{R}((x))[y]$, this yields $f = g_1^2 + g_2^2$. □

Lemma 4.4. *Suppose $f \in \mathbb{R}[x,y]$ is non-negative on the strip $[0,1] \times \mathbb{R}$, and the leading coefficient of f is positive on the interval $[0,1]$. Then:*

(1) For each $r \in (0,1)$, there exist g_1, g_2 polynomials in y with coefficients analytic functions in x in some neighborhood of r, such that $f = g_1^2 + g_2^2$ holds for x sufficiently close to r.

(2) There exist g_{ij}, $i,j = 1,2$, polynomials in y with coefficients analytic functions in x in some neighborhood of 0, such that $f = \sum_{i=1}^{2} g_{i1}^2 + \sum_{i=1}^{2} g_{i2}^2 x$ holds for x sufficiently close to 0.

(3) There exist g_{ij}, $i,j = 1,2$, polynomials in y with coefficients analytic functions in x in some neighborhood of 1, such that $f = \sum_{i=1}^{2} g_{i1}^2 + \sum_{i=1}^{2} g_{i2}^2(1-x)$ holds for x sufficiently close to 1.

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[1]If n is odd the real half-branches (one with $x > 0$, one with $x < 0$) are given by $y = \sum a_i \tau^i x^{\frac{i}{n}}$ where τ is the unique element of μ_n satisfying $\tau^2 = \omega$. If n is even there are two elements $\tau, -\tau \in \mu_{2n}$ satisfying $\tau^2 = (-\tau)^2 = \omega$. If $\tau, -\tau \in \mu_n$ the real half-branches (both with $x > 0$) are given by $y = \sum a_i \tau^i x^{\frac{i}{n}}$ and $y = \sum a_i (-\tau)^i x^{\frac{i}{n}}$. If $\tau, -\tau \notin \mu_n$ the real half-branches (both with $x < 0$) are given by $y = \sum a_i \tau^i (-x)^{\frac{i}{n}}$ and $y = \sum a_i (-\tau)^i (-x)^{\frac{i}{n}}$.

Proof. For (1), apply Lemma 4.3, viewing f as a polynomial in $x-r$ and y. For (2), apply Lemma 4.3, viewing f as a polynomial in $\sqrt{x}$ and y, to obtain $f = g_1^2+g_2^2$ with g_i a polynomial in y with coefficients analytic in $\sqrt{x}$, $i=1,2$. Decomposing each of the coefficients, using $\sum_k a_k x^{\frac{k}{2}} = \sum_\ell a_{2\ell}x^\ell + \sum_\ell a_{2\ell+1}x^\ell\sqrt{x}$, yields $g_i = g_{i1}+g_{i2}\sqrt{x}$, where the g_{ij} are polynomials in y with coefficients analytic functions of x near $x=0$. Expanding g_i^2, $i=1,2$ then yields $f=\sum_{i=1}^2 g_{i1}^2 + \sum_{i=1}^2 g_{i2}^2 x + 2\sum_{i=1}^2 g_{i1}g_{i2}\sqrt{x}$, so $f = \sum_{i=1}^2 g_{i1}^2 + \sum_{i=1}^2 g_{i2}^2 x$ and $\sum_{i=1}^2 g_{i1}g_{i2} = 0$. The proof of (3) is similar to the proof of (2). □

Proposition 4.5. *Suppose $\phi, \psi : [0,1] \to \mathbb{R}$ are continuous functions, $\phi(x) \le \psi(x)$ for all $x \in [0,1]$, and $\phi(x) < \psi(x)$ for all but finitely many $x \in [0,1]$. If ϕ and ψ are analytic at each point $a \in [0,1]$ where $\phi(a) = \psi(a)$ then $\exists$ a polynomial $p(x) \in \mathbb{R}[x]$ such that $\phi(x) \le p(x) \le \psi(x)$ holds for all $x \in [0,1]$.*

Proof. [2] Induct on the number of points $a \in [0,1]$ satisfying $\phi(a) = \psi(a)$. If there are no such points, existence of $p(x)$ follows from the Weierstrass Approximation Theorem. Suppose $a \in [0,1]$ is such that $\phi(a) = \psi(a)$. Let k be the vanishing order of $\psi - \phi$ at a. If $a \in (0,1)$ then k is even. In this case, $\phi(x) = f(x) + (x-a)^k\phi_1(x)$, $\psi(x) = f(x) + (x-a)^k\psi_1(x)$, where $f(x) \in \mathbb{R}[x]$, $\phi_1(x), \psi_1(x)$ are analytic at a, and $\phi_1(a) < \psi_1(a)$. Extend ϕ_1, ψ_1 to continuous functions $\phi_1, \psi_1 : [0,1] \to \mathbb{R}$ by defining $\phi_1(x) = \frac{\phi(x)-f(x)}{(x-a)^k}$, $\psi_1(x) = \frac{\psi(x)-f(x)}{(x-a)^k}$ for $x \ne a$. Then $\phi_1(x) \le \psi_1(x)$ for all $x \in [0,1]$, and, $\forall\ b \in [0,1]$, $\phi_1(b) = \psi_1(b)$ iff $\phi(b) = \psi(b)$ and $b \ne a$. By induction we have $p_1(x) \in \mathbb{R}[x]$ such that $\phi_1(x) \le p_1(x) \le \psi_1(x)$ on $[0,1]$. Take $p(x) = f(x) + (x-a)^k p_1(x)$. The case where $a=0$ and the case where $a=1$ are dealt with in a similar fashion. □

5. The end of the proof

Let $f(x,y) = \sum_{i=0}^{2d} a_i(x)y^i$, $d \ge 1$. By lemmas 2.1 and 2.2 we can assume $a_{2d}(x) > 0$ on the interval and $f(x,y)$ has only finitely many zeros in $[0,1]\times\mathbb{R}$. By Lemma 4.2, $\exists$ a polynomial $\epsilon(x) \in \mathbb{R}[x]$ such that $f(x,y) \ge \epsilon(x)(1+y^2)^d$ on the strip, $\epsilon(x) \ge 0$ on $[0,1]$, and $\epsilon(x) = 0$ iff $\exists\ y \in \mathbb{R}$ with $f(x,y) = 0$. Let $f_1(x,y) := f(x,y) - \epsilon(x)(1+y^2)^d$. Then f_1 is ≥ 0 on the strip. Replacing $\epsilon(x)$ by $\frac{\epsilon(x)}{N}$, $N > 1$, if necessary, we can assume f_1 has degree $2d$ (as a polynomial in y) and the leading coefficient of f_1 is positive on $[0,1]$.

By Lemma 4.4, for each $r \in [0,1]$, there exists an open neighborhood $U(r)$ of r in $\mathbb{R}$ such that f_1 decomposes as

$$f_1 = \sum_{j=1}^{2} g_{0j}(r)^2 + \sum_{j=1}^{2} g_{1j}(r)^2 x + \sum_{j=1}^{2} g_{2j}(r)^2(1-x)$$

on $U(r)\times\mathbb{R}$, where the $g_{ij}(r)$ are polynomials in y (of degree $\le d$) whose coefficients are analytic functions of x, for $x \in U(r)$.[3] By compactness of $[0,1]$, finitely many of the $U(r)$ cover $[0,1]$, say $U(r_1), \cdots, U(r_\ell)$ cover $[0,1]$. Choose a continuous partition of unity $1 = \nu_1 + \cdots + \nu_\ell$ on $[0,1]$, with $0 \le \nu_k \le 1$ on $[0,1]$ and $\overline{\mathrm{supp}(\nu_k)} \subseteq U(r_k)$ for $k = 1, \cdots, \ell$, having the additional property that, for each root r of $\epsilon(x)$ in

[2] Proposition 4.5 is probably well-known. The author only became aware of Proposition 4.5 and its simple proof through reading an unpublished manuscript of V. Powers.

[3] Applying Lemma 4.4, we can choose the $g_{ij}(r)$ so that $g_{2j}(r) = 0$, $j=1,2$, if $r=0$; $g_{1j}(r) = 0$, $j=1,2$, if $r=1$; and $g_{1j}(r) = g_{2j}(r) = 0$, $j=1,2$, if $0<r<1$.

$[0,1]$, there is just one k such that $\nu_k(x) \neq 0$ close to r (so $\nu_k(x) = 1$ for x close to r). One way to ensure the last property is to shrink the covering sets $U(r_k)$ ahead of time so that each root r of $\epsilon(x)$ in $[0,1]$ lies in some unique $U(r_k)$. Then f_1 decomposes as

$$f_1 = \sum_{k=1}^{\ell} \nu_k f_1 = \sum_{k=1}^{\ell} (\sum_{j=1}^{2} \phi_{0jk}^2 + \sum_{j=1}^{2} \phi_{1jk}^2 x + \sum_{j=1}^{2} \phi_{2jk}^2 (1-x))$$

on $[0,1] \times \mathbb{R}$, where ϕ_{ijk} denotes the polynomial of degree $\leq d$ in y whose coefficients are the functions from $[0,1]$ to $\mathbb{R}$ obtained by extending the corresponding coefficients of $\sqrt{\nu_k} g_{ij}(r_k)$ by zero off $U(r_k)$. The coefficients of the ϕ_{ijk} are continuous on $[0,1]$ and analytic at each of the roots of $\epsilon(x)$ in $[0,1]$ (since ν_k is constantly 0 or 1 in a neighborhood of each of these roots).

By Proposition 4.5, for each real $N > 0$, and each triple i, j, k, $\exists$ a polynomial h_{ijk} of degree $\leq d$ in y with coefficients in $\mathbb{R}[x]$, such that, for each coefficient u of ϕ_{ijk}, the corresponding coefficient w of h_{ijk} satisfies

$$u(x) - \frac{\epsilon(x)}{N} \leq w(x) \leq u(x) + \frac{\epsilon(x)}{N}, \text{ for each } x \in [0,1].$$

At this point we proceed as in the proof of [3, Th. 5.1], approximating the coefficients of the ϕ_{ijk} closely by polynomials (by taking N sufficiently large), to obtain polynomials h_{ijk} of degree $\leq d$ in y with coefficients in $\mathbb{R}[x]$ such that

$$f_1(x,y) =$$
$$\sum_{k=1}^{\ell} (\sum_{j=1}^{2} h_{0jk}(x,y)^2 + \sum_{j=1}^{2} h_{1jk}(x,y)^2 x + \sum_{j=1}^{2} h_{2jk}(x,y)^2 (1-x)) + \sum_{i=0}^{2d} b_i(x) y^i,$$

$b_i(x) \in \mathbb{R}[x]$, $|b_i(x)| \leq \frac{2}{5}\epsilon(x)$ on $[0,1]$, $i = 0, \cdots, 2d$. Combining this with $f(x,y) = f_1(x,y) + \epsilon(x)(1+y^2)^d$ yields $f(x,y) = s_1(x,y) + s_2(x,y) + s_3(x,y)$, where

$$s_1(x,y) := \sum_{k=1}^{\ell} (\sum_{j=1}^{2} h_{0jk}(x,y)^2 + \sum_{j=1}^{2} h_{1jk}(x,y)^2 x + \sum_{j=1}^{2} h_{2jk}(x,y)^2 (1-x)),$$

$$s_2(x,y) := \frac{2}{5}\epsilon(x)(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} b_i(x) y^i,$$

$$s_3(x,y) := \epsilon(x)[(1+y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d})].$$

Let T denote the preordering of $\mathbb{R}[x,y]$ generated by $x(1-x)$. As pointed out earlier, $x, 1-x \in T$. Clearly $s_1(x,y) \in T$. The argument in [3, Th. 5.1] shows that $s_2(x,y) \in T$. In more detail, since $|b_i(x)| \leq \frac{2}{5}\epsilon(x)$ on $[0,1]$, $\frac{2}{5}\epsilon(x) \pm b_i(x) \in T$, by [3, Th. 2.2] or [4, Prop. 2.7.3], for $i = 0, \cdots, 2d$. This yields

$$\frac{2}{5}\epsilon(x) y^i + b_i(x) y^i \in T, \text{ for } i \text{ even.} \tag{5.1}$$

For i odd, say $i = 2m+1$, use the identity $y^{2m+1} = \frac{1}{2} y^{2m}((y+1)^2 - y^2 - 1)$ plus the fact that $\frac{2}{5}\epsilon(x) y^{2m}(y+1)^2 + b_i(x) y^{2m}(y+1)^2$, $\frac{2}{5}\epsilon(x) y^{2m} y^2 - b_i(x) y^{2m} y^2$ and

$\frac{2}{5}\epsilon(x)y^{2m} - b_i(x)y^{2m}$ all belong to T to obtain

$$\frac{2}{5}\epsilon(x)(y^{i+1} + y^i + y^{i-1}) + b_i(x)y^i \in T, \text{ for } i \text{ odd.} \tag{5.2}$$

Adding together the various terms of type (5.1) and (5.2), for $i = 0, \cdots, 2d$, we see that $s_2(x, y) \in T$. The fact that $s_3(x, y)$ belongs to T follows from the identity

$$\begin{aligned}
&(1 + y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) \\
&= (1 + y^2)^d + \frac{1}{5}(1 + y^2 + \cdots + y^{2d-2})(1 - y)^2 \\
&- \frac{8}{5}(y^2 + y^4 + \cdots + y^{2d-2}) - (1 + y^{2d}) \\
&= \frac{1}{5}(1 + y^2 + \cdots + y^{2d-2})(1 - y)^2 + \sum_{i=1}^{d-1}(\binom{d}{i} - \frac{8}{5})y^{2i}.
\end{aligned}$$

This means, finally, that $f(x, y) = s_1(x, y) + s_2(x, y) + s_3(x, y) \in T$. $\square$

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