ERROR ESTIMATES IN THE OPTIMIZATION OF DEGREE TWO POLYNOMIALS ON A DISCRETE HYPERCUBE

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Abstract. The paper considers the distribution of values $Q(x)$, $x \in \{-1,1\}^n$, where Q is a quadratic form in n variables with real coefficients. Error estimates are established for approximations of the maximum and minimum value of Q on $\{-1,1\}^n$ which can be obtained by semidefinite programming. Bounds are given involving the sum of the absolute values of the off-diagonal entries. Other bounds are given which are useful in the case of extreme skewness. Used in conjunction with earlier bounds of Nesterov in [5], these new bounds lead to improvements on the bound given by the trace. The trigonometric description of the maximum and minimum given in [5], which is based on the rounding argument introduced by Goemans and Williamson in [1], is a major tool in obtaining these bounds.

1. introduction

Let Q be a polynomial in n variables with real coefficients, $S \subseteq \mathbb{R}^n$ a basic closed semialgebraic set. Lasserre's algorithm [3] produces an increasing sequence of lower bounds for Q on S computable via semidefinite programming which, in case S is compact, converges to the exact minimum of Q on S. In [4] a refinement of Lasserre's algorithm is described which takes into account the fact that S may have dimension less than n. This involves computation in the factor ring $\mathbb{R}[x]/\mathfrak{a}$ using Gröbner basis techniques where $\mathfrak{a}$ is the ideal of polynomials vanishing on S. Often the sequence converges rapidly. Often the first or second term in the sequence is already close to the exact minimum. Still, there is no general theory in this regard.

In particular, one would like to be able to estimate the accuracy of the first term in the sequence. In the present paper we consider this question in case Q is of degree 2 and S is the discrete hypercube $\{-1,1\}^n$. In this case encouraging results have been obtained already, by a number of people [5] [6] [8] [9] [10] [11], following up on the ground-breaking work of Goemans and Williamson in [1].

When S is $\{-1,1\}^n$ the ideal $\mathfrak{a}$ is generated by x_i^2-1, $i = 1,\dots,n$. The factor ring $\mathbb{R}[x]/\mathfrak{a}$ is 0-dimensional with basis as a vector space over $\mathbb{R}$ consisting of all products $\prod_{i\in I} x_i$, I a subset of $\{1,\dots,n\}$. We consider the problem of minimizing (or maximizing) a degree 2 polynomial $Q \in \mathbb{R}[x]$ on $\{-1,1\}^n$. More precisely, we consider an approximation to this minimum (or maximum) which, in terms of the algorithm described in [4], is just the first

1991 *Mathematics Subject Classification*. Primary 14P99 15A63 90C22.

This research was supported by NSERC of Canada and by the universities of Konstanz, Paris 6 and Paris 7. The author wishes to thank Markus Schweighofer for sharing his expertise.

term in the sequence of approximations converging to the exact value, and we examine the accuracy of this approximation.

Since minimizing (or maximizing) Q on $\{-1,1\}^n$ is equivalent to minimizing (or maximizing) the associated quadratic form $x_0^2Q(\frac{x_1}{x_0},\dots,\frac{x_n}{x_0})$ in $n+1$ variables $x_0,\dots,x_n$ on $\{-1,1\}^{n+1}$, we may as well assume from the start that Q is a quadratic form. We identify Q with its associated symmetric matrix, so $Q(x)=x^tQx$. We define

$$Q_* = \min\{Q(x) \mid x \in \{-1,1\}^n\}$$
$$Q^* = \max\{Q(x) \mid x \in \{-1,1\}^n\}.$$

One can associate to Q the graph with vertices $V=\{1,\dots,n\}$ and edges $E=\{(i,j): i<j, Q_{ij}\neq 0\}$. Computation of Q_* (resp., Q^*) can be viewed as a 'weighted' version of the MAX-CUT problem considered in [1], where Q_{ij} is the 'weight' attached to the edge (i,j).

The approximations of Q_* and Q^* that we are dealing with are

$$Q_+ := \min\{\langle Q,X\rangle \mid X \text{ is PSD, } X_{ii}=1, i=1,\dots,n\}$$
$$Q^+ := \max\{\langle Q,X\rangle \mid X \text{ is PSD, } X_{ii}=1, i=1,\dots,n\}.$$

Here, $\langle Q,X\rangle := \sum_{i,j} Q_{ij}X_{ij}$. It is clear that $Q_+ \le Q_*$ (and, similarly, that $Q^* \le Q^+$): If $x\in\{-1,1\}^n$ define X by $X_{ij}=x_ix_j$. Then X is PSD, $X_{ii}=1$ for $i=1,\dots,n$ and $\langle Q,X\rangle = \sum_{i,j} Q_{ij}x_ix_j = x^tQx = Q(x)$. In summary we have

$$Q_+ \le Q_* \le Q^* \le Q^+.$$

We also have the following dual description of Q_+ (and of Q^+):

1.1 Theorem. *$Q_+ = \overline{Q}_+$ and $Q^+ = \overline{Q}^+$ where*

$$\overline{Q}_+ = \max\{\lambda \mid \exists y \in \mathbb{R}^n,\ \lambda = \sum y_i,\ Q - \mathit{Diag}(y) \text{ is PSD}\}$$
$$\overline{Q}^+ = \min\{\lambda \mid \exists y \in \mathbb{R}^n,\ \lambda = \sum y_i,\ \mathit{Diag}(y) - Q \text{ is PSD}\}.$$

Here, $\mathrm{Diag}(y)$ denotes the diagonal matrix with diagonal entries $y_1,\dots,y_n$. Computation of Q_+ (resp., Q^+) is a semidefinite programming problem. Computation of $\overline{Q}_+$ (resp., of $\overline{Q}^+$) is the dual semidefinite programming problem. The inequality $Q_+ \ge \overline{Q}_+$ (resp., $\overline{Q}^+ \ge Q^+$) is based on the fact that if A,B are PSD then $\langle A,B\rangle \ge 0$: Theorem 1.1 asserts that the duality gap $Q_+ - \overline{Q}_+$ (resp., $\overline{Q}^+ - Q^+$) is zero. This can be proved in various ways, e.g., see [4] [5] or [6] for more general results.

In [5] [6] Nesterov obtains bounds for Q_* (resp., Q^*) in terms of Q_+, Q^+ and $\mathrm{tr}(Q) := \sum_{i=1}^n Q_{ii}$. We recall these bounds in Section 3; see Theorems 3.2 and 3.3. The bound given by Theorem 3.3 is always better than the bound given by Theorem 3.2. In Section 4 we give bounds which involve $\sum_{i\neq j}|Q_{ij}|$. The first, see Theorem 4.1, is a simple generalization of the result in [1]. A particular case of this appears already in [11]. The second, see Theorem 4.2, is new. The bound provided by Theorem 4.2 is better than the bound provided by Theorem 3.3 in cases where $Q^+ - \operatorname{tr}(Q)$ (resp., $\operatorname{tr}(Q) - Q_+$) is 'sufficiently close' to $\sum_{i \neq j} |Q_{ij}|$. This occurs, for example, if $Q_{ij} \geq 0$ (resp., $Q_{ij} \leq 0$) for $i \neq j$. We also compare Theorems 4.1 and 4.2 and explain how Theorem 4.2 predicts better accuracy of the MAX-CUT algorithm in [1] when the output is either more than $\approx 86.6\%$ of the total number of edges or less than $\approx 67.0\%$ of the total number of edges.

For Q not diagonal, the ratio $r = \frac{Q^+ - \operatorname{tr}(Q)}{Q^+ - Q_+}$ lies somewhere in the closed interval $[\frac{1}{n}, \frac{n-1}{n}]$. This follows from Theorem 2.6. When r is not too large, the bound for Q_* given by Theorem 3.3 is significantly better than the trivial bound $Q_* \leq \operatorname{tr}(Q)$. If $Q_{ij} \geq 0$ for $i \neq j$, one can improve on this, using Theorem 4.2. At the other extreme, when r is sufficiently close to $\frac{n-1}{n}$, other bounds come into play, see Corollary 5.5, which are also significantly better than the trivial bound. In the intermediate case nothing much seems to be known. It may be that no significant improvement on the trivial bound is possible in this case; see Question 5.3. The best we are able to show in general is that, for large n, $\frac{\operatorname{tr}(Q) - Q_*}{\operatorname{tr}(Q) - Q_+}$ is bounded away from zero by a function of the form $\frac{C}{\sqrt[3]{n}}$, where C is a constant; see Theorem 5.6.

2. ELEMENTARY OBSERVATIONS

In studying the distribution $Q(x)$, $x \in \{-1,1\}^n$, it is natural to consider the mean and standard deviation. Denote by $\operatorname{tr}(Q)$ the trace of Q, i.e., $\operatorname{tr}(Q) = \langle Q, I \rangle = \sum_i Q_{ii}$.

2.1 Lemma. *The mean value of Q on $\{-1,1\}^n$ is equal to* $\operatorname{tr}(Q)$.

Proof. The proof is trivial. In the sum

$$\sum_{x \in \{-1,1\}^n} Q(x) = \sum_{x \in \{-1,1\}^n} \sum_{i,j} Q_{ij} x_i x_j$$

the terms with $i \neq j$ cancel. $\square$

It follows from Lemma 2.1 (also see [5, Cor. 2.4]) that $Q_* \leq \operatorname{tr}(Q) \leq Q^*$. We refer to the bound $Q_* \leq \operatorname{tr}(Q)$ (resp., $Q^* \geq \operatorname{tr}(Q)$) as the *trivial bound* for Q_* (resp., Q^*).

2.2 Lemma. *The standard deviation of Q on $\{-1,1\}^n$ is*

$$2\sqrt{\sum_{i<j} Q_{ij}^2} = \sqrt{2\sum_{i \neq j} Q_{ij}^2}.$$

Proof. The proof is similar to the proof of Lemma 2.1 and is omitted. $\square$

We also have the following lower (resp., upper) bound for Q_+ (resp., Q^+):

2.3 Theorem.

$$Q_+ \geq \operatorname{tr}(Q) - \sum_{i \neq j} |Q_{ij}|$$
$$Q^+ \leq \operatorname{tr}(Q) + \sum_{i \neq j} |Q_{ij}|.$$

Proof. Adding $\frac{|Q_{ij}|}{2}(x_i^2+x_j^2)$ to $Q_{ij}x_ix_j$ for $j \neq i$ yields the perfect square

$$\frac{|Q_{ij}|}{2}(x_i \pm x_j)^2.$$

Consequently the quadratic form $Q(x) - \sum_{i=1}^n (Q_{ii} - \sum_{j\neq i}|Q_{ij}|)x_i^2$ is PSD, so $\overline{Q}_+ \geq \mathrm{tr}(Q) - \sum_{i\neq j}|Q_{ij}|$. The first assertion follows from this using $Q_+ \geq \overline{Q}_+$ (the easy half of Theorem 1.1). The second assertion follows from the first by replacing Q by $-Q$. □

2.4 Corollary. *See [11, Theorem 2]. If $Q_{ij} \geq 0$ (resp., $Q_{ij} \leq 0$) for $i \neq j$ then*

$$Q^* = Q^+ = \mathrm{tr}(Q) + \sum_{i\neq j}|Q_{ij}|$$
$$(\textit{resp.},\ Q_* = Q_+ = \mathrm{tr}(Q) - \sum_{i\neq j}|Q_{ij}|).$$

In particular, if Q is diagonal then $Q_+ = Q_* = \mathrm{tr}(Q) = Q^* = Q^+$.

Proof. Since $Q(1,\dots,1) = \sum_{i,j} Q_{ij}$, this is immediate from Theorem 2.3. □

2.5 Lemma.

$$Q_* \leq \mathrm{tr}(Q) - 2\max\{|Q_{ij}| \mid i \neq j\}$$
$$Q^* \geq \mathrm{tr}(Q) + 2\max\{|Q_{ij}| \mid i \neq j\}.$$

In particular, Q not diagonal $\Rightarrow Q_* < \mathrm{tr}(Q) < Q^*$.

Proof. We prove that if $Q_* \geq \mathrm{tr}(Q) - \delta$ (resp., $Q^* \leq \mathrm{tr}(Q) + \delta$), then $|Q_{ij}| \leq \frac{\delta}{2}$ for all $i \neq j$. Replacing Q by $Q' = Q - \mathrm{Diag}(Q_{11},\dots,Q_{nn})$, we can assume the diagonal entries of Q are 0. We can assume $n \geq 2$ and after re indexing that $i = 1$, $j = 2$. If $n = 2$ then $Q(x) = 2Q_{12}x_1x_2$ and the hypothesis $Q_* = -2|Q_{12}| \geq -\delta$ implies $|Q_{12}| \leq \frac{\delta}{2}$ as required. If $n \geq 3$ use the identity

$$Q(x_1,\dots,x_{n-1},0) = \frac{1}{2}(Q(x_1,\dots,x_{n-1},x_n) + Q(x_1,\dots,x_{n-1},-x_n))$$

and proceed by induction on n. □

The invariants Q_+ and Q^+ provide not only upper and lower bounds for the distribution $Q(x)$, $x \in \{-1,1\}^n$, but also, by comparing the relative magnitude of $Q^+ - \mathrm{tr}(Q)$ and $\mathrm{tr}(Q) - Q_+$, they provide some rough measure of the skewness of the distribution. Lemma 2.5 implies that the ratio $\frac{Q^+ - \mathrm{tr}(Q)}{\mathrm{tr}(Q) - Q_+}$ is well-defined and positive, for Q non-diagonal.

2.6 Theorem. *For Q non-diagonal,*

$$\frac{1}{n-1} \leq \frac{Q^+ - \mathrm{tr}(Q)}{\sum_{i\neq j}|Q_{ij}|} \leq \frac{Q^+ - \mathrm{tr}(Q)}{\mathrm{tr}(Q) - Q_+} \leq \frac{\sum_{i\neq j}|Q_{ij}|}{\mathrm{tr}(Q) - Q_+} \leq n-1.$$

Proof. The middle inequalities are immediate from Theorem 2.3. The first inequality follows from the last, replacing Q by $-Q$, so we concentrate on the last inequality. One reduces easily to the case where $Q_+ = 0$ and Q is PSD. This is just a matter of replacing Q by $Q' = Q - \mathrm{Diag}(y)$ where $y \in \mathbb{R}^n$ is chosen so that $\sum_i y_i = Q_+$ and Q' is PSD. Scaling, we can assume $\mathrm{tr}(Q) = 1$. We want to show $\sum_{i\neq j} |Q_{ij}| \leq n-1$. Since Q is PSD there exist vectors $w_1, \ldots, w_n$ in $\mathbb{R}^n$ such that $Q_{ij} = \langle w_i, w_j \rangle$. (Use the Spectral Theorem to decompose Q as $Q = D^t D$ and take $w_1, \ldots, w_n$ to be the columns of D.) Thus $\sum_i \|w_i\|^2 = \sum_i Q_{ii} = \mathrm{tr}(Q) = 1$. By the Cauchy-Schwartz Inequality, $|\langle w_i, w_j\rangle| \leq \|w_i\|\|w_j\|$. Thus

$$\sum_{i\neq j} |Q_{ij}| \leq \sum_{i\neq j} \|w_i\|\|w_j\| = (\sum_i \|w_i\|)^2 - \sum_i \|w_i\|^2 = (\sum_i \|w_i\|)^2 - 1.$$

We know from calculus that the maximum value of $f(x) = (\sum_i x_i)^2$ on the sphere $\sum_i x_i^2 = 1$ is n. The maximum is achieved at $x = \pm(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$. This proves $\sum_{i\neq j} |Q_{ij}| \leq n-1$ and completes the proof. $\square$

Later we establish similar bounds for the ratio $\frac{Q^* - \mathrm{tr}(Q)}{\mathrm{tr}(Q) - Q_*}$; see Section 5, Theorem 5.4, for the precise statement. The following example shows that the bounds given by Theorem 2.6 and Theorem 5.4 are best possible.

2.7 Example. Take

$$Q(x) = (x_1 + \cdots + x_n)^2.$$

Then $\mathrm{tr}(Q) = n$, $Q^+ = Q^* = n^2$ and Q_* is either 0 or 1 depending on whether n is even or odd. We claim that $Q_+ = 0$. Since Q is PSD we see that $Q_+ \geq 0$. Choose vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ such that $\|v_i\| = 1$ and $v_1 + \cdots + v_n = 0$ (always possible if $n \geq 2$) and define X by $X_{ij} = \langle v_i, v_j \rangle$. Then X is PSD, $X_{ii} = 1$ and $\langle Q, X\rangle = \sum_{i,j} \langle v_i, v_j \rangle = \|v_1 + \cdots + v_n\|^2 = 0$. This proves the upper bounds given by Theorem 2.6 and Theorem 5.4 are best possible. Similarly, looking at

$$Q(x) = -(x_1 + \cdots + x_n)^2,$$

we see that the lower bounds are also best possible.

It follows by continuity that each number $t \in [\frac{1}{n-1}, n-1]$ is equal to $\frac{Q^+ - \mathrm{tr}(Q)}{\mathrm{tr}(Q) - Q_+}$ for some non-diagonal symmetric $n \times n$ matrix Q. For example, if $t \in [1, n-1]$ we can choose Q of the form $Q(x) = a(x_1 + \cdots + x_n)^2 + (1-a)(x_1 + x_2)^2$ for suitable $a \in [0,1]$. An analogous result holds for the ratio $\frac{Q^* - \mathrm{tr}(Q)}{\mathrm{tr}(Q) - Q_*}$.

3. NESTEROV'S ERROR BOUNDS

The main technical tool used in establishing bounds for Q_* and Q^* is an alternate description of Q_* and Q^* proved in [5], which, in turn, is motivated by the probabilistic argument in [1]:

3.1 Theorem. *See* [*5, Theorem 3.1*].

$$Q_* = \min\{\frac{2}{\pi}\langle Q, \arcsin[X]\rangle \mid X \text{ is PSD, } X_{ii}=1, i=1,\dots,n\}$$
$$Q^* = \max\{\frac{2}{\pi}\langle Q, \arcsin[X]\rangle \mid X \text{ is PSD, } X_{ii}=1, i=1,\dots,n\}.$$

Here, $\arcsin[X]$ denotes the matrix with ij entry $\arcsin(X_{ij})$.

In [5] and [6] Nesterov uses Theorem 3.1 to obtain the following result as a special case of a more general result:

3.2 Theorem. *See* [*5, Theorem 3.3*].

$$Q_* \leq (1-\frac{2}{\pi})Q^+ + \frac{2}{\pi}Q_+$$
$$Q^* \geq (1-\frac{2}{\pi})Q_+ + \frac{2}{\pi}Q^+.$$

As explained in [5] and [6], it is possible to improve on Theorem 3.2 with a bound that takes the trivial bound into account. In the case we are considering here, the improvement reads as follows. For Q non-diagonal, define

$$r := \frac{Q^+ - \mathrm{tr}(Q)}{Q^+ - Q_+}, \quad r' := \frac{\mathrm{tr}(Q) - Q_+}{Q^+ - Q_+}. \tag{1}$$

Note: $r + r' = 1$ and $\frac{r}{r'} = \frac{Q^+ - \mathrm{tr}(Q)}{\mathrm{tr}(Q) - Q_+}$. By Theorem 2.6, $\frac{1}{n-1} \leq \frac{r}{r'} \leq n-1$, so $r, r' \in [\frac{1}{n}, \frac{n-1}{n}]$.

3.3 Theorem. *See* [*5, Theorem 3.5*]. *For Q not diagonal,*

$$Q_* \leq (1-\omega(r))Q^+ + \omega(r)Q_+$$
$$Q^* \geq (1-\omega(r'))Q_+ + \omega(r')Q^+.$$

Here r, r' are defined by equation (1), and $\omega(y) := \frac{2}{\pi}(\sqrt{1-y^2} + y\arcsin(y))$.

The function ω is increasing on $[0,1]$, $\omega(0) = \frac{2}{\pi}$, $\omega(1) = 1$. In particular, the bound given by Theorem 3.3 is always better than that given by Theorem 3.2. The bound given by Theorem 3.3 also improves on the trivial bound. This follows from the proof of Theorem 3.3 given in [5]. Also see Section 5, Corollary 5.1 (1).

4. Error bounds involving $\sum_{i\neq j} |Q_{ij}|$

Denote by μ the minimal value of the function $\frac{2}{\pi}\frac{x}{1-\cos x}$ on the interval $(0,\pi)$. $\mu \approx 0.8786$ is the well-known Goemans-Williamson approximation ratio. The proof of the following bound copies the argument given in [1].

4.1 Theorem.

$$
\begin{aligned}
Q_* \leq& (1-\mu)(\operatorname{tr}(Q) + \sum_{i\neq j} |Q_{ij}|) + \mu Q_+ \\
Q^* \geq& (1-\mu)(\operatorname{tr}(Q) - \sum_{i\neq j} |Q_{ij}|) + \mu Q^+.
\end{aligned}
$$

Proof. Replacing Q by $Q' = Q - \operatorname{Diag}(Q_{11}, \ldots, Q_{nn})$ we are reduced to the case where the diagonal entries of Q are zero. We apply Theorem 3.1. Fix X PSD with $X_{ii} = 1$, $i = 1, \ldots, n$ such that $\langle Q, X\rangle = Q_+$. Choose $\epsilon_{ij} \in \{-1, 1\}$ such that $Q_{ij} = \epsilon_{ij}|Q_{ij}|$. Then

$$
\begin{aligned}
Q_* \leq \frac{2}{\pi}\langle Q, \arcsin[X]\rangle =& \frac{2}{\pi}\sum_{i\neq j} Q_{ij}\arcsin(X_{ij}) \\
=& \frac{2}{\pi}\sum_{i\neq j} |Q_{ij}|\arcsin(\epsilon_{ij}X_{ij}) \\
=& -\frac{2}{\pi}\sum_{i\neq j} |Q_{ij}|(\frac{\pi}{2} - \arcsin(\epsilon_{ij}X_{ij})) + \sum_{i\neq j}|Q_{ij}| \\
\leq& -\mu\sum_{i\neq j} |Q_{ij}|(1 - \cos(\frac{\pi}{2} - \arcsin(\epsilon_{ij}X_{ij}))) + \sum_{i\neq j}|Q_{ij}| \\
=& -\mu\sum_{i\neq j} |Q_{ij}|(1 - \epsilon_{ij}X_{ij}) + \sum_{i\neq j}|Q_{ij}| \\
=& -\mu\sum_{i\neq j} |Q_{ij}| + \mu\sum_{i\neq j} Q_{ij}X_{ij} + \sum_{i\neq j}|Q_{ij}| \\
=& (1-\mu)(\sum_{i\neq j}|Q_{ij}|) + \mu Q_+.
\end{aligned}
$$

This proves the first assertion. The second assertion follows from the first by replacing Q by $-Q$. $\square$

See [11, Theorem 3] for a proof of Theorem 4.1 in the special case where $Q_{ij} \geq 0$ (resp., $Q_{ij} \leq 0$) for $i \neq j$. In this case, $\operatorname{tr}(Q) + \sum_{i\neq j}|Q_{ij}|$ (resp., $\operatorname{tr}(Q) - \sum_{i\neq j}|Q_{ij}|$) coincides with Q^+ (resp., Q_+); see Corollary 2.4. One obtains the Goemans-Williamson result in [1] by applying Theorem 4.1 to the quadratic form $Q(x) := \sum_{(i,j)\in E} \frac{(x_i - x_j)^2}{4}$, where E is some set of ordered pairs (i, j), $i, j \in \{1, \ldots, n\}$ with $i < j$ (the set of edges of a graph with vertices $1, \ldots, n$). The examples considered by Karloff in [2] show that the constant μ in Theorem 4.1 is best possible.

We now give another bound involving $\sum_{i\neq j}|Q_{ij}|$, in some sense complementary to Theorem 4.1, which takes the trivial bound into account. For Q non-diagonal, define

$$
s := \frac{\sum_{i\neq j}|Q_{ij}|}{\operatorname{tr}(Q) + \sum_{i\neq j}|Q_{ij}| - Q_+}, \quad s' := \frac{\sum_{i\neq j}|Q_{ij}|}{Q^+ - \operatorname{tr}(Q) + \sum_{i\neq j}|Q_{ij}|}. \tag{2}
$$

Note: $\frac{s}{1-s} = \frac{\sum_{i\neq j}|Q_{ij}|}{\operatorname{tr}(Q)-Q_+}$. By Theorems 2.3 and 2.6, $1 \le \frac{s}{1-s} \le n-1$, so $s \in [\frac{1}{2}, \frac{n-1}{n}]$. Also, $\frac{s}{1-s} \ge \frac{Q^+-\operatorname{tr}(Q)}{\operatorname{tr}(Q)-Q_+} = \frac{r}{r'} = \frac{r}{1-r}$, so $s \ge r$ and $s = r$ iff $Q^+ - \operatorname{tr}(Q) = \sum_{i\neq j}|Q_{ij}|$. A similar argument shows that $s' \in [\frac{1}{2}, \frac{n-1}{n}]$, $s' \ge r'$ and $s' = r'$ iff $\operatorname{tr}(Q) - Q_+ = \sum_{i\neq j}|Q_{ij}|$.

4.2 Theorem. *For Q not diagonal,*

$$\begin{aligned} Q_* \le& (1-\beta(s))\operatorname{tr}(Q) + \beta(s)Q_+ \\ Q^* \ge& (1-\beta(s'))\operatorname{tr}(Q) + \beta(s')Q^+. \end{aligned}$$

Here s, s' are defined by equation (2), and $\beta(y) := \frac{2}{\pi}\max_{t\in(0,1]}\{\arcsin(t) - g(t)\frac{y}{1-y}\}$ where $g(t) := \sqrt{(\frac{\arcsin(t)}{t})^2-1} - \arctan(\sqrt{(\frac{\arcsin(t)}{t})^2-1})$.

Proof. Replacing Q by $Q' = Q - \operatorname{Diag}(Q_{11},\dots,Q_{nn})$ we are reduced to the case where the diagonal entries of Q are zero. Pick X PSD with $X_{ii} = 1$, $i = 1,\dots,n$, and $\langle Q, X\rangle = Q_+$. Let $X_t := tX + (1-t)I$, $t \in [0,1]$. Then $\arcsin[X_t] = (\frac{\pi}{2} - \arcsin(t))I + \arcsin[tX]$ and, by Theorem 3.1,

$$\begin{aligned} Q_* \le& \frac{2}{\pi}\langle Q, \arcsin[X_t]\rangle \\ =& (1 - \frac{2}{\pi}\arcsin(t))\langle Q, I\rangle + \frac{2}{\pi}\langle Q, \arcsin[tX]\rangle \\ =& \frac{2}{\pi}\langle Q, \arcsin[tX]\rangle \\ =& \frac{2}{\pi}\langle Q, \arcsin[tX] - \arcsin(t)X\rangle + \frac{2}{\pi}\arcsin(t)\langle Q, X\rangle \\ =& \frac{2}{\pi}\sum_{i\neq j} Q_{ij}(\arcsin(tX_{ij}) - \arcsin(t)X_{ij}) + \frac{2}{\pi}\arcsin(t)Q_+. \end{aligned}$$

Write $Q_{ij} = \epsilon_{ij}|Q_{ij}|$, $\epsilon_{ij} \in \{-1, 1\}$, and consider the individual terms

$$Q_{ij}(\arcsin(tX_{ij}) - \arcsin(t)X_{ij}) = |Q_{ij}|(\arcsin(t\epsilon_{ij}X_{ij}) - \arcsin(t)\epsilon_{ij}X_{ij})$$

in the sum. Since $\arcsin(t)x \ge \arcsin(tx)$ for $x \in (0,1]$, the terms with $\epsilon_{ij}X_{ij} \ge 0$ contribute negatively. Terms with $\epsilon_{ij}X_{ij} < 0$ contribute at most $g(t)|Q_{ij}|$ where $g(t)$ denotes the maximum of the function

$$h_t(x) = \arcsin(t)x - \arcsin(tx)$$

on the interval $(0,1]$. One checks that the maximum is achieved at $x = \frac{\sqrt{(\frac{\arcsin(t)}{t})^2-1}}{\arcsin(t)}$, so

$$\begin{aligned} g(t) =& \arcsin(t)x - \arcsin(tx) \\ =& \sqrt{(\frac{\arcsin(t)}{t})^2-1} - \arcsin(\frac{\sqrt{(\frac{\arcsin(t)}{t})^2-1}}{\frac{\arcsin(t)}{t}}) \\ =& \sqrt{(\frac{\arcsin(t)}{t})^2-1} - \arctan(\sqrt{(\frac{\arcsin(t)}{t})^2-1}). \end{aligned}$$

This proves
$$Q_* \leq \frac{2}{\pi} g(t) \sum_{i \neq j} |Q_{ij}| + \frac{2}{\pi} \arcsin(t) Q_+.$$
Finally, using $\frac{s}{1-s} = \frac{\sum_{i\neq j} |Q_{ij}|}{\mathrm{tr}(Q) - Q_+} = \frac{\sum_{i\neq j} |Q_{ij}|}{-Q_+}$, this yields
$$\begin{aligned} Q_* \leq & \frac{2}{\pi} g(t) \sum_{i \neq j} |Q_{ij}| + \frac{2}{\pi} \arcsin(t) Q_+ \\ = & -\frac{2}{\pi} g(t) \frac{s}{1-s} Q_+ + \frac{2}{\pi} \arcsin(t) Q_+ = \frac{2}{\pi} (\arcsin(t) - g(t) \frac{s}{1-s}) Q_+. \end{aligned}$$
Since this holds for any $t \in (0,1]$, the first assertion is now clear. The second assertion follows from the first. □

Theorem 4.2 can also be formulated as follows:

4.3 Corollary. *For Q not diagonal,*
$$\begin{aligned} Q_* \leq & (1 - \delta(s))(\mathrm{tr}(Q) + \sum_{i \neq j} |Q_{ij}|) + \delta(s) Q_+ \\ Q^* \geq & (1 - \delta(s'))(\mathrm{tr}(Q) - \sum_{i \neq j} |Q_{ij}|) + \delta(s') Q^+, \end{aligned}$$
where $\delta(y) := y + \beta(y)(1-y)$ and $\beta(y)$ is defined as in Theorem 4.2.

Proof. By Theorem 4.2,
$$\begin{aligned} \frac{\mathrm{tr}(Q) + \sum_{i\neq j} |Q_{ij}| - Q_*}{\mathrm{tr}(Q) + \sum_{i\neq j} |Q_{ij}| - Q_+} \geq & \frac{\mathrm{tr}(Q) + \sum_{i\neq j} |Q_{ij}| - (1 - \beta(s))\,\mathrm{tr}(Q) - \beta(s) Q_+}{\mathrm{tr}(Q) + \sum_{i\neq j} |Q_{ij}| - Q_+} \\ = & \frac{\sum_{i\neq j} |Q_{ij}| + \beta(s)(\mathrm{tr}(Q) - Q_+)}{\mathrm{tr}(Q) + \sum_{i\neq j} |Q_{ij}| - Q_+} = s + \beta(s)(1-s). \end{aligned}$$
This proves the first assertion. The second assertion follows from the first. □

Theorem 4.2 improves on Theorem 3.3 when Q^+ (resp., Q_+) is 'sufficiently close' to $\mathrm{tr}(Q) + \sum_{i\neq j} |Q_{ij}|$ (resp., $\mathrm{tr}(Q) - \sum_{i\neq j} |Q_{ij}|$); see Corollary 4.3 and Figure 1. This occurs, for example, when $Q_{ij} \geq 0$ (resp., $Q_{ij} \leq 0$) for $i \neq j$; see Corollary 2.4.

It is possible to show that $\delta(y) \leq \mu \Leftrightarrow y \in [a, b]$, where $a \approx 0.5777$, $b \approx 0.7457$, so Theorem 4.2 improves on Theorem 4.1 when $s \in [0.5, 0.5777) \cup (0.7457, 1]$ (resp., when $s' \in [0.5, 0.5777) \cup (0.7457, 1]$). It is important to note that this does not contradict the result in [2]. The examples in [2] showing that the Goemans-Williamson approximation ratio is best possible, have s (resp., s') close to 0.5920, i.e., well within the interval $[0.5777, 0.7457]$.

Thus Theorem 4.2 allows one to predict better accuracy of the output of (the natural generalization of) the Goemans-Williamson MAX-CUT algorithm in certain cases, depending on s. Specifically, in terms of the original Goemans-Williamson MAX-CUT algorithm,

FIGURE 1.

y	$\omega(y)$	$\delta(y)$
0.50	0.718	0.895
0.55	0.736	0.884
0.60	0.755	0.875
0.65	0.777	0.871
0.70	0.800	0.873
0.75	0.826	0.879
0.80	0.854	0.891
0.85	0.885	0.908
0.90	0.919	0.931
0.95	0.957	0.961

since $\frac{1}{(2)(0.5777)} \approx 0.8655$ and $\frac{1}{(2)(0.7467)} \approx 0.6705$ one can predict better accuracy when the output value is either $\geq 86.6\%$ of the total number of edges or $\leq 67.0\%$ of the total number of edges. By way of comparison, $\frac{1}{(2)(0.5920)} \approx 0.8446$, so the Karloff examples have cut size $\approx 84.5\%$ of the total number of edges.

5. Improvements on the trivial bound

To simplify the presentation, we focus our attention now on Q_*. The reader will have no difficulty at this point in formulating the corresponding results for Q^*.

If n is large and r (resp., s) is relatively close to $\frac{n-1}{n}$ then $\operatorname{tr}(Q) - Q_+$ is relatively small compared to $Q^+ - \operatorname{tr}(Q)$ (resp., $\sum_{i\neq j} |Q_{ij}|$) so, in this sense, the trivial bound $Q_* \leq \operatorname{tr}(Q)$ for Q_* is already a good bound. Whether it is best possible is another question. This latter question is the one we are concerned with in this section.

To allow comparison with the trivial bound, we note that Theorems 3.3 and 4.1 can also be formulated as follows:

5.1 Corollary. *For Q not diagonal,*

(1) $Q_* \leq (1 - \alpha(r)) \operatorname{tr}(Q) + \alpha(r) Q_+$,

(2) $Q_* \leq (1 - \gamma(s)) \operatorname{tr}(Q) + \gamma(s) Q_+$,

where r is defined by equation (1), s is defined by equation (2), $\alpha(y) := \frac{2}{\pi} \frac{\sqrt{1-y^2} - y \arccos(y)}{1-y}$ and $\gamma(y) := \frac{\mu - y}{1-y}$.

Proof. (1) By Theorem 3.3,

$$\begin{aligned}\frac{\mathrm{tr}(Q)-Q_*}{\mathrm{tr}(Q)-Q_+} \geq& \frac{\mathrm{tr}(Q)-(1-\omega(r))Q^+-\omega(r)Q_+}{\mathrm{tr}(Q)-Q_+}\\ =&\omega(r)\frac{Q^+-Q_+}{\mathrm{tr}(Q)-Q_+}-\frac{Q^+-\mathrm{tr}(Q)}{\mathrm{tr}(Q)-Q_+}\\ =&\omega(r)\frac{1}{1-r}-\frac{r}{1-r}\\ =&\frac{\omega(r)-r}{1-r}\\ =&\frac{\frac{2}{\pi}(\sqrt{1-r^2}+r\arcsin(r))-r}{1-r}\\ =&\frac{2}{\pi}\frac{\sqrt{1-r^2}-r(\frac{\pi}{2}-\arcsin(r))}{1-r}\\ =&\frac{2}{\pi}\frac{\sqrt{1-r^2}-r\arccos(r)}{1-r}.\end{aligned}$$

(2) By Theorem 4.1,

$$\begin{aligned}\frac{\mathrm{tr}(Q)-Q_*}{\mathrm{tr}(Q)-Q_+} \geq& \frac{\mathrm{tr}(Q)-(1-\mu)(\mathrm{tr}(Q)+\sum_{i\neq j}|Q_{ij}|)-\mu Q_+}{\mathrm{tr}(Q)-Q_+}\\ =&\mu\frac{\mathrm{tr}(Q)+\sum_{i\neq j}|Q_{ij}|-Q_+}{\mathrm{tr}(Q)-Q_+}-\frac{\sum_{i\neq j}|Q_{ij}|}{\mathrm{tr}(Q)-Q_+}\\ =&\mu\frac{1}{1-s}-\frac{s}{1-s}=\frac{\mu-s}{1-s}. \quad \square\end{aligned}$$

Clearly Theorem 4.1 improves on the trivial bound iff $s<\mu$. The function α is positive and decreasing on $[0,1)$, $\alpha(0)=\frac{2}{\pi}$, $\lim_{y\to 1^-}\alpha(y)=0$. The function β defined in the statement of Theorem 4.2 is positive and decreasing on $[\frac{1}{2},1)$, $\beta(\frac{1}{2})\approx 0.7895$, $\lim_{y\to 1^-}\beta(y)=0$. Thus Theorems 3.3 and 4.2 both improve on the trivial bound, but for r (resp., s) close to $\frac{n-1}{n}$ with n large, the improvement is only marginal. See Figure 2. The next result describes more exactly the behaviour of $\alpha(y)$ and $\beta(y)$ for y close to 1.

5.2 Theorem.

(1) $\lim_{y\to 1^-}\frac{\alpha(y)}{\sqrt{1-y}}=(\frac{2}{\pi})(\frac{2\sqrt{2}}{3})\approx(\frac{2}{\pi})(0.9428)$.

(2) $\lim_{y\to 1^-}\frac{\beta(y)}{\sqrt{1-y}}=(\frac{2}{\pi})(\frac{2}{3^{1/4}})\approx(\frac{2}{\pi})(1.5197)$.

Proof. The proof of (1) is straightforward, e.g., use l'Hôpital's rule. For (2), denote by $t=t_y$ the value of t that maximizes the function in the formula for $\beta(y)$. Clearly $t_y\to 0$ as $y\to 1^-$. Using power series approximations, one checks that, for y close to 1, $t_y\approx 3^{3/4}\sqrt{1-y}$. The rest of the computation is standard. $\square$

Figure 2.

y	$\alpha(y)$	$\beta(y)$	$\gamma(y)$
0.50	0.436	0.789	0.757
0.55	0.412	0.743	0.730
0.60	0.388	0.688	0.696
0.65	0.361	0.632	0.653
0.70	0.334	0.576	0.595
0.75	0.304	0.517	0.514
0.80	0.271	0.456	0.393
0.85	0.234	0.389	0.190
0.90	0.191	0.314	−0.214
0.95	0.135	0.219	−1.429

For fixed n, there exists a positive real number $\epsilon \le 1$ (depending on n), such that $\frac{\operatorname{tr}(Q)-Q_*}{\operatorname{tr}(Q)-Q_+} \ge \epsilon$ holds for all non-diagonal symmetric $n\times n$ matrices Q, i.e., such that $Q_* \le (1-\epsilon)\operatorname{tr}(Q)+\epsilon Q_+$ holds for *all* symmetric $n\times n$ matrices Q. For example, we can take $\epsilon = \alpha(\frac{n-1}{n})$ or $\beta(\frac{n-1}{n})$. Alternatively, one can prove the result directly, using Lemma 2.5 and a compactness argument.

Denote by ρ_n the largest $\epsilon \le 1$ such that $Q_* \le (1-\epsilon)\operatorname{tr}(Q)+\epsilon Q_+$ holds for all symmetric $n\times n$ matrices Q. One checks easily that $\rho_1=\rho_2=1$. Recent computations by Pereira [7] show that $\rho_3=\rho_4=\rho_5=\rho_6=\frac{2}{3}$. Of course, ρ_n is a non-increasing function, so $\rho_n \le \frac{2}{3}$ for $n \ge 7$. Nothing else seems to be known about upper bounds for ρ_n. The quadratic form $Q(x)=(x_1+x_2+x_3)^2$ gives $\frac{\operatorname{tr}(Q)-Q_*}{\operatorname{tr}(Q)-Q_+} = \frac{2}{3}$; see Example 2.7. The author knows of no example worse than this.[1] The following question appears to be open.

5.3 Question. Is it true that $\lim_{n\to\infty} \rho_n = 0$?

Our goal here is to find a better lower bound for ρ_n. According to Theorem 5.2 (2), for large n,

$$\beta(\frac{n-1}{n}) \approx \frac{2}{\pi}1.5197\sqrt{1-\frac{n-1}{n}} \approx \frac{0.9675}{\sqrt{n}},$$

i.e., we have a lower bound for ρ_n which approaches zero like $\frac{1}{\sqrt{n}}$. We proceed to improve on this, obtaining a lower bound for ρ_n which approaches zero like $\frac{1}{\sqrt[3]{n}}$. We begin by proving the analog of Theorem 2.6 referred to earlier:

5.4 Theorem. *For Q non-diagonal,*

(1) *If n is even then $\frac{1}{n-1} \le \frac{Q^*-\operatorname{tr}(Q)}{\operatorname{tr}(Q)-Q_*} \le n-1$.*
(2) *If n is odd then $\frac{1}{n} \le \frac{Q^*-\operatorname{tr}(Q)}{\operatorname{tr}(Q)-Q_*} \le n$.*

[1] If Conjecture 2.12 at the end of Karloff's paper [2] is true, then one can find examples with $\frac{\operatorname{tr}(Q)-Q_*}{\operatorname{tr}(Q)-Q_+}$ close to $\frac{2}{\pi}$ (i.e., just slightly worse than $\frac{2}{3}$), for large n.

Proof. We can assume the diagonal entries of Q are 0. For any subset I of $\{1,\dots,n\}$, denote by $Q_I(x)$ the quadratic form obtained from $Q(x)$ by replacing x_i by $-x_i$ for each $i \in I$. For each pair of indices $i < j$, if either both of i and j are in I or both of i and j are not in I then the coefficient of x_ix_j in $Q_I(x)$ is $2Q_{ij}$. In the remaining cases, i.e., where one of i,j is in I and the other is not, the coefficient of x_ix_j in $Q_I(x)$ is $-2Q_{ij}$. For $1 \le k < n$ denote by $Q_k(x)$ the sum of the $Q_I(x)$, I running through all k-element subsets of $\{1,\dots,n\}$. For fixed $i < j$, x_ix_j appears with coefficient $-2Q_{ij}$ in $2\binom{n-2}{k-1}$ of the $Q_I(x)$ and with coefficient $2Q_{ij}$ in the remainder of the $Q_I(x)$. It follows that

$$Q_k(x) = \binom{n}{k} Q(x) - 4\binom{n-2}{k-1} Q(x) = \frac{(n-2k)^2-n}{n(n-1)}\binom{n}{k} Q(x).$$

Let $Q_* = -\delta$. Then $Q_I(x) \ge -\delta$ for each $x \in \{-1,1\}^n$ so $Q_k(x) \ge -\binom{n}{k}\delta$, i.e., $-Q_k(x) \le \binom{n}{k}\delta$. If $n > (n-2k)^2$ this yields $Q(x) \le \frac{n(n-1)}{n-(n-2k)^2}\delta$. For n odd, say $n = 2\ell - 1$, apply this with $k = \ell$ to obtain $Q(x) \le n\delta$. Similarly, for n even, say $n = 2\ell$, apply this with $k = \ell$ to obtain $Q(x) \le (n-1)\delta$. A similar argument shows that if $Q^* = \delta$ then for any $x \in \{-1,1\}^n$, $Q(x) \ge -n\delta$ if n is odd and $Q(x) \ge -(n-1)\delta$ if n is even. □

5.5 Corollary.

(1) *If Q is non-diagonal then $Q_* \le (1 - \frac{r\alpha(r')}{r'(2\ell-1)})\operatorname{tr}(Q) + \frac{r\alpha(r')}{r'(2\ell-1)}Q_+$.*

(2) *If, in addition, $Q_{ij} \ge 0$ for $i \ne j$, then $Q_* \le (1 - \frac{r}{r'(2\ell-1)})\operatorname{tr}(Q) + \frac{r}{r'(2\ell-1)}Q_+$.*

Here, r, r' are defined by equation (1) and ℓ is defined by $n = 2\ell$ if n is even, $n = 2\ell - 1$ if n is odd.

Proof. (1) After shifting and scaling, we can assume that $\operatorname{tr}(Q) = 0$ and $Q_+ = -1$, so $Q^+ = \frac{r}{r'}$. Applying Corollary 5.1 (1) to $-Q$ yields $Q^* \ge (1-\alpha(r'))\operatorname{tr}(Q) + \alpha(r')Q^+ = \alpha(r')Q^+$. By Theorem 5.4, $2\ell - 1 \ge \frac{Q^*}{-Q_*}$. This implies $Q_* \le -\frac{Q^*}{2\ell-1} \le -\frac{\alpha(r')Q^+}{2\ell-1} = -\frac{r\alpha(r')}{r'(2\ell-1)}$. The proof of (2) is similar, except that now, by Corollary 2.4, we have $Q^* = Q^+$. □

Note: As y increases to $\frac{n-1}{n}$, $\frac{y\alpha(1-y)}{(1-y)(2\ell-1)}$ increases to $\alpha(\frac{1}{n})$ if n is even and to $\frac{n-1}{n}\alpha(\frac{1}{n})$ if n is odd, and, similarly, $\frac{y}{(1-y)(2\ell-1)}$ increases to 1 if n is even and to $\frac{n-1}{n}$ if n is odd. In particular, the bounds provided by Corollary 5.5 are appreciably better than the trivial bound if r is sufficently close to $\frac{n-1}{n}$.

Now consider

$$\overline{\alpha}_n := \min_{y \in [\frac{1}{n}, \frac{n-1}{n}]} \{\max\{\alpha(y), \frac{y\alpha(1-y)}{(1-y)(2\ell-1)}\}\},$$

where ℓ is defined as in Corollary 5.5. This is a lower bound for ρ_n. See Figure 3 for a comparison of the lower bounds $\overline{\alpha}_n$, $\beta(\frac{n-1}{n})$ and $\gamma(\frac{n-1}{n})$. The bound given by $\beta(\frac{n-1}{n})$ is best for n in the range $2 \le n \le 23$, $n \ne 3$. For $n = 3$ the bound given by $\gamma(\frac{n-1}{n})$ is best.

For $n \geq 24$ the bound given by $\overline{\alpha}_n$ is best. It is pretty clear that these lower bounds are nowhere near optimal.

The 5th column in Figure 3 gives lower bounds for the function $\rho'_n :=$ the maximum $\epsilon \leq 1$ such that $Q_* \leq (1-\epsilon)\operatorname{tr}(Q) + \epsilon Q_+$ holds for all symmetric $n \times n$ matrices Q such that $Q_{ij} \geq 0$ for $i \neq j$. The function $\overline{\beta}_n$ is defined by

$$\overline{\beta}_n := \min_{y \in [\frac{1}{2}, \frac{n-1}{n}]} \{\max\{\beta(y), \frac{y}{(1-y)(2\ell-1)}, \gamma(y)\}\}.$$

Again, it is pretty clear that this lower bound for ρ'_n is nowhere near optimal.

FIGURE 3.

n	$\overline{\alpha}_n$	$\beta(\frac{n-1}{n})$	$\gamma(\frac{n-1}{n})$	$\overline{\beta}_n$
2	0.4360	0.7895	0.7572	0.7895
3	0.3526	0.6134	0.6358	0.6440
4	0.3497	0.5174	0.5144	0.6440
5	0.3093	0.4560	0.3930	0.5467
10	0.2653	0.3137		0.4524
15	0.2300	0.2539		0.3857
20	0.2148	0.2190		0.3579
25	0.1980	0.1954		0.3278
50	0.1615	0.1389		0.2639
100	0.1293	0.0975		0.2097
200	0.1035	0.0685		0.1667

5.6 Theorem. *For large* n,

(1) $\overline{\alpha}_n \approx \frac{0.6121}{\sqrt[3]{n}}$.

(2) $\overline{\beta}_n \approx \frac{0.9782}{\sqrt[3]{n}}$.

Proof. (1) On the interval $[\frac{1}{n}, \frac{n-1}{n}]$, $\alpha(y)$ is decreasing and $\frac{y\alpha(1-y)}{(1-y)(2\ell-1)}$ is increasing. Thus the minimum occurs when $\alpha(y) = \frac{y\alpha(1-y)}{(1-y)(2\ell-1)}$. It follows that $y \to 1$ as $n \to \infty$ and

$$\begin{aligned} n\alpha(y)^3 =& n\frac{y\alpha(1-y)}{(1-y)(2\ell-1)}\alpha(y)^2 \\ =& \frac{n}{2\ell-1}y\alpha(1-y)(\frac{\alpha(y)}{\sqrt{1-y}})^2 \to (1)(1)(\frac{2}{\pi})(\frac{2}{\pi})^2(\frac{2\sqrt{2}}{3})^2 \text{ as } n \to \infty. \end{aligned}$$

Here, we use Theorem 5.2 (1). This implies $\lim_{n\to\infty} \sqrt[3]{n}\overline{\alpha}_n = \frac{2}{\pi}(\frac{2\sqrt{2}}{3})^{2/3} \approx 0.6121$.

(2) The proof is similar. For large n, the minimum on the interval $[\frac{1}{2}, \frac{n-1}{n}]$ occurs when $\beta(y) = \frac{y}{(1-y)(2\ell-1)}$ and $\gamma(y)$ is negative. Also $y \to 1$ as $n \to \infty$ and

$$\begin{aligned} n\beta(y)^3 =& n\frac{y}{(1-y)(2\ell-1)}\beta(y)^2 \\ =& \frac{n}{2\ell-1} y (\frac{\beta(y)}{\sqrt{1-y}})^2 \to (1)(1)(\frac{4}{3^{1/4}\pi})^2 \text{ as } n \to \infty. \end{aligned}$$

Here, we use Theorem 5.2 (2). This implies $\lim_{n\to\infty} \sqrt[3]{n}\beta_n = (\frac{4}{3^{1/4}\pi})^{2/3} \approx 0.9782$. □

6. Conclusion

The paper gives error estimates when Q_* is approximated by Q_+. The bounds provided by Theorems 3.2 and 3.3 come from [5]. Theorem 4.1 is a generalization of the result in [1]. A special case of Theorem 4.1 appears already in [11]. The bound given in Theorem 4.2 is new, improves on Theorem 3.3 when s is close to r, and provides improved estimates in the MAX-CUT algorithm in [1], for $s \notin [0.5777, 0.7457]$. The bounds given by Corollary 5.5 are also new, but are only useful when r is sufficiently close to $\frac{n-1}{n}$. The situation regarding ρ_n is unsatisfactory. One would like to know the limiting value of ρ_n as $n \to \infty$. Is it zero, or is it positive? The best we are able to show is that, for large n, ρ_n is bounded away from zero by a function of the form $\frac{C}{\sqrt[3]{n}}$, where C is a constant.

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