CLOSURES OF QUADRATIC MODULES

JAKA CIMPRIČ, MURRAY MARSHALL, TIM NETZER

Abstract. We consider the problem of determining the closure $\overline{M}$ of a quadratic module M in a commutative $\mathbb{R}$-algebra with respect to the finest locally convex topology. This is of interest in deciding when the moment problem is solvable [28] [29] and in analyzing algorithms for polynomial optimization involving semidefinite programming [12]. The closure of a semiordering is also considered, and it is shown that the space $\mathcal{Y}_M$ consisting of all semiorderings lying over M plays an important role in understanding the closure of M. The result of Schmüdgen for preorderings in [29] is strengthened and extended to quadratic modules. The extended result is used to construct an example of a non-archimedean quadratic module describing a compact semialgebraic set that has the strong moment property. The same result is used to obtain a recursive description of $\overline{M}$ which is valid in many cases.

The *moment problem* from functional analysis asks which linear functionals on certain vector spaces of functions are integration with respect to a measure. For example, for each positive linear functional L on the Banach space $C(X)$ of continuous real-valued functions on a compact space X, there exists some Borel measure μ on X such that $L(f) = \int_X f d\mu$ for all $f \in C(X)$. This is a version of the famous Riesz Representation Theorem.

One also considers other vector spaces or algebras instead of $C(X)$. Haviland's Theorem, explained in detail in [15], asserts that a linear functional L on the real polynomial algebra $\mathbb{R}[x_1, \dots, x_n]$ is integration with respect to a Borel measure on some closed set $K \subseteq \mathbb{R}^n$ if and only if L maps polynomials that are nonnegative on K to nonnegative reals. This result links the functional analytic moment problem to an important problem from real algebraic geometry: describe all polynomials that are nonnegative on a given set K.

For example, every globally nonnegative polynomial in one variable is a sum of squares of polynomials. Since sums of squares are much easier to deal with than general nonnegative polynomials (from the point of

---

Date: September 2, 2009.

2000 *Mathematics Subject Classification.* 12D15, 14P99, 44A60.

Key words and phrases. moment problem, positive polynomials, sums of squares.

view of semidefinite optimization for example), this is a result quite in the spirit one likes to have. Unfortunately, the problem is harder in higher dimensions. Not every globally nonnegative polynomial in two or more variables is a sum of squares of polynomials, the first explicit example was given by Motzkin in 1967.

However, if a set $K \subseteq \mathbb{R}^n$ is described by finitely many polynomial inequalities $p_1(x) \geq 0, \ldots, p_r(x) \geq 0$, and if K is *compact*, then the situation is much nicer again: at least every polynomial that is *strictly positive* on K can be represented by products and sums of squares and the defining inequalities p_i, i.e. it belongs to the *preordering* generated by $p_1, \ldots, p_r$. This is Schmüdgen's famous result from 1991 [28], which triggered comprehensive developments in the whole area of real algebraic geometry. See [27] for an overview of the most recent results and [13] for their application to polynomials optimization.

Going back to the moment problem, one takes the following approach: instead of testing nonnegativity of a linear functional on *all* polynomials that are nonnegative on a set $K \subseteq \mathbb{R}^n$, one would like to test nonnegativity on a certain finitely generated preordering only, and this should be enough to obtain an integral representation for the functional. This would significantly reduce the complexity of the problem and allow to apply semidefinite optimization procedures. So the question is the following: is there a finitely generated preordering whose double dual cone consists of all polynomials nonnegative on K? Schmüdgen's above mentioned result gives a positive answer in the case of a compact basic closed semialgebraic set K. For non-compact sets, his Fibre Theorem from 2003 [29] provides a method to apply a dimension reduction when considering the problem.

In all of the above results one is also interested in finitely generated *quadratic modules* instead of preorderings. The elements from the quadratic module arise by multiplying the generators with sums of squares and adding them, but no pairwise multiplications of the generators is involved. The quadratic module is smaller than the preordering.

Since the double dual cone of a cone in a vector space equals its closure with respect to the finest locally convex topology, we begin this work by investigating this closure in a general setting. In Section 1 we consider the general relationship between the closure $\overline{C}$ and the sequential closure $C^{\ddagger}$ of a subset C of a real vector space V in the finest locally convex topology. We are mainly interested in the case where C is a cone in V. We consider cones with non-empty interior and cones satisfying $C \cup -C = V$ (semiorderings have this property).

In Section 2 we begin our investigation of the closure $\overline{M}$ of a quadratic module M of a commutative $\mathbb{R}$-algebra A; the focus is on finitely generated quadratic modules of finitely generated algebras. The closure of a semiordering Q of A is also considered, and it is shown that the space $\mathcal{Y}_M$ consisting of all semiorderings of A lying over M plays an important role in understanding the closure of M; see Propositions 2.2, 2.3 and 2.4. The result of Schmüdgen for preorderings in [29] is strengthened and extended to quadratic modules; see Theorem 2.8.

In Section 3 we consider the case of quadratic modules that describe compact semialgebraic sets. We use Theorem 2.8 to deduce various results; see Theorems 3.1 and 3.4; and also to construct an example where the semialgebraic set $\mathcal{K}_M$ is compact, M satisfies the strong moment property (SMP), but M is not archimedean; see Example 3.7.

Theorem 2.8 is also used in Section 4, to obtain a recursive description of $\overline{M}$ which although it is not valid in general; see Example 4.3; is valid in many cases; see Theorem 4.7.

In Section 5, which is an appendix to Section 1, we give an example of a cone C where the increasing sequence of iterated sequential closures

$$C \subseteq C^{\ddagger} \subseteq (C^{\ddagger})^{\ddagger} \subseteq \cdots$$

terminates after precisely n steps. In the case of quadratic modules and preorderings, nothing much is known about the sequence of iterated sequential closures beyond the example with $M^{\ddagger} \neq \overline{M}$ given in [19].

1. Closures of Cones

In this section we consider general convex cones, their closures and interiors. The results will be used in the following sections, when considering closures of quadratic modules.

Consider a real vector space V. A convex set $U \subseteq V$ is called *absorbent*, if for every $x \in V$ there exists $\lambda > 0$ such that $x \in \lambda U$. U is called *symmetric*, if $\lambda U \subseteq U$ for all $|\lambda| \leq 1$. The set of all convex, absorbent and symmetric subsets of V forms a zero neighborhood base of a vector space topology on V (see [4, II.25] or [25]). This topology is called the *finest locally convex topology* on V. V endowed with this topology is hausdorff, each linear functional on V is continuous, and each finite dimensional subspace of V inherits the euclidean topology.

Let C be a subset of V and denote by $C^{\ddagger}$ the set of all elements of V which are expressible as the limit of some sequence of elements of C. By [25, Ch. 2, Example 7(b)], every converging sequence in V lies in a finite dimensional subspace of V, so $C^{\ddagger}$ is just the union of the $\overline{C \cap W}$, W running through the set of all finite dimensional subspaces of V. (Observe: Each such W is closed in V, so $\overline{C \cap W}$ is just the closure of $C \cap W$ in W.) We refer to $C^{\ddagger}$ as the *sequential closure* of C. Clearly $C \subseteq C^{\ddagger} \subseteq \overline{C}$, where $\overline{C}$ denotes the closure of C. For any subset C of V we have a transfinite increasing sequence of subsets $(C_\lambda)_{\lambda \geq 0}$ of V defined by $C_0 = C$, $C_{\lambda^+} = (C_\lambda)^{\ddagger}$, and $C_\mu = \cup_{\lambda < \mu} C_\lambda$ if μ is a limit ordinal. Question: Can one say anything at all about when this sequence terminates? We return to this point later; see the appendix at the end of the paper.

We are in particular interested in the case where the dimension of V is countable. In this case, a subset C of V is closed if and only if $C \cap W$ is closed in W for each finite dimensional subspace W of V [3, Proposition 1]. So $C^{\ddagger} = C$ if and only if C is closed. Thus the sequence of iterated sequential closures of C terminates precisely at $\overline{C}$.

For the time being, we drop the assumption that V is of countable dimension. We are in particular interested in the case when C is a cone of V, i.e. if $C + C \subseteq C$ and $\mathbb{R}^+ \cdot C \subseteq C$ holds. In this case $C^{\ddagger}$ and $\overline{C}$ are also cones. Every cone is a convex set. If U is any convex open set in V such that $U \cap C = \emptyset$ then, by the Separation Theorem [4, II.39, Corollary 5] (or [15, Theorem 3.6.3] in the case of countable dimension), there exists a linear map $L : V \to \mathbb{R}$ such that $L \geq 0$ on C and $L < 0$ on U. This implies $\overline{C} = C^{\vee\vee}$. Here, $C^{\vee}$ is the set of all linear functionals $L : V \to \mathbb{R}$ such that $L(v) \geq 0$ for all $v \in C$ and $C^{\vee\vee}$ is the set of all $v \in V$ such that $L(v) \geq 0$ for all $L \in C^{\vee}$. $C^{\vee\vee}$ is also called the *double dual cone* of C.

Proposition 1.1. *Let C be a cone in V and let $v \in V$. The following are equivalent:*

1. *v is the limit of a sequence of elements of C.*
2. *there is some $q \in V$ such that $v + \epsilon q \in C$ for each real $\epsilon > 0$.*

Proof. (2) $\Rightarrow$ (1). Let $v_i = v + \frac{1}{i} q$, $i = 1, 2, \cdots$. Then $v_i \in C$ and $v_i \to v$ as $i \to \infty$. (1) $\Rightarrow$ (2). Let $v = \lim_{i\to\infty} v_i$, $v_i \in C$. As explained earlier, the subspace of V spanned by $v_1, v_2, \cdots$ is finite dimensional. Let $w_1, \dots, w_N \in C$ be a basis for this subspace. Then $v_i = \sum_{j=1}^N r_{ij} w_j$, $v = \sum_{j=1}^N r_j w_j$, $r_{ij}, r_i \in \mathbb{R}$, $r_j = \lim_{i\to\infty} r_{ij}$. Let $q := \sum_{j=1}^N w_j$. Then, for any real $\epsilon > 0$, $r_{ij} < r_j + \epsilon$ for i sufficiently large, so $v + \epsilon q = \sum_{j=1}^N (r_j + \epsilon) w_j = \sum_{j=1}^N r_{ij} w_j + \sum_{j=1}^N (r_j + \epsilon - r_{ij}) w_j = v_i + \sum_{j=1}^N (r_j + \epsilon - r_{ij}) w_j \in C$. □

Corollary 1.2. *If C is a cone of V then*

$$C^{\ddagger} = \{v \in V \mid \exists q \in V \text{ such that } v + \epsilon q \in C \text{ for all real } \epsilon > 0\}.$$

The proof of Proposition 1.1 shows we can always choose $q \in C$. In fact, we can find a finite dimensional subspace W of V (namely, the subspace of V spanned by $w_1, \dots, w_N$) such that $q \in W$ and q is an interior point of $C \cap W$.

Cones with non-empty interior are of special interest. For a subset C of V, a vector $v \in C$ is called an *algebraic interior point of* C if for all $w \in V$ there is a real $\epsilon > 0$ such that $v + \epsilon w \in C$.

Proposition 1.3.

1. *Let C be a convex set in V. A vector $v \in C$ is an interior point of C iff v is an algebraic interior point of C.*
2. *Let q be an interior point of a cone C of V. If $v \in \overline{C}$ then $v + \epsilon q$ is an interior point of C for all real $\epsilon > 0$.*
3. *If C is a cone of V with non-empty interior, then $C^{\ddagger} = \overline{C} = \overline{\operatorname{int}(C)} = \operatorname{int}(C)^{\ddagger}$.*

Proof. (1) Let $v \in C$ be an algebraic interior point. Translating, we can assume $v = 0$. Fix a basis v_i, $i \in I$ for V and real $\epsilon_i > 0$ such that $\epsilon_i v_i$ and $-\epsilon_i v_i$ belong to C. Take U to be the convex hull of the set $\{\epsilon_i v_i, -\epsilon_i v_i \mid i \in I\}$. U is convex, absorbent and symmetric and $0 \in U \subseteq C$. The converse is clear.

(2) If $q \in \operatorname{int}(C)$ and $v \in \overline{C}$ then $\lambda v + (1-\lambda) q \in \operatorname{int}(C)$ for all $0 \leq \lambda < 1$, by [5, chapter III, Lemma 2.4] or [25, page 38, 2.1.1]. Applying this with $\lambda = \frac{1}{1+\epsilon}$ and multiplying by $1 + \epsilon$ yields $v + \epsilon q \in \operatorname{int}(C)$ for all real $\epsilon > 0$.

(3) This is immediate from (2), by Corollary 1.2. □

Here is more folklore concerning cones with non-empty interior:

Proposition 1.4. *Suppose that C is a cone of V, q is an interior point of C, and $v \in V$. Then the following are equivalent:*

1. *v is an interior point of C,*
2. *there exist $\epsilon > 0$ such that $v - \epsilon q \in C$,*
3. *for every nonzero $L \in C^{\vee}$, $L(v) > 0$.*

Proof. (1) implies (2) by the easy direction of assertion (1) in Proposition 1.3. To prove that (2) implies (3), pick $L \in C^{\vee}$ and $w \in V$ such that $L(w) \neq 0$. Since q is an interior point of C, there exists a $\delta > 0$ such that $q \pm \delta w \in C$. It follows that $L(q) \geq \delta |L(w)| > 0$. Hence, $L(v) \geq \epsilon L(q) > 0$. Finally, we prove that (3) implies (1) by contradiction. Note that $\operatorname{int}(C)$ is an open convex set. If $v \notin \operatorname{int}(C)$, there exists by the Separation Theorem a functional L on V such that $L(v) \leq 0$ and $L(\operatorname{int}(C)) > 0$. It follows that $L(\overline{\operatorname{int}(C)}) \geq 0$. But $\overline{\operatorname{int}(C)} = \overline{C}$ by assertion (3) of Proposition 1.3, hence $L(C) \geq 0$. □

We are also interested in cones satisfying $C \cup -C = V$. Note: For any cone C of V, $C \cap -C$ is a subspace of V.

Proposition 1.5. *Let C be a cone of V satisfying $C \cup -C = V$. The following are equivalent:*

1. *C is closed in V.*
2. *The vector space $\frac{V}{C\cap -C}$ has dimension ≤ 1.*

Proof. (2) $\Rightarrow$ (1). Replacing V by $V/(C\cap -C)$ and C by $C/(C\cap -C)$, we are reduced to the case $C \cap -C = \{0\}$. If V is 0-dimensional then $V = \{0\} = C$, so C is closed in V. If V is 1-dimensional, fix $v \in C$, $v \neq 0$. Then $V = \mathbb{R}v$ and $C = \mathbb{R}^+ v$, so C is closed in V. (1) $\Rightarrow$ (2). Suppose C is closed and $\frac{V}{C\cap -C}$ has dimension ≥ 2. Fix $v_1, v_2 \in V$ linearly independent modulo $C \cap -C$. Let W be denote the subspace of V spanned by v_1, v_2. Then $C \cap W$ is closed in W, $(C \cap W) \cup -(C\cap W) = W$, and $v_1, v_2 \in W$ are linearly independent modulo $(C\cap W) \cap -(C\cap W)$. In this way, replacing V by W and C by $C \cap W$, we are reduced to the case where $V = \mathbb{R}v_1 \oplus \mathbb{R}v_2$. Replacing v_i by $-v_i$, if necessary, we can suppose $v_i \in -C$, $i = 1, 2$. Then $v := v_1 + v_2$ is an interior point of $-C$. In particular, $\operatorname{int}(-C) \neq \emptyset$. Since v_1 and v_2 are linearly independent modulo $C \cap -C$, we find $C \cap -C = \{0\}$ and $C \cap \operatorname{int}(-C) = \emptyset$. By the Separation Theorem, there exists a linear map $L : V \to \mathbb{R}$ with $L \geq 0$ on C, $L < 0$ on $\operatorname{int}(-C)$ (so $L \leq 0$ on $-C$). Since V is 2-dimensional, there exists $w \in V$, $L(w) = 0$, $w \neq 0$. Replacing w by $-w$ if necessary, we may assume $w \in -C$ (so $w \notin C$). Consider the line through v and w. Since $L(v) < 0$ and $L(w) = 0$, there are points u on this line arbitrarily close to w satisfying $L(u) > 0$ (so $u \in C$). This proves $w \in \overline{C}$ for all such points w, so C is not closed, a contradiction. □

Corollary 1.6. *Suppose C is a cone of V satisfying $C \cup -C = V$. Then $C^\ddagger$ is closed, i.e., $\overline{C} = C^\ddagger$.*

Proof. According to Proposition 1.5 it suffices to show that $\frac{V}{C^\ddagger \cap -C^\ddagger}$ has dimension at most one. Suppose this is not the case, so we have $v_1, v_2 \in V$ linearly independent modulo $C^\ddagger \cap -C^\ddagger$. Let $W = \mathbb{R}v_1 \oplus \mathbb{R}v_2$ and consider the closed cone $\overline{C\cap W}$ in W. Since $\overline{C\cap W} \cup -\overline{C\cap W} = W$, Proposition 1.5 applied to the cone $\overline{C\cap W}$ of W implies that v_1, v_2 are linearly dependent modulo $\overline{C\cap W} \cap -\overline{C\cap W}$. On the other hand, $\overline{C\cap W} \subseteq C^\ddagger$, so $\overline{C\cap W} \cap -\overline{C\cap W} \subseteq C^\ddagger \cap -C^\ddagger$. This contradicts the assumption that v_1, v_2 are linearly independent modulo $C^\ddagger \cap -C^\ddagger$. □

2. Closures of Quadratic Modules

We introduce basic terminology, also see [15] or [23]. Let A be a commutative ring with 1. For the rest of this work we assume $\frac{1}{2} \in A$. For $f_1, \dots, f_t \in A$, $(f_1, \dots, f_t)$ denotes the ideal of A generated by

$f_1, \dots, f_t$. For any prime ideal $\mathfrak{p}$ of A, $\kappa(\mathfrak{p})$ denotes the residue field of A at $\mathfrak{p}$, i.e., $\kappa(\mathfrak{p})$ is the field of fractions of the integral domain $\frac{A}{\mathfrak{p}}$. We denote by $\dim(A)$ the krull dimension of the ring A.

A *quadratic module* of A is a subset Q of A satisfying $Q + Q \subseteq Q$, $f^2 \cdot Q \subseteq Q$ for all $f \in A$ and $1 \in Q$. If Q is a quadratic module of A, then $Q \cap -Q$ in an ideal of A (since $\frac{1}{2} \in A$). $Q \cap -Q$ is referred to as the *support* of Q. The quadratic module Q is said to be *proper* if $Q \neq A$. Since $\frac{1}{2} \in A$, this is equivalent to $-1 \notin Q$ (using the identity $a = (\frac{a+1}{2})^2 - (\frac{a-1}{2})^2$). A *semiordering* of A is a quadratic module Q of A satisfying $Q \cup -Q = A$ and $Q \cap -Q$ is a prime ideal of A. A *preordering* (resp., *ordering*) of A is a quadratic module (resp., semiordering) of A which is closed under multiplication. $\sum A^2$ denotes the set of (finite) sums of squares of elements of A. This is the smallest quadratic module of A, and also a preordering.

We assume always that our ring A is an $\mathbb{R}$-algebra. Then A comes equipped with the topology described in Section 1. Any quadratic module Q of A is a cone, so $Q^{\ddagger}$ and $\overline{Q}$ are cones. But actually, if Q is a quadratic module (resp., preordering) of A, then $Q^{\ddagger}$ and $\overline{Q}$ are quadratic modules (resp. preorderings) of A. For $Q^{\ddagger}$ this is easy to see, for $\overline{Q}$ it is proven as in [6, Lemma 1], using that multiplication by a fixed element is a linear map on A, and therefore continuous.

In case A is finitely generated, say $x_1, \dots, x_n$ generate A as an $\mathbb{R}$-algebra, then the set of monomials $x_1^{d_1} \cdots x_n^{d_n}$ is countable and generates A as a vector space over $\mathbb{R}$. In that case, the multiplication of A is continuous. This is another way to prove that closures of quadratic modules (preorderings) are again quadratic modules (preorderings) in that case. We denote the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$ by $\mathbb{R}[\underline{x}]$ for short.

A quadratic module Q is said to be *archimedean* if for every $f \in A$ there is an integer $k \geq 0$ such that $k + f \in Q$.

Proposition 2.1. *For any quadratic module Q of A, the following are equivalent:*

(1) *Q is archimedean,*
(2) *1 belongs to the interior of Q,*
(3) *Q has non-empty interior.*

Proof. Clearly, Q is archimedean iff 1 is an algebraic interior point of Q, hence (1) $\Leftrightarrow$ (2) follows from the first assertion of Proposition 1.3. It remains to show (3) $\Rightarrow$ (2). Every functional $L \in Q^{\vee}$ satisfies the Cauchy-Schwartz inequality, $L(a)^2 \leq L(1)L(a^2)$. If follows that every nonzero $L \in Q^\vee$ satisfies $L(1) > 0$. Since Q has non-empty interior, it follows by Proposition 1.4 that 1 is an interior point of Q. $\square$

The simplest example of a non-archimedean quadratic module is the quadratic module $Q = \sum \mathbb{R}[x]^2$ of the algebra $A = \mathbb{R}[x]$. By Proposition 2.1, 1 is not an interior point of Q and, by its proof, $L(1) > 0$ for every nonzero $L \in Q^\vee$. So, the implication (3) $\Rightarrow$ (1) of Proposition 1.4 is in general not valid for cones without interior points.

Proposition 2.2. *Let Q be a semiordering of A. If Q is not archimedean then $Q^\ddagger = \overline{Q} = A$. If Q is archimedean, then there exists a unique ring homomorphism $\alpha : A \to \mathbb{R}$ with $Q \subseteq \alpha^{-1}(\mathbb{R}^+)$, and $Q^\ddagger = \overline{Q} = \alpha^{-1}(\mathbb{R}^+)$.*

Proof. If Q is not archimedean there exists $q \in A$ with $k + q \notin Q$ for all real $k > 0$. Then $-k - q \in Q$, i.e., $-1 - \frac{1}{k}q \in Q$, for all real $k > 0$. This proves $-1 \in Q^\ddagger$, so $Q^\ddagger = \overline{Q} = A$.

Suppose Q is archimedean. According to [15, Theorem 5.2.5] there exists a ring homomorphism $\alpha : A \to \mathbb{R}$ such that $Q \subseteq \alpha^{-1}(\mathbb{R}^+)$. Since α is linear and therefore continuous, $\alpha^{-1}(\mathbb{R}^+)$ is closed, and so $\overline{Q} \subseteq \alpha^{-1}(\mathbb{R}^+)$. If $f \in \alpha^{-1}(\mathbb{R}^+)$ then for any real $\epsilon > 0$, $\alpha(f + \epsilon) > 0$ so $f + \epsilon \in Q$. (If $f + \epsilon \notin Q$ then $-(f + \epsilon) \in Q$ so $-(f + \epsilon) \in \alpha^{-1}(\mathbb{R}^+)$, which contradicts our assumption.) It follows that $Q^\ddagger = \overline{Q} = \alpha^{-1}(\mathbb{R}^+)$. (This can also be deduced from Proposition 1.5.) Uniqueness of α follows easily from the fact that α is the identity on $\mathbb{Q}$, see for example [15, Lemma 5.2.6]. $\square$

Proposition 2.3. *Let A be finitely generated. For any set of semiorderings $\mathcal{Y}$ of A,*

$$(\cap_{Q\in\mathcal{Y}} Q)^\ddagger = \overline{\cap_{Q\in\mathcal{Y}} Q} = \cap_{Q\in\mathcal{Y}} \overline{Q}.$$

Proof. Suppose $f \in \cap_{Q\in\mathcal{Y}} \overline{Q}$. Fix generators $x_1, \dots, x_n$ of A as an $\mathbb{R}$-algebra and let d denote the degree of f viewed as a polynomial in $x_1, \dots, x_n$ with coefficients in $\mathbb{R}$. Let $g = 1 + \sum_{i=1}^n x_i^2$ and fix an integer e with $2e > d$. We claim that for any real $\epsilon > 0$ and any $Q \in \mathcal{Y}$, $f + \epsilon g^e \in Q$. This will prove that $f + \epsilon g^e \in \cap_{Q\in\mathcal{Y}} Q$ for any real $\epsilon > 0$, so $f \in (\cap_{Q\in\mathcal{Y}} Q)^\ddagger$, which will complete the proof. Let $\mathfrak{p} := Q \cap -Q$, let Q' denote the extension of Q to the residue field $\kappa(\mathfrak{p})$, and let v denote the natural valuation of $\kappa(\mathfrak{p})$ associated to Q' (the valuation ring of v is the convex hull of the integers with respect to Q', see for example [15, Theorem 5.3.3]). To prove the claim we consider two cases. Suppose first that $v(x_i + \mathfrak{p}) < 0$ for some i. Reindexing we may suppose $v(x_1 + \mathfrak{p}) \le v(x_i + \mathfrak{p})$ for all i. Then $v(g^e + \mathfrak{p}) = ev(g + \mathfrak{p}) = 2ev(x_1 + \mathfrak{p}) < dv(x_1 + \mathfrak{p}) \le v(f + \mathfrak{p})$. It follows that the sign of $f + \epsilon g^e$ at Q is the same as the sign of g^e at Q in this case, i.e., $f + \epsilon g^e \in Q$. In the remaining case $v(x_i + \mathfrak{p}) \geq 0$ for all i so $\frac{A}{\mathfrak{p}}$ is a subring of the valuation ring B_v in this case. Since the residue field of v is $\mathbb{R}$, we have a ring homomorphism $\alpha : A \to \mathbb{R}$ defined by the composition $A \to \frac{A}{\mathfrak{p}} \subseteq B_v \to \mathbb{R}$. Then $\overline{Q} \subseteq \alpha^{-1}(\mathbb{R}^+)$ so $\alpha(f) \geq 0$ and $\alpha(f + \epsilon g^e) > 0$. This implies that $f + \epsilon g^e \in Q$ also holds in this case. □

We assume always that M is a quadratic module of A. For some results we need that M and/or A are finitely generated, some results hold in general. Let $\mathcal{Y}_M$ denote the set of all semiorderings of A containing M, $\mathcal{X}_M$ the set of all orderings of A containing M and $\mathcal{K}_M$ the set of geometric points of $\mathcal{X}_M$, i.e., the orderings of A having the form $\alpha^{-1}(\mathbb{R}^+)$ for some ring homomorphism $\alpha : A \to \mathbb{R}$ with $M \subseteq \alpha^{-1}(\mathbb{R}^+)$. If A is finitely generated, then $\mathcal{K}_M$ can be identified with a certain subset of the real points of the affine variety corresponding to A.

For any set of semiorderings $\mathcal{Y}$ of A, define $\operatorname{Pos}(\mathcal{Y}) := \cap_{Q\in\mathcal{Y}} Q$, i.e., $\operatorname{Pos}(\mathcal{Y}) := \{f \in A \mid f \geq 0 \text{ on } \mathcal{Y}\}$. Since $\mathcal{K}_M \subseteq \mathcal{X}_M \subseteq \mathcal{Y}_M$ it follows that

$$\operatorname{Pos}(\mathcal{K}_M) \supseteq \operatorname{Pos}(\mathcal{X}_M) \supseteq \operatorname{Pos}(\mathcal{Y}_M) \supseteq M. \tag{2.1}$$

Proposition 2.4. *Let A be finitely generated, M an arbitrary quadratic module in A. Then*

$$\operatorname{Pos}(\mathcal{K}_M) = \operatorname{Pos}(\mathcal{Y}_M)^{\ddagger} = \overline{\operatorname{Pos}(\mathcal{Y}_M)}.$$

Proof. Immediate from Proposition 2.2 and 2.3. □

One can improve upon (2.1) and Proposition 2.4 in important cases:

Proposition 2.5.

(1) *If A and M are finitely generated, then* $\operatorname{Pos}(\mathcal{K}_M) = \operatorname{Pos}(\mathcal{X}_M)$.
(2) *If either M is a preordering of A, or A is finitely generated and* $\dim(\frac{A}{M\cap -M}) \leq 1$*, then* $\operatorname{Pos}(\mathcal{X}_M) = \operatorname{Pos}(\mathcal{Y}_M)$.

Proof. (1) is a standard result from Real Algebra, it follows from Tarski's Transfer Principle.

(2) If A is finitely generated and $\dim(\frac{A}{M\cap -M}) \leq 1$ then every semiordering lying over M is an ordering, e.g., by [15, Theorem 7.4.1], so the result is clear in this case. Suppose now that M is a preordering, $f \geq 0$ on $\mathcal{X}_M$ and $Q \in \mathcal{Y}_M$. Let $\mathfrak{p} := Q \cap -Q$ and let M' be the extension of M to $\kappa(\mathfrak{p})$. M' is a preordering of $\kappa(\mathfrak{p})$, so it is the intersection of the orderings of $\kappa(\mathfrak{p})$ lying over M', by the Artin-Schreier Theorem [15, Lemma 1.4.4]. Since $f \geq 0$ on $\mathcal{X}_M$ this forces $f + \mathfrak{p} \in M'$. Since

M' is a subset of the extension of Q to $\kappa(\mathfrak{p})$, this implies in turn that $f \in Q$. □

$\mathrm{Pos}(\mathcal{Y}_M)$ can also be described in other ways, which make no explicit mention of $\mathcal{Y}_M$:

$$\begin{aligned}\mathrm{Pos}(\mathcal{Y}_M) &= \{f \in A \mid pf = f^{2m} + q \text{ for some } p \in \textstyle\sum A^2, q \in M, m \geq 0\} \\ &= \{f \in A \mid f + \mathfrak{p} \text{ belongs to the extension of } M \text{ to } \kappa(\mathfrak{p}) \\ &\qquad\qquad \text{for all primes } \mathfrak{p} \text{ of } A\}.\end{aligned}$$

This is well-known and is a consequence of the abstract Positivstellensatz for semiorderings, e.g., see [7] or [15, Theorem 5.3.2]. Typically one uses ideas from quadratic form theory and valuation theory to decide when $f + \mathfrak{p}$ lies in the extension of M to $\kappa(\mathfrak{p})$; see [8] and [15]. Note that one needs only consider primes $\mathfrak{p}$ satisfying $f \notin \mathfrak{p}$ and $(M + \mathfrak{p}) \cap -(M + \mathfrak{p}) = \mathfrak{p}$.

We turn now to $\overline{M}$. One has the obvious commutative diagram:

$$\begin{tikzcd}
\overline{M} \arrow[r] & \mathrm{Pos}(\mathcal{K}_M) \\
M \arrow[r] \arrow[u] & \mathrm{Pos}(\mathcal{Y}_M), \arrow[u]
\end{tikzcd}$$

The arrows here denote inclusions. Interest in $\overline{M}$ stems from the Moment Problem:

Proposition 2.6. *Let A be finitely generated and M an arbitrary quadratic module of A. Then the following are equivalent:*

1. $\overline{M} = \mathrm{Pos}(\mathcal{K}_M)$.
2. *For each $L \in M^\vee$ there exists a positive Borel measure μ on $\mathcal{K}_M$ such that $L(f) = \int f\, d\mu$ for all $f \in A$.*

Proof. If $A = \mathbb{R}[\underline{x}]$ is the polynomial algebra, this is Haviland's Theorem as stated in [15, Theorem 3.1.2] (using $\overline{M} = M^{\vee\vee}$, as explained above). For an arbitrary finitely generated algebra A, write A as a quotient $A = \mathbb{R}[\underline{x}]/I$ and let $M' \subseteq \mathbb{R}[\underline{x}]$ be the inverse image of M under the canonical projection from $\mathbb{R}[\underline{x}]$ to A. Then every functional from $(M')^\vee$ factors through A, and $\overline{M} = \mathrm{Pos}(\mathcal{K}_M)$ and $\overline{M'} = \mathrm{Pos}(\mathcal{K}_{M'})$ are equivalent. So the result follows from the case $A = \mathbb{R}[\underline{x}]$. □

See [15, Theorem 3.2.2] for an extended version of Haviland's Theorem, which does not assume that A is finitely generated.

For arbitrary A and M, we say M satisfies the *strong moment property* (SMP) if condition (1) of Proposition 2.6 holds.

In computing $\overline{M}$ it seems there are only two basic tools available, which are the following two theorems.

Theorem 2.7. *Let A and M be finitely generated. If M is stable then $M^{\ddagger} = \overline{M} = M + \sqrt{M \cap -M}$ and $\overline{M}$ is stable.*

See [26] or [15, Theorem 4.1.2] for the proof of Theorem 2.7. Here, $\sqrt{M \cap -M}$ denotes the radical of the ideal $M \cap -M$. Recall: M is said to be *stable* [18] [22] [26] if for each finite dimensional subspace V of A there exists a finite dimensional subspace W of A such that each $f \in M \cap V$ is expressible as $f = \sigma_0 + \sigma_1 g_1 + \cdots + \sigma_s g_s$ where $g_1, \dots, g_s$ are the fixed generators of M and the σ_i are sums of squares of elements of W. See [15] for an equivalent definition.

Interest in stability arose in the search for examples where (SMP) fails. The quadratic module $\sum \mathbb{R}[\underline{x}]^2$ of the polynomial ring $\mathbb{R}[\underline{x}]$ is stable. Theorem 2.7 was proved first in this special case in [1], and the result was then used to show that $\sum \mathbb{R}[\underline{x}]^2$ does not satisfy (SMP) if $n \geq 2$. More recently, in [26, Theorem 5.4], it is shown that if M is stable and $\dim(\mathcal{K}_M) \geq 2$ then M does not satisfy (SMP). See [1] [10] [15] [18] [21] [22] for examples where stability holds.

The second basic tool is the following result, which is both a strengthening and an extension to quadratic modules of Schmüdgen's fibre theorem in [29]; also see [17].

Theorem 2.8. *Let $f \in A$, $a, b \in \mathbb{R}$.*

1. *If $a < f < b$ on $\mathcal{Y}_M$ then $b - f, f - a \in M^{\ddagger}$.*
2. *If A has countable vector space dimension and $b - f, f - a \in \overline{M}$ then $\overline{M} = \cap_{a \leq \lambda \leq b} \overline{M_\lambda}$, where $M_\lambda := M + (f - \lambda)$.*

From part (1) one can immediately deduce that $b' - f, f - a' \in \overline{M}$ where $a' := \sup\{a \in \mathbb{R} \mid a \leq f \text{ on } \mathcal{Y}_M\}$, $b' := \inf\{b \in \mathbb{R} \mid f \leq b \text{ on } \mathcal{Y}_M\}$. In fact one even gets $b' - f, f - a' \in (M^{\ddagger})^{\ddagger}$, the second cone in the sequence of iterated sequential closures of M.

Part (1) is useful in conjunction with part (2). If M and A are finitely generated and either M is a preordering or $\dim(\frac{A}{M \cap -M}) \leq 1$, then the assumption that $a \leq f \leq b$ on $\mathcal{Y}_M$ is equivalent to the assumption that $a \leq f \leq b$ on $\mathcal{K}_M$; see Proposition 2.5. In particular, parts (1) and (2) taken together yield Schmüdgen's result in [29] as a special case.

Part (1) is also of independent interest. It is an improvement of the corresponding result in [29], not only because of the extension from preorderings to quadratic modules, but also because the conclusion $b - f, f - a \in \overline{M}$ has been replaced by the stronger conclusion $b - f, f - a \in M^{\ddagger}$.

The proof of (2) for finitely generated algebras and finitely generated quadratic modules is given already in [15, Theorem 4.4.1]. The general case of an algebra of countable vector space dimension and arbitrary M is almost the same, see [20, Theorem 2.6].

As also explained in [15] [28] [29], to prove (1), one is reduced to showing Lemma 2.9 below. In fact if $a < f < b$ on $\mathcal{Y}_M$, then either $a < b$ or $M = A$, in which case the whole result is trivially true. So we assume $a < b$. By replacing a by $a - \frac{a+b}{2}$, b by $b - \frac{a+b}{2}$ and f by $f - \frac{a+b}{2}$ we can also assume $a = -b$ and $b > 0$. Then $b^2 - f^2 > 0$ on $\mathcal{Y}_M$ follows from the identity

$$b^2 - f^2 = \frac{1}{2b}\left[(b-f)^2(b+f) + (b+f)^2(b-f)\right].$$

By Lemma 2.9 then $b^2 - f^2 \in M^\ddagger$, and the identity

$$b \pm f = \frac{1}{2b}\left[(b \pm f)^2 + (b^2 - f^2)\right]$$

then finally yields (1).

Lemma 2.9. *Suppose $f \in A$, $\ell \in \mathbb{R}$, $\ell^2 - f^2 > 0$ on $\mathcal{Y}_M$. Then $\ell^2 - f^2 \in M^\ddagger$.*

Proof. By the abstract Positivstellensatz for semiorderings, see [15, Theorem 5.3.2], the hypothesis implies $(\ell^2 - f^2)p = 1 + q$ for some $p \in \sum A^2$, $q \in M$. Now one starts with Schmüdgen's argument involving Hamburger's Theorem (also see the proof of [15, Theorem 3.5.1]), i.e. one proceeds as follows:

Claim 1: $\ell^{2i}p - f^{2i}p \in M$ for all $i \geq 1$. Since $\ell^2 p - f^2 p = (\ell^2 - f^2)p = 1 + q$, this is clear when $i = 1$. Since

$$\ell^{2i+2}p - f^{2i+2}p = \ell^2(\ell^{2i}p - f^{2i}p) + f^{2i}(\ell^2 p - f^2 p),$$

the result follows, by induction on i.

Claim 2: $\ell^{2i+2}p - f^{2i} \in M$ for all $i \geq 1$. Since

$$\ell^{2i+2}p - f^{2i} = \ell^2(\ell^{2i}p - f^{2i}p) + f^{2i}(\ell^2 p - 1),$$

and $\ell^2 p - 1 = q + f^2 p \in M$, this follows from Claim 1.

Now we use a little technical trick. Define $V := \mathbb{R}[f] + \mathbb{R}p$, a vector subspace of A. Write $M_V := M \cap V$, so M_V is a cone in V. We claim that $p + 1$ is an interior point of M_V in V. Indeed,

$$\ell^{2i+2}p \pm 2f^i + 1 = \ell^{2i+2}p - f^{2i} + (f^i \pm 1)^2 \in M_V$$

for all $i \geq 1$, using Claim 2. So with $N := \max\{1, \ell\}$ we have for every $i \geq 1$

$$(p+1) \pm \frac{2}{N^{2i+2}} \cdot f^i \in M_V.$$

Clearly also

$$(p+1) \pm 1 \in M_V \text{ and } (p+1) \pm p \in M_V$$

holds, which proves the claim, using Proposition 1.3.

We now claim that $\ell^2 - f^2$ belongs to $\overline{M_V} = (M_V)^{\vee\vee}$ in V. Therefore fix $L \in (M_V)^\vee$ and consider the linear map $L_1 : \mathbb{R}[Y] \to \mathbb{R}$ defined by $L_1(r(Y)) = L(r(f))$. Here, $r(f)$ denotes the image of $r(Y)$ under the algebra homomorphism from $\mathbb{R}[Y]$ to V defined by $Y \mapsto f$. Since $r(f)^2$ is a square in A, and M contains all squares, and L is ≥ 0 on M_V, we see that $L_1(r^2) = L(r(f)^2) \geq 0$ for all $r \in \mathbb{R}[Y]$. By Hamburger's Theorem [15, Corollary 3.1.4], there exists a Borel measure ν on $\mathbb{R}$ such that

$$L(r(f)) = L_1(r) = \int r \, d\nu,$$

for each $r \in \mathbb{R}[Y]$. Let $\lambda > 0$ and let $\mathcal{X}_\lambda$ denote the characteristic function of the set $(-\infty, -\lambda) \cup (\lambda, \infty)$. Then

$$\lambda^{2i} \int \mathcal{X}_\lambda \, d\nu \leq \int Y^{2i} \, d\nu = L_1(Y^{2i}) = L(f^{2i}) \leq \ell^{2i+2} L(p).$$

The first inequality follows from the fact that $\lambda^{2i} \mathcal{X}_\lambda \leq Y^{2i}$ on $\mathbb{R}$. The last inequality follows from Claim 2. Since this holds for any $i \geq 1$, it clearly implies that $\int \mathcal{X}_\lambda \, d\nu = 0$, for any $\lambda > \ell$. This implies, in turn, that $\int \mathcal{X}_\ell \, d\nu = 0$ i.e., the set $(-\infty, -\ell) \cup (\ell, \infty)$ has ν measure zero. Since $Y^2 \leq \ell^2$ holds on the interval $[-\ell, \ell]$, this yields

$$L(f^2) = \int Y^2 \, d\nu \leq \int \ell^2 \, d\nu = L(\ell^2).$$

This proves $L(\ell^2 - f^2) \geq 0$. Since this is true for any $L \in (M_V)^\vee$, this proves $\ell^2 - f^2 \in (M_V)^{\vee\vee} = \overline{M_V}$.

Now finally, since M_V has an interior point in V, $\overline{M_V} = (M_V)^\ddagger$ by Proposition 1.3. Therefore, $\ell^2 - f^2 \in M_V^\ddagger \subseteq M^\ddagger$. □

Theorem 2.8 can be used to produce examples where (SMP) holds, see [15] [17] [28] [29]. Assuming the hypothesis of Theorem 2.8 (2), M satisfies (SMP) iff each M_λ satisfies (SMP). The implication ($\Leftarrow$) is immediate from Theorem 2.8 (2) and the easy fact

$$\operatorname{Pos}(\mathcal{K}_M) = \bigcap_{a \leq \lambda \leq b} \operatorname{Pos}(\mathcal{K}_{M_\lambda}).$$

The implication ($\Rightarrow$) is a consequence of the following observation, due to Scheiderer:

Lemma 2.10. *If A is finitely generated and M satisfies (SMP) then so does $M + I$, for each ideal I of A.*

See [26, Proposition 4.8] for the proof of Lemma 2.10. Theorem 2.8 has also been used to construct an example where $M^\ddagger \neq \overline{M}$; see [19]. The reader will encounter additional applications of Theorem 2.8 in Sections 3 and 4.

3. The Compact Case

We recall basic facts concerning archimedean quadratic modules. We characterize archimedean quadratic modules in various ways.

Theorem 3.1. *Suppose M is archimedean. Then $f \geq 0$ on $\mathcal{K}_M \Rightarrow f + \epsilon \in M$ for all real $\epsilon > 0$. In particular, $M^\ddagger = \overline{M} = \operatorname{Pos}(\mathcal{K}_M)$.*

Theorem 3.1 is Jacobi's Representation Theorem [7]. See [15, Theorem 5.4.4] for an elementary proof. There is no requirement that A or M be finitely generated. We give another proof of Theorem 3.1, based on Theorem 2.8 (1).

Proof. Suppose $f \in A$, $f \geq 0$ on $\mathcal{K}_M$, $\epsilon \in \mathbb{R}$, $\epsilon > 0$. For each $Q \in \mathcal{Y}_M$, Q is archimedean (because M is) so, arguing as in the proof of Proposition 2.2, there is a ring homomorphism $\alpha : A \to \mathbb{R}$ such that $\alpha^{-1}(\mathbb{R}^+) \supseteq Q$ and $f + \epsilon \in Q$. This proves $f \geq -\epsilon$ on $\mathcal{Y}_M$. Since M is archimedean, there is some $b \in \mathbb{R}$, $b - f \in M$, so $b \geq f \geq -\epsilon$ on $\mathcal{Y}_M$. According to Theorem 2.8 (1) this implies $f + \epsilon \in \overline{M}$ for each real $\epsilon > 0$, so $f \in \overline{M}$. Since M is archimedean, 1 is an algebraic interior point of M. By Proposition 1.3, $f + \epsilon \in M$ for all real $\epsilon > 0$. □

The following result is proved in [23, Theorem 5.1.18]:

Theorem 3.2. *If M is archimedean, then every maximal semiordering Q of A lying over M is clearly also archimedean. If A is a finitely generated $\mathbb{R}$-algebra, then the converse is also true.*

There is no requirement here that M be finitely generated. Note: Maximal semiorderings and maximal proper quadratic modules are the same thing, e.g., see [7] or [15, Sect. 5.3]. By [15, Theorem 5.2.5], every maximal semiordering Q which is archimedean has the form $Q = \alpha^{-1}(\mathbb{R}^+)$ for some (unique) ring homomorphism $\alpha : A \to \mathbb{R}$.

Corollary 3.3. *Suppose $x_1, \dots, x_n$ generate A as an $\mathbb{R}$-algebra. The following are equivalent:*

1. *M is archimedean.*
2. *$\sum_{i=1}^n x_i^2$ is bounded on $\mathcal{Y}_M$.*

If M is a finitely generated preordering then, by Proposition 2.5, $\sum x_i^2$ is bounded on $\mathcal{Y}_M \Leftrightarrow \sum x_i^2$ is bounded on $\mathcal{K}_M \Leftrightarrow \mathcal{K}_M$ is compact. In this case, Corollary 3.3 is just "Wörmann's Trick"; see [15] [30].

Proof. (1) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (1). Fix a positive constant k such that $k - \sum x_i^2 > 0$ on $\mathcal{Y}_M$. By [15, Corollary 5.2.4], each maximal semiordering Q of A lying over M is archimedean. Now apply Theorem 3.2. $\square$

The second assertion of Theorem 3.2 is not true for general A. In [14] an example is given of a countably infinite dimensional $\mathbb{R}$-algebra A such that every maximal proper quadratic module Q of A is archimedean (so has the form $\alpha^{-1}(\mathbb{R}^+)$ for some ring homomorphism $\alpha : A \to \mathbb{R}$), but $\sum A^2$ itself is not archimedean. In fact, in this example, the only elements $h \in A$ satisfying $\ell \pm h \in \sum A^2$ for some integer $\ell \geq 1$ are the elements of $\mathbb{R}$. But there is a certain weak version of the second assertion of Theorem 3.2 which does hold for general A:

Theorem 3.4. *If every maximal semiordering of A lying over M is archimedean, then $\mathcal{K}_M$ is compact and $(M^\ddagger)^\ddagger = \overline{M} = \operatorname{Pos}(\mathcal{K}_M)$. In particular, $(M^\ddagger)^\ddagger$ is archimedean.*

There is no requirement here that A or M be finitely generated.

Proof. The result follows from Theorem 2.8 (1) once we prove that $\mathcal{K}_M$ is compact (using the fact that $f \geq 0$ on $\mathcal{K}_M \Rightarrow f > -\epsilon$ on $\mathcal{Y}_M$, for all real $\epsilon > 0$). Fix $f \in A$ and let $M_\ell = M - \sum A^2(\ell^2 - f^2)$. Then $M_\ell \subseteq M_{\ell+1}$. If $-1 \notin \cup_{\ell \geq 1} M_\ell$ then we would have a maximal semiordering Q containing $\cup_{\ell \geq 1} M_\ell$. Then $-(\ell^2 - f^2) \in Q$ for all $\ell \geq 1$, so $(\ell - 1)^2 - f^2 \notin Q$ for all $\ell \geq 1$. This is a contradiction. Thus $-1 = s - p(\ell^2 - f^2)$ for some $s \in M$, $p \in \sum A^2$ and some integer $\ell \geq 1$. This implies $-\ell < \alpha(f) < \ell$ for all $\alpha \in \mathcal{K}_M$, for some integer $\ell \geq 1$ (depending on f), say $\ell = \ell_f$. Then $\mathcal{K}_M$ is identified with a closed subspace of the compact space $\prod_{f \in A}[-\ell_f, \ell_f]$. $\square$

Note: Instead of arguing with the quadratic modules M_ℓ, one could exploit the compactness of the spectral space Semi-Sper(A), as was done in the proof of [23, Theorem 5.1.18]. This shows that if $\mathcal{Y}$ is any set of archimedean semiorderings in Semi-Sper(A) which is closed in the constructible topology then $\cap_{Q \in \mathcal{Y}} Q$ is archimedean.

Corollary 3.5. *The following are equivalent:*

1. *$\overline{M}$ is archimedean.*
2. *Every maximal semiordering of A lying over $\overline{M}$ is archimedean.*
3. *$\mathcal{K}_M$ is compact and $\overline{M} = \operatorname{Pos}(\mathcal{K}_M)$.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious. If $\alpha \in \mathcal{K}_M$ then $\alpha^{-1}(\mathbb{R}^+)$ is closed and $M \subseteq \alpha^{-1}(\mathbb{R}^+)$, so $\overline{M} \subseteq \alpha^{-1}(\mathbb{R}^+)$. This proves that $\mathcal{K}_M = \mathcal{K}_{\overline{M}}$. The implication (2) $\Rightarrow$ (3) follows from this observation, by applying Theorem 3.4 to the quadratic module $N = \overline{M}$. $\square$

Note: Since $\mathcal{K}_M = \mathcal{K}_{\overline{M}}$, one sees now that Corollary 3.5 is not really a statement about the quadratic module M, but rather it is a statement about the closed quadratic module $\overline{M}$.

Clearly M archimedean $\Rightarrow$ $\overline{M}$ archimedean $\Rightarrow$ $\mathcal{K}_M$ compact. We conclude by giving concrete examples to show that $\mathcal{K}_M$ compact $\not\Rightarrow$ $\overline{M}$ archimedean and $\overline{M}$ archimedean $\not\Rightarrow$ M archimedean:

Example 3.6. Let $A := \mathbb{R}[\underline{x}]$, $n \geq 2$, and let M be the quadratic module of $\mathbb{R}[\underline{x}]$ generated by

$$x_1 - 1, \dots, x_n - 1, c - \prod_{i=1}^{n} x_i,$$

where c is a positive real constant. Then $\mathcal{K}_M$ is compact (possibly empty, depending on the value of c), but, as explained in [8], M is not archimedean. As pointed out in [18] (also see [15]) M is also stable, so $\overline{M} = M$, by Theorem 2.7.

Example 3.7. Let $A := \mathbb{R}[\underline{x}]$, $n \geq 2$, and let M be the quadratic module of $\mathbb{R}[\underline{x}]$ generated by

$$1 - x_1, \dots, 1 - x_n, \prod_{i=1}^{n} x_i - c, x_1 x_n^2, x_1 x_2 x_n^2, \dots, x_1 \cdots x_{n-1} x_n^2,$$

where c is a positive real constant. In this example, $\mathcal{K}_M$ is compact, M is not archimedean, but $\overline{M} = \operatorname{Pos}(\mathcal{K}_M)$, so $\overline{M}$ is archimedean. One checks that $0 < x_1 \leq 1$ on $\mathcal{Y}_M$ so, by Theorem 2.8, $\overline{M} = \cap_{0 \leq \lambda \leq 1} \overline{M_\lambda}$ where $M_\lambda := M + (x_1 - \lambda)$. So to prove M satisfies (SMP) it suffices to prove each M_λ satisfies (SMP). Exploiting the natural isomorphism $\frac{\mathbb{R}[\underline{x}]}{(x_1 - \lambda)} \cong \mathbb{R}[x_2, \dots, x_n]$, this reduces to showing that the quadratic module N_λ of $\mathbb{R}[x_2, \dots, x_n]$ generated by

$$1 - \lambda, 1 - x_2, \dots, 1 - x_n, \lambda \prod_{i=2}^{n} x_i - c, \lambda x_n^2, \lambda x_2 x_n^2, \dots, \lambda x_2 \cdots x_{n-1} x_n^2$$

satisfies (SMP). If $\lambda = 0$ then $-1 \in N_\lambda$ so this is true for trivial reasons. If $0 < \lambda \leq 1$ then N_λ is generated by

$$1 - x_2, \dots, 1 - x_n, \prod_{i=2}^{n} x_i - \frac{c}{\lambda}, x_2 x_n^2, \dots, x_2 \cdots x_{n-1} x_n^2,$$

and N_λ satisfies (SMP) by induction on n. This proves M satisfies (SMP). To show that M is not archimedean it suffices to show $k^2 - x_1^2 \notin M$ for each real k. Taking $x_2 = \cdots = x_{n-1} = 1$, this reduces to the case $n = 2$ and, in this case, it can be verified by an easy degree argument (considering terms of highest degree). But actually, one can say more:

Claim 1. The only elements of $\mathbb{R}[\underline{x}]$ which are bounded on $\mathcal{Y}_M$ are the elements in $\mathbb{R}[x_1]$. For suppose $f \in \mathbb{R}[\underline{x}]$, $f \notin \mathbb{R}[x_1]$ and $k \in \mathbb{R}$. We must show $\exists\ Q \in \mathcal{Y}_M$ such that $f^2 \geq k^2$ at Q. Consider the diagonal quadratic form

$$\rho = \langle 1, 1-x_1, \ldots, 1-x_n, \prod x_i - c, x_1x_n^2, \ldots, x_1\cdots x_{n-1}x_n^2, f^2-k^2\rangle,$$

defined over the rational function field $\mathbb{R}(\underline{x})$, and the valuation v on $\mathbb{R}(\underline{x})$ having value group $\mathbb{Z}^n$ ordered lexicographically, defined by

$$\begin{aligned} v(x_1) &= (0,\ldots,0,1),\\ v(x_2) &= (0,\ldots,0,-1,0),\\ &\cdots\\ v(x_n) &= (-1,0,\ldots,0). \end{aligned}$$

The residue field of v is $\mathbb{R}$. The quadratic form ρ has $2n$ residue forms with respect to v, one of them 3-dimensional, the rest 1-dimensional, all of them strongly anisotropic. It follows that ρ is strongly anisotropic, even strongly v-anisotropic [15, Prop. 8.2.3]. Using this and [15, Lem. 8.3.1], the entries of ρ generate a proper quadratic module of $\mathbb{R}(\underline{x})$. Since any proper quadratic module is contained in a semiordering [15, Th. 5.2.5(1)], this yields $Q \in \mathcal{Y}_M$ with $f^2 - k^2 \in Q$.

Claim 2. The only elements f of $\mathbb{R}[\underline{x}]$ such that $k^2 - f^2 \in M$ for some real constant k are the elements of $\mathbb{R}$. For suppose

$$\begin{aligned} k^2 - f^2 &= \sigma_0 + \sigma_1(1-x_1) + \cdots + \sigma_n(1-x_n)\\ &\quad + \sigma_{n+1}(\prod x_i - c) + \sigma_{n+2}x_1x_n^2 + \cdots + \sigma_{2n}x_1\cdots x_{n-1}x_n^2 \end{aligned}$$

with $\sigma_i \in \sum \mathbb{R}[\underline{x}]^2$, $i = 0,\ldots,2n$. If x_n actually appears in one of the terms, consider the terms of highest degree in x_n. These terms cannot cancel with each other. It follows that x_n cannot appear in any term. This forces $f \in \mathbb{R}[x_1,\ldots,x_{n-1}]$, $\sigma_i \in \sum \mathbb{R}[x_1,\ldots,x_{n-1}]^2$ for $i = 0,\ldots,n-1$, $\sigma_i = 0$ for $i = n,\ldots,2n$, and

$$k^2 - f^2 = \sigma_0 + \sigma_1(1-x_1) + \cdots + \sigma_{n-1}(1-x_{n-1}).$$

This implies f is bounded on the affine cone $(-\infty,1]^{n-1}$, so f is constant.

Note: There is a valuation-theoretic criterion for deciding when a finitely generated quadratic module M is archimedean, given that $\mathcal{K}_M$ is compact; see [8] or [15]. But typically this does not apply to $\overline{M}$, because $\overline{M}$ is not finitely generated.

4. COMPUTATION OF $\overline{M}$ IN SPECIAL CASES

If $\dim(\frac{A}{M\cap -M}) \leq 1$, Theorems 2.7 and 2.8 combine to yield a recursive description of $\overline{M}$. This is a consequence of the following result:

Theorem 4.1. *Let A be finitely generated and suppose the finitely generated quadratic module M fulfills* $\dim(\frac{A}{M\cap -M}) \leq 1$. *If the only elements of A bounded on $\mathcal{K}_M$ are the elements of* $\mathbb{R} + M \cap -M$, *then M is stable.*

A preordering version of Theorem 4.1 appears already in [21, Corollary 2.11].

Proof. First we reduce to the case $M \cap -M = \{0\}$. Therefore let $A' := A/(M \cap -M)$ and let M' be the image of M under the canonical projection from A to $A/(M \cap -M)$. Since every ring homomorphism from $\mathcal{K}_M$ factors through A', we find that the only elements from A' bounded on $\mathcal{K}_{M'}$ are the reals (note that $M' \cap -M' = \{0\}$). So assume we already know that M' is stable. Then by [26, Lemma 3.9], M is also stable. So we only have to proof the theorem for $M \cap -M = \{0\}$ now.

Since A is noetherian there are just finitely many minimal primes of A. Let $\mathfrak{p}$ be a minimal prime of A, $\kappa(\mathfrak{p}) := \mathrm{ff}(\frac{A}{\mathfrak{p}})$. According to [15, Proposition 2.1.7], $(M + \mathfrak{p}) \cap -(M + \mathfrak{p}) = \mathfrak{p}$, i.e., M extends to a proper preordering of $\kappa(\mathfrak{p})$. $\dim(\frac{A}{\mathfrak{p}})$ is either 0 or 1. For $\dim(\frac{A}{\mathfrak{p}}) = 1$ let $S_{\infty,\mathfrak{p}}$ denote the set of valuations $v \neq 0$ of $\kappa(\mathfrak{p})$ compatible with some ordering of $\kappa(\mathfrak{p})$ lying over the extension of M to $\kappa(\mathfrak{p})$ and such that $\frac{A}{\mathfrak{p}} \not\subseteq B_v$, where $B_v \subseteq \kappa(\mathfrak{p})$ is the valuation ring of v. By Noether Normalization there is some $t \in \frac{A}{\mathfrak{p}}$ transcendental over $\mathbb{R}$ such that $\frac{A}{\mathfrak{p}}$ is integral over $\mathbb{R}[t]$. Since B_v is integrally closed, $\frac{A}{\mathfrak{p}} \not\subseteq B_v$ is equivalent to $t \notin B_v$, and this implies that v is one of the extensions of the discrete valuation v_∞ of $\mathbb{R}(t)$. Since $[\kappa(\mathfrak{p}) : \mathbb{R}(t)] < \infty$, the set $S_{\infty,\mathfrak{p}}$ is finite and each $v \in S_{\infty,\mathfrak{p}}$ is discrete with residue field $\mathbb{R}$. Let S_∞ be the union of the sets $S_{\infty,\mathfrak{p}}$, $\mathfrak{p}$ running through the mimimal primes of A with $\dim(\frac{A}{\mathfrak{p}}) = 1$. Thus S_∞ is finite. View elements of S_∞ as functions $v : A \to \mathbb{Z} \cup \{\infty\}$ by defining $v(f) = v(f + \mathfrak{p})$ if $v \in S_{\infty,\mathfrak{p}}$.

If $\mathcal{K}_{M+\mathfrak{p}}$ is compact then every $f \in A$ is bounded on $\mathcal{K}_{M+\mathfrak{p}}$, so either $\dim(\frac{A}{\mathfrak{p}}) = 0$ or $\dim(\frac{A}{\mathfrak{p}}) = 1$ and $S_{\infty,\mathfrak{p}} = \emptyset$. If $\mathcal{K}_{M+\mathfrak{p}}$ is not compact then $\dim(\frac{A}{\mathfrak{p}}) = 1$, $S_{\infty,\mathfrak{p}} \neq \emptyset$, and $f \in A$ is bounded on $\mathcal{K}_{M+\mathfrak{p}}$ iff $v(f) \geq 0$ for all $v \in S_{\infty,\mathfrak{p}}$. (This uses the compactness of the real spectrum.) Anyway, since $\mathcal{K}_M$ is the union of the $\mathcal{K}_{M+\mathfrak{p}}$, we have established the following:

Claim 1: $f \in A$ is bounded on $\mathcal{K}_M$ iff $v(f) \geq 0$ holds for all $v \in S_\infty$.

If $S_\infty = \emptyset$ then every element of A is bounded on $\mathcal{K}_M$, by Claim 1, so, by hypothesis, $A = \mathbb{R} \cdot 1$ (i.e., either $A = M = \{0\}$ or $A = \mathbb{R}$ and $M = \mathbb{R}^+$). In this case M is obviously stable. Thus we may assume $S_\infty \neq \emptyset$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the minimal primes of A with $\dim(\frac{A}{\mathfrak{p}_i}) = 1$ and $S_{\infty,\mathfrak{p}_i} \neq \emptyset$. Since $S_\infty \neq \emptyset$, $k \geq 1$. If $f \in \cap_{i=1}^k \mathfrak{p}_i$ then $v(f) = \infty$ for all $v \in S_\infty$ so, by Claim 1, f is bounded on $\mathcal{K}_M$, so, by hypothesis, $f \in \mathbb{R}$. Since $k \geq 1$, this forces $f = 0$. This proves $\cap_{i=1}^k \mathfrak{p}_i = \{0\}$, i.e., $\sqrt{\{0\}} = \{0\}$ and $\{\mathfrak{p}_i \mid i = 1, \dots, k\}$ is the complete set of minimal primes of A.

For any non-empty subset S of S_∞ and any integer d, let

$$V_{S,d} := \{f \in A \mid v(f) \geq d \ \forall v \in S\}.$$

$V_{S,d}$ is clearly an $\mathbb{R}$-subspace of A.

Claim 2: If $d < e$ then $V_{S,d}/V_{S,e}$ is finite dimensional.

Consider all pairs (T, n) where T is a non-empty subset of S and $d \leq n < e$ such that there exists an element $g \in A$ with $v(g) = n$ for all $v \in T$ and $v(g) > n$ for $v \in S \backslash T$. Fix such an element $g = g_{T,n}$ for each such pair. To prove Claim 2 it suffices to show that these elements generate $V_{S,d}$ modulo $V_{S,e}$. This is pretty clear. Suppose $f \in V_{S,d}$. Let $n = \min\{v(f) \mid v \in S\}$, so $n \geq d$. If $n \geq e$ then $f \in V_{S,e}$. Suppose $n < e$. Let $T = \{v \in S \mid v(f) = n\}$. Thus $T \neq \emptyset$. Fix $v_0 \in T$. Since the residue field of v_0 is $\mathbb{R}$, there is some $a \in \mathbb{R}$ such that $v_0(f - a g_{T,n}) > n$. Let $f' = f - a g_{T,n}$, i.e., $f = a g_{T,n} + f'$. Now repeat the process, working with f' instead of f. Either $\min\{v(f') \mid v \in S\} > n$ or $\min\{v(f') \mid v \in S\} = n$ and $T' = \{v \in S \mid v(f') = n\}$ is non-empty and properly contained in T (because $v_0 \in T$, $v_0 \notin T'$). Anyway, the process terminates after finitely many steps. This proves Claim 2.

By Claim 1, $V_{S_\infty,0} = \mathbb{R}$. Combining this with Claim 2, we see that $V_{S_\infty,d}$ is finite dimensional for each $d \leq 0$. Clearly $A = \cup_{d \leq 0} V_{S_\infty,d}$.

Fix generators $g_1, \dots, g_t$ for M. We may assume each g_i is non-zero. Complications arise from the fact that k may be strictly greater than 1, so some of the g_i may be divisors of zero. We need some notation: Let $g_0 := 1$. For $0 \leq i \leq s$, denote by $S_\infty^{(i)}$ the union of the sets $S_{\infty,\mathfrak{p}_j}$ such that $1 \leq j \leq k$ and $g_i \notin \mathfrak{p}_j$. Thus $S_\infty^{(0)} = S_\infty$. Let $e_i := \max\{v(g_i) \mid v \in S_\infty^{(i)}\}$. Let $e'_i := \min\{v(g_i) \mid v \in S_\infty^{(i)}\}$. Note that $S_\infty^{(i)} \neq \emptyset$ so $e_i \neq +\infty$. Fix $d \geq 0$ and let $W^{(i)}$ be a f.d. vector subspace of A which generates $V_{S_\infty^{(i)}, \frac{-d-e_i}{2}}$ modulo $V_{S_\infty^{(i)}, -e'_i+1}$. This exists by Claim 2. To complete the proof it suffices to prove:

Claim 3: Each $f \in M \cap V_{S_\infty,-d}$ is expressible in the form $f = \sum_{i=0}^{s} \tau_i g_i$ where τ_i a sum of squares of elements of $W^{(i)}$, $i = 0, \dots, s$.

Let $f \in V_{S_\infty,-d}$, $f = \sum_{i=0}^{s} \sigma_i g_i$, $\sigma_i \in \sum A^2$. Then $-d \leq v(f) = \min\{v(\sigma_i g_i) \mid i = 0, \dots, s\}$, i.e., $v(\sigma_i g_i) \geq -d$ for all $v \in S_\infty$. For $v \in S_\infty^{(i)}$ this yields $v(\sigma_i) \geq -d - v(g_i) \geq -d - e_i$, by definition of e_i. Express σ_i as $\sigma_i = \sum_p h_{ip}^2$. Then $v(h_{ip}^2) \geq -d - e_i$, i.e., $v(h_{ip}) \geq \frac{-d-e_i}{2}$ for all $v \in S_\infty^{(i)}$. Decompose h_{ip} as $h_{ip} = t_{ip} + u_{ip}$ with $t_{ip} \in W^{(i)}$, $u_{ip} \in V_{S_\infty^{(i)},-e_i'+1}$. Then $h_{ip}^2 = t_{ip}^2 + 2t_{ip}u_{ip} + u_{ip}^2 = t_{ip}^2 + (2t_{ip} + u_{ip})u_{ip}$, so $h_{ip}^2 g_i = t_{ip}^2 g_i + (2t_{ip} + u_{ip})u_{ip} g_i$. One checks that $v(u_{ip} g_i) > 0$ for all $v \in S_\infty$: if $v \notin S_\infty^{(i)}$ then $v(g_i) = \infty$, $v(u_{ip} g_i) = \infty$, so this is clear. If $v \in S_\infty^{(i)}$, then $v(u_{ip}) > -e_i'$, so $v(u_{ip} g_i) > -e_i' + v(g_i) \geq 0$ by definition of e_i'. According to Claim 1 and our hypothesis this implies $u_{ip} g_i = 0$. Thus $h_{ip}^2 g_i = t_{ip}^2 g_i$ and $\sigma_i g_i = \tau_i g_i$ where $\tau_i := \sum t_{ip}^2$. $\square$

The conclusion of Theorem 4.1 is false if $\dim(\frac{A}{M \cap -M}) \geq 2$:

Example 4.2. Let $A = \mathbb{R}[x, y]$, let M be the quadratic module of $\mathbb{R}[x, y]$ generated by $(1 - x)xy^2$, and let N be the quadratic module of $\mathbb{R}[x, y]$ generated by $(1 - x)x$ (note that both M and N are preorderings). $\mathcal{K}_N$ is the strip $[0, 1] \times \mathbb{R}$. $\mathcal{K}_M$ is the strip together with the x-axis. Applying Schmüdgen's fibre theorem (Theorem 2.8) we see that $\overline{N} = \mathrm{Pos}(\mathcal{K}_N)$. In fact, one even has $N = \mathrm{Pos}(\mathcal{K}_N)$; see [16]. According to [26, Theorem 5.4], this implies that N is not stable. On the other hand, $y^2 N \subseteq M$, so if M were stable then N would also be stable: if $f \in N$ then $y^2 f \in M$. If M were stable we would have $y^2 f = \sigma + \tau(1-x)xy^2$, $\sigma, \tau \in \sum \mathbb{R}[x, y]^2$ with degree bounds on σ and τ depending only on $\deg(f)$. Clearly $\sigma = y^2 \sigma_1$ for some $\sigma_1 \in \sum \mathbb{R}[x, y]^2$, so this would yield $f = \sigma_1 + \tau(1 - x)x$ with degree bounds on σ_1, τ depending only on $\deg(f)$. This proves that M is not stable. On the other hand, the elements of $\mathbb{R}[x, y]$ bounded on $\mathcal{K}_N$ are precisely the elements of $\mathbb{R}[x]$. Since the only elements of $\mathbb{R}[x]$ bounded on the x-axis are the elements of $\mathbb{R}$, this proves that the only elements of $\mathbb{R}[x, y]$ bounded on $\mathcal{K}_M$ are the elements of $\mathbb{R}$. We remark that even though M is not stable, it might still be closed.

We can strengthen the example in the following way:

Example 4.3. Let $A = \mathbb{R}[x, y]$, let M be the quadratic module of $\mathbb{R}[x, y]$ generated by $(1 - x)x^3y^2$ and let N be the quadratic module of $\mathbb{R}[x, y]$ generated by $(1-x)x^3$. Again $\mathcal{K}_N$ is the strip $[0, 1] \times \mathbb{R}$ and $\mathcal{K}_M$ is the strip together with the x-axis. Theorem 2.8 again shows that $\overline{N} = \mathrm{Pos}(\mathcal{K}_N)$. However, we have $x \in \mathrm{Pos}(\mathcal{K}_N) \setminus N$. Indeed writing down a possible representation of x in N and evaluating in $y = 0$ gives such a representation for x in $\mathbb{R}[x]$; evaluating in $x = 0$ then shows that x^2 divides x, a contradiction. So N can not be closed.

We have $N = \{f \in \mathbb{R}[x, y] \mid y^2 f \in M\}$, with the same argument as in the preceding example. Now since N is not closed and the mapping $f \mapsto y^2 f$ is linear and therefore continuous, M can not be closed (so in view of Theorem 2.7, M can also not be stable). On the other hand the only polynomials bounded on $\mathcal{K}_M$ (or $\mathcal{Y}_M$) are the reals.

Open problem 1 in [21, p. 85] should be mentioned in this context. It is asked there whether the absence of nontrivial bounded polynomials implies stability of the quadratic module, at least if the semialgebraic set is regular at infinity. Our example does not answer the question, since $\mathcal{K}_M$ is not regular at infinity, i.e. it is not the union of a compact set and a set that is the closure of its interior. So the question is still open.

For polyhedra however, the following result is true:

Theorem 4.4. *Let $A = \mathbb{R}[\underline{x}]$ and suppose the quadratic module M is generated by finitely many linear polynomials. Suppose the only linear polynomials that are bounded on $\mathcal{K}_M$ are from $\mathbb{R} + M \cap -M$. Then M is stable.*

Proof. If $\mathcal{K}_M$ has empty interior, then it lies in a strict affine subspace of $\mathbb{R}^n$. Any linear polynomial vanishing on this subspace belongs to $M \cap -M$, by [15, Lemma 7.1.5]. So by going over to the polynomial ring modulo some linear polynomials defining this affine subspace we retain the condition on bounded linear polynomials. If the quadratic module in this quotient is now stable, then so is M, by [26, Lemma 3.9]. So we can assume that $\mathcal{K}_M$ has non-empty interior and $M \cap -M = \{0\}$.

Without loss of generality $0 \in \mathcal{K}_M$. Group the non constant linear generators of M so that

$$p_1(0), \ldots, p_r(0) > 0$$

and

$$q_1(0) = \cdots = q_s(0) = 0.$$

Write $p_i = c_i + \widetilde{p}_i$ with $c_i \in \mathbb{R}_{>0}$ and $\widetilde{p}_i(0) = 0, \widetilde{p}_i \neq 0$. All $\widetilde{p}_i$ and q_j are homogeneous polynomials of degree one. We claim that $\widetilde{p}_1, \ldots, \widetilde{p}_r, q_1, \ldots, q_s$ are positively linear independent. So assume

$$\sum_i \lambda_i \widetilde{p}_i + \sum_j \gamma_j q_j = 0$$

for some nonnegative coefficients λ_i, γ_j, not all zero. Then some λ_i must be nonzero, since $M \cap -M = \{0\}$. Assume $\lambda_1 > 0$. With $N := \sum_i \lambda_i c_i$

we have $\lambda_1 p_1, N - \lambda_1 p_1 \in M$. So by our assumption on the bounded linear polynomials we find $p_1 \in \mathbb{R}$, a contradiction. This proves the claim.

So there must be a point $d \in \mathbb{R}^n$ where all $\widetilde{p}_1, \dots, \widetilde{p}_r, q_1, \dots, q_s$ are strictly positive (Theorem of alternatives for strict linear inequalities [5, Example 2.21]). Thus $\mathcal{K}_M$ contains a full dimensional cone, and so M is stable (see [10] or [18] or [22]). $\square$

We define the *weak closure* $\widetilde{M}$ of a quadratic module M of A. Informally, $\widetilde{M}$ is the part of $\overline{M}$ that can be 'seen' by applying Theorem 2.8 recursively. Formally, we define $\widetilde{M}$ as follows:

1. If $M = A$ then $\widetilde{M} = M$.
2. If the only elements of A bounded on $\mathcal{Y}_M$ are the elements of $\mathbb{R} + M \cap -M$, then $\widetilde{M} = M$.
3. If some $f \in A$ is bounded on $\mathcal{Y}_M$, say $a \le f \le b$ on $\mathcal{Y}_M$, and $f \notin \mathbb{R} + M \cap -M$, then $\widetilde{M} = \cap_{a \le \lambda \le b} \widetilde{M_\lambda}$, where $M_\lambda := M + (f - \lambda)$.

Note: Although case (1) is included for clarity, it can also be viewed as a special case of (2). It is also important to note that the description of $\widetilde{M}$ given in (3) holds trivially if $f \in \mathbb{R} + M \cap -M$ (in the sense that if $f = \lambda_0 + g$, $\lambda_0 \in \mathbb{R}$, $g \in M \cap -M$, then $M_\lambda = M$ if $\lambda = \lambda_0$ and $M_\lambda = A$ if $\lambda \neq \lambda_0$).

Theorem 4.5. *Let A be finitely generated. Then for every quadratic module M of A,*

1. *$\widetilde{M}$ is a well-defined quadratic module of A.*
2. *$M \subseteq \widetilde{M} \subseteq \overline{M}$.*

Proof. A is noetherian, so if the above notion of $\widetilde{M}$ was not well defined, then there is some quadratic module M with $M \cap -M$ maximal such that $\widetilde{M}$ is not a well-defined quadratic module. Obviously we are not in case (1) or (2), i.e., we are in case (3). Suppose we have $f, g \in A$ bounded on $\mathcal{Y}_M$, say $a \le f \le b$ and $c \le g \le d$ on $\mathcal{Y}_M$, $f, g \notin \mathbb{R} + M \cap -M$. By the maximality of $M \cap -M$, $\widetilde{M + (f - \lambda)}$ and $\widetilde{M + (g - \mu)}$ are well-defined, for $a \le \lambda \le b$, $c \le \mu \le d$. We need to show

$$\cap_{a \le \lambda \le b} \widetilde{M + (f - \lambda)} = \cap_{c \le \mu \le d} \widetilde{M + (g - \mu)}.$$

This follows easily from $\widetilde{M + (f - \lambda)} = \cap_{c \le \mu \le d} \widetilde{M + (f - \lambda, g - \mu)}$ and $\widetilde{M + (g - \mu)} = \cap_{a \le \lambda \le b} \widetilde{M + (f - \lambda, g - \mu)}$. These latter facts hold either by definition or for trivial reasons.

Statement (2) is proven similar. To prove $\widetilde{M} \subseteq \overline{M}$ one of course needs to use Theorem 2.8. $\square$

In [26, Lemma 3.13] it is shown that $\sqrt{M\cap -M}\subseteq \overline{M}$, for arbitrary A and M. One can improve on this as follows:

Lemma 4.6. $\sqrt{M\cap -M}\subseteq \widetilde{M}$.

Proof. Let $f\in\sqrt{M\cap -M}$. If $f=\lambda_0+g$, $\lambda_0\in\mathbb{R}$, $g\in M\cap -M$, then $\lambda_0=f-g\in\sqrt{M\cap -M}$. Then either $M\cap -M=A$ (so $f\in\widetilde{M}$) or $\lambda_0=0$ and $f\in M\cap -M\subseteq M\subseteq\widetilde{M}$. If $f\notin\mathbb{R}+M\cap -M$, then, since $0\le f\le 0$ on $\mathcal{Y}_M$, $\widetilde{M}=\cap_{0\le\lambda\le 0}\widetilde{M_\lambda}=\widetilde{M_0}$, where $M_\lambda:=M+(f-\lambda)$. Anyway, since $f\in M_0$, this means $f\in\widetilde{M_0}=\widetilde{M}$. $\square$

Note that example 4.3 above shows that $\widetilde{M}$ and $\overline{M}$ are not the same in general. In the example the only polynomials bounded on $\mathcal{Y}_M$ are the reals, so $M=\widetilde{M}$. But we have shown that M is not closed, so $M=\widetilde{M}\subsetneq M^{\ddagger}\subseteq\overline{M}$ holds.

Note also that the inclusion $\widetilde{M}\subseteq M^{\ddagger}$ is not always true. Let M be the preordering of $\mathbb{R}[x,y]$ generated by $y^3, x+y, 1-xy, 1-x^2$. This is the example from [19] with $M^{\ddagger}\subsetneq\overline{M}$. One easily checks that $\widetilde{M}=\overline{M}$ holds, so $M^{\ddagger}\subsetneq\widetilde{M}$ in this example.

On the other hand, in many simple cases where we are able to compute $\widetilde{M}$, we find $\widetilde{M}=\overline{M}$:

Theorem 4.7. *Let A be finitely generated. $\overline{M}=\widetilde{M}$ holds in the following cases:*

1. *M finitely generated and stable.*
2. *M finitely generated and* $\dim(\frac{A}{M\cap -M})\le 1$.
3. *M archimedean.*
4. *$A=\mathbb{R}[\underline{x}]$ and M generated by finitely many linear polynomials.*

Proof. (1) $\overline{M}=M+\sqrt{M\cap -M}$, by Theorem 2.7. By Lemma 4.6 this implies $\overline{M}\subseteq\widetilde{M}$.

(2) Choose M with $M\cap -M$ maximal such that $\dim(\frac{A}{M\cap -M})\le 1$ and $\overline{M}\ne\widetilde{M}$. In view of Theorems 2.8 and 4.1 and (1) and the recursive description of $\widetilde{M}$, we again have $\overline{M}=\widetilde{M}$, which is a contradiction.

(3) Choose M with $M\cap -M$ maximal such that M is archimedean, $\overline{M}\ne\widetilde{M}$. Since M is archimedean every element of A is bounded on $\mathcal{Y}_M$. If $A=\mathbb{R}+M\cap -M$ then $\overline{M}=M=\widetilde{M}$, a contradiction. If there is some $f\in A$, $f\notin\mathbb{R}+M\cap -M$, then by Theorem 2.8 and the recursive description of $\widetilde{M}$, $\overline{M}=\widetilde{M}$, again a contradiction.

(4) Take such M with $M\cap -M$ maximal such that $\widetilde{M}\ne\overline{M}$. So the only linear polynomials bounded on $\mathcal{Y}_M$ are the elements from $\mathbb{R}+M\cap -M$. Any linear polynomial that is bounded on $\mathcal{K}_M$ is also

bounded on $\mathcal{Y}_M$. So by Theorem 4.4, M is stable, and we are in case (1). $\square$

5. Appendix

In this section we construct a cone for which the sequence of iterated sequential closures terminated after n steps. Therefore let

$$E = \bigoplus_{i=0}^{\infty} \mathbb{R} \cdot e_i = \{(f_i)_{i\in\mathbb{N}} \mid f_i \in \mathbb{R},\ \text{only finitely many } f_i \neq 0\}$$

be a countable dimensional $\mathbb{R}$-vector space. For $m \in \mathbb{N} \setminus \{0\}$ we write

$$W_m := \bigoplus_{i=0}^{m-1} \mathbb{R} \cdot e_i,$$

so the increasing sequence $(W_m)_{m\in\mathbb{N}}$ of finite dimensional subspaces exhausts the whole space E. For $n \in \{1, 2, \ldots\}$ and $l = (l_0, l_1, \ldots, l_n) \in (\mathbb{N} \setminus \{0\})^{n+1}$ define

$$V(l) := \underbrace{[\frac{1}{l_1}, 1] \times \cdots \times [\frac{1}{l_1}, 1]}_{l_0 \text{ times}} \times \underbrace{[\frac{1}{l_2}, 1] \times \cdots \times [\frac{1}{l_2}, 1]}_{l_1 \text{ times}} \times \cdots$$
$$\times \underbrace{[\frac{1}{l_n}, 1] \times \cdots \times [\frac{1}{l_n}, 1]}_{l_{n-1} \text{ times}}.$$

$V(l)$ is a compact subset of $W_{l_0+\cdots+l_{n-1}}$. Let

$$U(l) := V(l) \times \bigoplus_{i=l_0+\cdots+l_{n-1}}^{\infty} [0,1] \cdot e_i,$$

so $U(l) \subseteq E$ and $U(l) \cap W_m$ is compact for every $m \in \mathbb{N}$; indeed non-empty if and only if $m \geq l_0 + \cdots + l_{n-1}$. Now define

$$M_n := \bigcup_{l\in(\mathbb{N}\setminus\{0\})^{n+1}} U(l).$$

The intention behind this is that M_n contains n "steps", and taking the sequential closure removes one at a time.

We have for $m \geq n \geq 2$

$$\overline{M_n \cap W_m} \subseteq M_{n-1}.$$

To see this take a converging sequence $(x_i)_i$ from $M_n \cap W_m$. So for each x_i there is some $l^{(i)} \in (\mathbb{N} \setminus \{0\})^{n+1}$ such that $x_i \in U(l^{(i)})$. As $U(l) \cap W_m$

is only non-empty if $l_0 + \cdots + l_{n-1} \leq m$, we can assume without loss of generality (by choosing a subsequence), that the $l^{(i)}$ coincide in all but the last component. This shows that the limit of the sequence $(x_i)_i$ belongs to M_{n-1} (indeed to $U(l_0^{(i)}, \ldots, l_{n-1}^{(i)}) \cap W_m$).

So $(M_n)^\ddagger \subseteq M_{n-1}$, and the other inclusion is obvious. We thus have for $n \geq 2$:

$$(M_n)^\ddagger = M_{n-1}.$$

In addition,

$$M_1 \subsetneq M_1^\ddagger = \bigoplus_{i=0}^{\infty} [0,1] \cdot e_i,$$

which is closed. This shows that the sequence of sequential closures for M_n terminates precisely after n steps at $\overline{M_n} = \bigoplus_{i=0}^{\infty} [0,1] \cdot e_i$.

Let $\mathrm{cc}(M_n)$ denote the cone generated by M_n, i.e. $\mathrm{cc}(M_n)$ consists of all finite positive combinations of elements from M_n, including 0. We have for $n \geq 2$

$$\mathrm{cc}(M_n)^\ddagger = \mathrm{cc}(M_{n-1}).$$

To see "$\subseteq$" suppose $x \in \mathrm{cc}(M_n)^\ddagger$. Then we have a sequence $(x_i)_i$ in some $\mathrm{cc}(M_n) \cap W_m = \mathrm{cc}(M_n \cap W_m)$ that converges to x in W_m. Write

$$x_i = \lambda_1^{(i)} a_1^{(i)} + \cdots \lambda_N^{(i)} a_N^{(i)}$$

with all $a_j^{(i)} \in M_n \cap W_m$ and all $\lambda_j^{(i)} \geq 0$. We can choose the same sum length N for all x_i, by the conic version of Carathéodory's Theorem (see for example [2], Problem 6, p. 65). By choosing a subsequence of $(x_i)_i$ we can assume that for all $j \in \{1, \ldots, N\}$ the sequence $(a_j^{(i)})_i$ converges to some element a_j. This uses $M_n \cap W_m \subseteq [0,1]^m$. All elements a_j lie in $M_n^\ddagger = M_{n-1}$. As $n \geq 2$, the first component of each element $a_j^{(i)}$ is at least $\frac{1}{m}$. So all the sequences $(\lambda_j^{(i)})_i$ are bounded and therefore without loss of generality also convergent. This shows that x belongs to $\mathrm{cc}(M_{n-1})$.

To see "$\supseteq$" note that $M_n^\ddagger \subseteq \mathrm{cc}(M_n)^\ddagger$ and $\mathrm{cc}(M_n)^\ddagger$ is a cone. So

$$\mathrm{cc}(M_{n-1}) = \mathrm{cc}(M_n^\ddagger) \subseteq \mathrm{cc}(M_n)^\ddagger.$$

For $n = 1$ we have

$$\mathrm{cc}(M_1) = \left\{ f = (f_i)_i \in \bigoplus_{i=0}^{\infty} \mathbb{R}_{\geq 0} \cdot e_i \mid f_0 = 0 \Rightarrow f = 0 \right\},$$

so

$$\mathrm{cc}(M_1) \subsetneq \mathrm{cc}(M_1)^\ddagger = \bigoplus_{i=0}^{\infty} \mathbb{R}_{\geq 0} \cdot e_i,$$

which is closed. So all in all we have proven:

For $n \in \{1, 2, \dots\}$ and the cone $\mathrm{cc}(M_n)$, the sequence of iterated sequential closures terminates precisely after n steps at

$$\overline{\mathrm{cc}(M_n)} = \bigoplus_{i=0}^{\infty} \mathbb{R}_{\geq 0} \cdot e_i.$$

References

[1] C. Berg, J. Christensen, C. Jensen, A remark on the multidimensional moment problem, Math. Ann. **243** (1979), 609–169.

[2] A. Barvinok, A Course in Convexity, Graduate studies in mathematics 54, American Mathematical Society, Providence (2002).

[3] T.B. Bisgaard, The topology of finitely open sets is not a vector space topology, Arch. Math. **60** (1993), 546–552.

[4] N. Bourbaki, Topological vector spaces, Chapters 1-5, English edition, Springer Verlag, Masson (1987).

[5] S. Boyd, L. Vandenberghe, Convex optimization, Cambridge University Press, Cambridge (2004).

[6] J. Cimprič, M. Marshall, T. Netzer, On the real multidimensional rational K-moment problem, to appear.

[7] T. Jacobi, A representation theorem for certain partially ordered commutative rings, Math. Zeit. **237** (2001), 223–235.

[8] T. Jacobi, A. Prestel, Distinguished representations of strictly positive polynomials, J. reine angew. Math. **532** (2001), 223–235.

[9] G. Köthe, Topological Vector Spaces I, Springer, Berlin (1969).

[10] S. Kuhlmann, M. Marshall, Positivity, sums of squares and the multidimensional moment problem, Trans. Amer. Math. Soc. **354** (2002), 4285–4301.

[11] S. Kuhlmann, M. Marshall, N. Schwartz, Positivity, sums of squares and the multidimensional moment problem II, Adv. Geom. **5** (2005), 583–606.

[12] J.P. Lasserre, Global optimization with polynomials and the problem of moments, SIAM J. Optim. **11** (2001), 796–817.

[13] M. Laurent, Sums of squares, moment matrices and optimization over polynomials. In: Emerging Applications of Algebraic Geometry (M. Putinar, S. Sullivant, eds.), IMA Volumes in Math. Appl. 149, Springer (2009) , pp. 157-270.

[14] M. Marshall, A real holomorphy ring without the Schmüdgen property, Canad. Math. Bull. **42** (1999), 354–358.

[15] M. Marshall, Positive polynomials and sums of squares, Amer. Math. Soc. Surveys and Monographs **146** (2008).

[16] M. Marshall, Polynomials non-negative on a strip, Proc. Amer. Math. Soc., to appear.

[17] T. Netzer, An Elementary Proof of Schmüdgen's Theorem on the Moment Problem of Closed Semi-algebraic Sets, Proc. of the Amer. Math. Soc. 136 (2008), 529-537.

[18] T. Netzer, Stability of quadratic modules, to appear in Manuscr. Math.

[19] T. Netzer, Positive polynomials and sequential closures of quadratic modules, to appear in Trans. Amer. Math. Soc.

[20] T. Netzer, Positive Polynomials, Sums of Squares and the Moment Problem, Ph.D. Thesis, Konstanz (2008). `http://kops.ub.uni-konstanz.de/volltexte/2008/6737`

[21] D. Plaumann, Bounded polynomials, sums of squares, and the moment problem, Ph.D. Thesis, Konstanz (2008). `http://kops.ub.uni-konstanz.de/volltexte/2008/5579/`

[22] V. Powers, C. Scheiderer, The moment problem for non-compact semialgebraic sets, Adv. Geom. **1** (2001), 71–88.

[23] A. Prestel, C. Delzell, Positive polynomials – from Hilbert's 17th problem to real algebra, Springer Monographs in Mathematics, Springer, Berlin (2001).

[24] M. Putinar, Positive polynomials on compact semialgebraic sets, Indiana Univ. Math. J. **43** (1993), 969–984.

[25] H.H. Schaefer, Topological vector spaces, Springer (1971).

[26] C. Scheiderer, Non-existence of degree bounds for weighted sums of squares representations, J. of Complexity **21** (2005), 823–844.

[27] C. Scheiderer, Positivity and sums of squares: A guide to recent results. In: Emerging Applications of Algebraic Geometry (M. Putinar, S. Sullivant, eds.), IMA Volumes Math. Appl. 149, Springer (2009), pp. 271-324.

[28] K. Schmüdgen, The K-moment problem for compact semialgebraic sets, Math. Ann. **289** (1991), 203–206.

[29] K. Schmüdgen, The moment problem for closed semialgebraic sets, J. reine angew. Math. **558** (2003), 225–234.

[30] T. Wörmann, Strikt positive Polynome in der semialgebraischen Geometrie, Dissertation, Univ. Dortmund (1998).

Cimprič, Jaka
Faculty of Mathematics and Physics
University of Ljubljana
Jadranska 21, SI-1000 Ljubljana, Slovenija
email: cimpric@fmf.uni-lj.si

Marshall, Murray
Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon, SK S7N 5E6, Canada
email: marshall@math.usask.ca

Netzer, Tim
Fachbereich Mathematik und Statistik
Universität Konstanz
D-78457 Konstanz, Germany
email: Tim.Netzer@uni-konstanz.de