The pp conjecture for the space of orderings of the field $\mathbb{R}(x,y)$

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Abstract

The paper considers the space of orderings $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$ of the field of rational functions over $\mathbb{R}$ in two variables. It is shown that the pp conjecture fails to hold for such a space; an example of a positive primitive formula which is not product-free and one-related is investigated and it is proven, that although the formula holds true for every finite subspace of $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$, it is false in general. This provides a negative answer to one of the questions raised in: M. Marshall, *Open questions in the theory of spaces of orderings*, J. Symbolic Logic 67 (2002), 341-352. This work is a sequel of previous results presented in: P. Gładki, M. Marshall, *The pp conjecture for spaces*

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of orderings of rational conics, to appear in J. Algebra Appl.; both spaces of orderings of conic sections and the space $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$ are important examples of spaces of stability index 2 that are in the scope of our research.

Keywords: quadratic forms, spaces of orderings.

Throughout this paper (X, G) denotes a space of orderings in the sense of [6, pp. 21-22]. We will be mostly dealing with spaces of orderings of the form (X_K, G_K), where K is a formally real field, X_K denotes the set of all orderings of K and $G_K = K^*/(\Sigma K^2 \setminus \{0\})$, ΣK^2 being the set of sums of squares of K [6, Theorem 2.1.4]. In such a case G_K is identified with a subgroup of the group $\{1, -1\}^{X_K}$ [6, Lemma 2.1.1]. With a slight abuse of the notation we shall use the same symbol to denote an element of K^*, a coset in G_K and a function in $\{-1, 1\}^{X_K}$.

For a fixed space of orderings (X, G) and $a \in G$ let

$$U(a) = \{x \in X : a(x) = 1\}.$$

As a subspace of (X, G) we understand a pair $(Y, G|_Y)$, where $Y \neq \emptyset$ is some intersection of sets of the form $U(a)$ and $G|_Y$ is the group of all restrictions $a|_Y$, $a \in G$ [6, pp. 32-33]. A subspace of a space of orderings is a space of orderings itself [6, Theorem 2.4.3]. While considering subspaces, we will usually use the same notation for elements $a \in G$ and their restrictions $a|_Y$.

If (Y, H) is a subspace of (X, G) and $a, b \in H$, we define the value set

$$D_Y(a, b) = \{c \in H : \forall x \in Y (c(x) = a(x) \text{ or } c(x) = b(x))\}.$$

In the case when $Y = X$ or when it is clear in which subspace we work, we shall write $D(a, b)$ instead of $D_Y(a, b)$.

With the notion of value sets we define positive primitive (pp for short) formulae as the ones of the form

$$P(\underline{a}) = \exists \underline{t} \bigwedge_{j=1}^{m} p_j(\underline{t}, \underline{a}) \in D(1, q_j(\underline{t}, \underline{a})),$$

where $\underline{t} = (t_1, \ldots, t_n)$, $\underline{a} = (a_1, \ldots, a_k)$, for $t_i, a_l \in G$, $i \in \{1, \ldots, n\}$, $l \in \{1, \ldots, k\}$, and $p_j(\underline{t}, \underline{a})$, $q_j(\underline{t}, \underline{a})$ are $\pm$ products of some of the t_i's and a_l's, $i \in \{1, \ldots, n\}$, $l \in \{1, \ldots, k\}$. Clearly, when we speak of a pp formula $P(\underline{a})$ in a subspace (Y, H), we think of all parameters a_l as their restrictions $a_l|_H$ and of all value sets $D(1, q_j(\underline{t}, \underline{a}))$ as value sets $D_Y(1, q_j(\underline{t}, \underline{a}))$.

The following problem, known as the pp conjecture, has been posed in [7]: *Is it true that every pp formula* $P(\underline{a})$ *with parameters* $\underline{a}$ *in* G *which holds in every finite subspace of* (X, G) *necessarily holds in* (X, G)*?* The answer to the problem is affirmative for numerous pp formulae describing important properties of quadratic forms over spaces of orderings (see [7] for details) and for - introduced in [8] - product-free and one-related formulae in spaces of finite stability index. The class of spaces for which the conjecture is true contains spaces of finite chain length, spaces of stability index 1 and is closed under direct sum and group extension [7]. As to spaces of stability index 2, the following examples are of our interest: spaces of orderings of formally real finitely generated extensions of $\mathbb{Q}$ of transcendence degree 1 (in particular $\mathbb{Q}(x)$ and function fields of conic sections) ([1, Proposition VI.3.5]), spaces of orderings of formally real finitely generated extensions of real closed fields of transcendence degree 2 (in particular $\mathbb{R}(x, y)$ and its finitely generated algebraic extensions) ([1, Proposition VI.3.2]), and spaces of orderings of a field of formal power series $R((x, y))$ in two variables, or a field of algebraic power series $R((x, y))_{alg}$, or a field of analytic power series $R\{\{x, y\}\}$ over a real closed field R (in particular $\mathbb{R}((x, y))$, $\mathbb{R}((x, y))_{alg}$, and $\mathbb{R}\{\{x, y\}\}$) ([1, Example VII.2.3 b), c), Remark VII.5.6]). The pp conjecture holds true for the space of orderings of the field $\mathbb{Q}(x)$ [4]. For spaces of orderings of conic sections the complete classification with respect to the conjecture is given in [5]. Due to rather complicated real valuations of the field $\mathbb{R}(x, y)$, methods used in [4] and [5] could not be applied to the space $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$. This paper circumvents this obstacle and here new, 'valuation theory free' methods are developed and used. Our main result is the following theorem:

Theorem 1. *The pp conjecture fails for the space of orderings* $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$.

Proof. For $n \in \mathbb{N} \setminus \{0\}$ consider the subspaces (X_n, G_n), where

$$X_n = U(x^2 + y^2 - 1) \cap U(1 + \frac{1}{n} - x^2 - y^2)$$

and $G_n = G_{\mathbb{R}(x,y)}|_{X_n}$. Define the subspace (X, G), where

$$X = \bigcap_{n \in \mathbb{N} \setminus \{0\}} X_n$$

and $G = G_{\mathbb{R}(x,y)}|_X$. It is sufficient to show that the conjecture fails in the space (X, G) [2, Proposition 6]. For $n \in \mathbb{N} \setminus \{0\}$ denote

$$A_n = \{(a, b) \in \mathbb{R}^2 : 1 < a^2 + b^2 < 1 + \frac{1}{n}\}$$

and let $\pi_1, \ldots, \pi_6 \in \mathbb{R}(x, y)$ be linear irreducibles which, for n large enough intersect with rings A_n as follows:

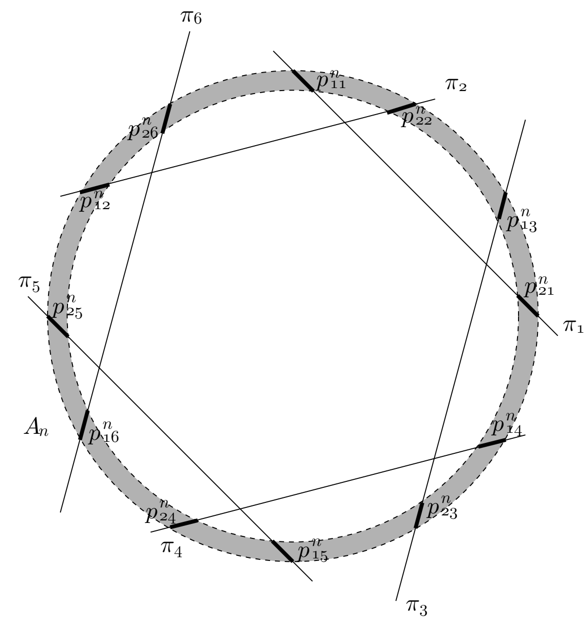

Fig. 1

Here p_{1i}^n, p_{2i}^n denote the two connected components of $\mathcal{Z}(\pi_i) \cap A_n$, $i \in \{1, \ldots, 6\}$, $n \in \mathbb{N} \setminus \{0\}$, and are arranged in the above order, where $\mathcal{Z}(\pi_i)$ is the set of real zeros of π_i. Replacing π_i by $-\pi_i$ we may assume that every π_i is positive at the origin. For two sets $p_{i_1 j_1}^n$ and $p_{i_2 j_2}^n$, $i_1, i_2 \in \{1, 2\}$, $j_1, j_2 \in \{1, \ldots, 6\}$, denote also by $A_n^{i_1 j_1, i_2 j_2}$ the ring sector starting at $p_{i_1 j_1}^n$ and, when moving clockwise along A_n, ending at $p_{i_2 j_2}^n$.

Let $a_1 = \pi_1 \pi_6$, $a_2 = \pi_1 \pi_4$ and $d = -\pi_1 \pi_2 \pi_3 \pi_5$. Consider the following pp formula:

$$P(a_1, a_2, d) = \exists t_1 \exists t_2 (t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge d t_1 t_2 \in D(1, a_1 a_2)).$$

We shall show that $P(a_1, a_2, d)$ fails to hold in the space (X, G).

Suppose, *a contrario*, that the formula holds true in (X, G) with certain $t_1, t_2 \in G$ verifying it. Without loss of generality we may assume that t_1, t_2 are square-free polynomials. Let

$$S = \{\sigma : \sigma \text{ is irreducible and } \sigma | t_1 \text{ or } t_2, \text{ or } \sigma = \pi_i \text{ for some } i \in \{1, \ldots, 6\}\}.$$

Observe, that there exists $N_1 \in \mathbb{N} \setminus \{0\}$ such that for $n \geq N_1$:

> for each $\sigma \in S$ the set $\mathcal{Z}(\sigma) \cap A_n$ is a finite disjoint union of smooth arcs $\gamma : (0,1) \to \mathbb{R}^2$ homeomorphic to an open line segment and such that $\lim_{t\to 0} \gamma(t)$ is a point on the circle $x^2 + y^2 = 1$, whilst $\lim_{t\to 1} \gamma(t)$ is a point on $x^2 + y^2 = 1 + \frac{1}{n}$,

and

> for $\sigma, \tau \in S$, $\sigma \neq \tau$:[1]
>
> $$\mathcal{Z}(\sigma) \cap \mathcal{Z}(\tau) \cap A_n = \emptyset.$$

This is intuitively clear, however if one wants to prove it formally, one should use the 'half-branches' theorem [3, Proposition 9.5.1] and the fact that we may restrict ourselves to those $\sigma \in S$ for which ideals (σ) are real (see [3, Theorem 4.5.1]).

Observe also that for n sufficiently large (say, $n \geq N_2$ for some $N_2 \in \mathbb{N} \setminus \{0\}$) $P(a_1, a_2, d)$ already holds in the subspace (X_n, G_n). Indeed, consider the open set

$$U = (U(-a_1) \cup U(t_1)) \cap (U(-a_2) \cup U(t_2)) \cap (U(-a_1 a_2) \cup U(dt_1 t_2)),$$

viewed as a subset in $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$. Since

$$t_1 \in D(1, a_1) \wedge t_2 \in D(1, a_2) \wedge dt_1 t_2 \in D(1, a_1 a_2)$$

holds true in (X, G), $X \subset U$. But $X = \bigcap_{n \in \mathbb{N}\setminus\{0\}} X_n$, where $X_1 \supset X_2 \supset \ldots$ is a chain of closed subsets, and $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$ is compact [6, Theorem 2.1.5], so for n large enough $X_n \subset U$. That means that $P(a_1, a_2, d)$ holds true in (X_n, G_n).

Fix $n \in \mathbb{N} \setminus \{0\}$ satisfying all of the above conditions (that is $n \geq \max\{N_1, N_2\}$) and consider the space (X_n, G_n). By looking at number of sign changes of each irreducible factor σ of t_1 or t_2 when we travel along the circle $x^2 + y^2 = 1 + \frac{1}{n}$ we see, that each such $\mathcal{Z}(\sigma)$ intersects with A_n in an even number of connected components [3, Theorem 4.5.1].

Furthermore, the signs of a_1, a_2 and d on the ring sectors between the successive p_{ij}^n, $i \in \{1, 2\}$, $j \in \{1, \ldots, 6\}$, are the following:

[1] Note that some of $\pi_1, \ldots, \pi_6$ might be also divisors of t_1 or t_2

	$A_n^{11,22}$	$A_n^{22,13}$	$A_n^{13,21}$	$A_n^{21,14}$	$A_n^{14,23}$	$A_n^{23,15}$	$A_n^{15,24}$	$A_n^{24,16}$	$A_n^{16,25}$	$A_n^{25,12}$	$A_n^{12,26}$	$A_n^{26,11}$
a_1	−	−	−	+	+	+	+	+	−	−	−	+
a_2	−	−	−	+	−	−	−	+	+	+	+	+
d	−	+	−	+	+	−	+	+	+	−	+	+

Tab. 1

We yield a contradiction by investigating the behaviour of t_1 and t_2 on A_n. The following criterion for representativity of binary forms shall be of constant use:

$$f \in D_{X_n}(1, g) \Leftrightarrow \forall (a, b) \in A_n (f(a, b) \geq 0 \text{ or } f(a, b) \cdot g(a, b) \geq 0)$$

(see [4, Corollary 3.2]).

On $A_n^{21,14}$, $A_n^{24,16}$ and $A_n^{26,11}$ both a_1 and a_2 are positive, so t_1 and t_2 are nonnegative. Moreover, since t_1 and t_2 are square-free and since there are no singular points of irreducible factors of t_1, t_2 inside of A_n, by the Sign Changing Criterion [3, Theorem 4.5.1], t_1 and t_2 are, in fact, positive.

Near p_{23}^n a_1 is positive, so t_1 is positive. It follows that $\mathcal{Z}(t_1)$ (from now on we shall simply write t_1) does not intersect with A_n along p_{13}^n: if it did, then π_3 would divide t_1 (since they would have infinitely many points in common), so $t_1 = 0$ on p_{23}^n.

Furthermore, $a_1 a_2 > 0$ near p_{13}^n, so $d t_1 t_2$ is nonnegative. Since d changes sign between $A_n^{22,13}$ and $A_n^{13,21}$ and t_1 does not intersect with A_n along p_{13}^n, t_2 has to pass A_n at p_{13}^n. Thus $\pi_3 | t_2$ and t_2 also cuts across A_n at p_{23}^n.

Similarly, $a_2 > 0$ near p_{12}^n, so $t_2 > 0$ and, as before, t_2 does not intersect with A_n along p_{22}^n. Close to p_{22}^n $a_1 a_2 > 0$, so $d t_1 t_2 \geq 0$ and thus t_1 passes A_n at p_{22}^n and also at p_{12}^n.

Next, near p_{11}^n $a_1 a_2 > 0$, so $d t_1 t_2 \geq 0$, whilst d changes sign between $A_n^{26,11}$ and $A_n^{11,22}$. Thus $t_1 t_2$ changes sign, so either t_1 intersects with A_n along p_{11}^n and t_2 does not, or t_2 does and t_1 does not.

Similarly, near p_{21}^n $a_1 a_2 > 0$, so $d t_1 t_2 \geq 0$. d changes sign at p_{21}^n and so does $t_1 t_2$, which implies that either t_1 crosses A_n at p_{21}^n and t_2 does not, or t_1 does not cross and t_2 does.

Of course if t_1 passes A_n at p_{11}^n, then $\pi_1 | t_1$, so t_1 also passes A_n at p_{21}^n. Therefore t_1 cuts across A_n at p_{11}^n if and only if it cuts across A_n at p_{21}^n and, similarly, t_2 traverses A_n at p_{11}^n if and only if it traverses A_n at p_{21}^n.

On $A_n^{11,22}$ $a_1a_2 > 0$, so $dt_1t_2 \geq 0$. Since $d < 0$, $t_1t_2 \leq 0$, so t_1 intersects with A_n if and only if t_2 does - say, there are m_1 such intersections within $A_n^{11,22}$.

Similarly, on $A_n^{13,21}$ $a_1a_2 > 0$, so $dt_1t_2 \geq 0$. At the same time $d < 0$, so $t_1t_2 \leq 0$. Thus t_1 intersects with A_n if and only if t_2 does; there are m_2 such intersections within $A_n^{13,21}$.

Finally, on $A_n^{22,13}$ $a_1a_2 > 0$ and $d > 0$, so $dt_1t_2 \geq 0$ and $t_1t_2 \geq 0$. Therefore t_1 intersects with A_n if and only if t_2 does and we have m_3 such simultaneous intersections within $A_n^{22,13}$.

To sum up, there are $m_1+m_2+m_3$ simultaneous intersections of t_1 and t_2 with A_n in $A_n^{11,21}$. Furthermore, t_1 crosses through both p_{22}^n and t_2 through p_{13}^n. And finally, exactly one of t_1, t_2 crosses through both p_{11}^n and p_{21}^n: say t_i does and t_j does not. Then t_j changes sign $m_1 + m_2 + m_3 + 1$ from $A_n^{26,11}$ to $A_n^{21,14}$, to go from positive to positive, hence $m_1 + m_2 + m_3 + 1$ is even and $m_1 + m_2 + m_3$ is odd.

Note now that the only simultaneous intersections of t_1 and t_2 with A_n are the $m_1 + m_2 + m_3$ listed above; on all other sectors of A_n at least one of a_1, a_2 is positive, forcing either t_1 or t_2 to be positive as well.

Simultaneous intersections may occur only at the common irreducible factors of t_1, t_2. According to our assumptions, each such factor has an even number of crossings with A_n - so $m_1 + m_2 + m_3$ is even, which is a contradiction. This finishes the first half of the proof.

It remains to show that $P(a_1, a_2, d)$ holds true on every finite subspace of (X, G). Suppose then that there is a finite subspace (Y, H) of (X, G) on which $P(a_1, a_2, d)$ fails to hold. Without loss of generality we may assume that (Y, H) is minimal with such property. We need to consider two cases.

Firstly, suppose that $d \notin D((1, a_1) \otimes (1, a_2))$ holds on (Y, H). We shall use the following desription of value sets of Pfister forms: for any $f_1, \ldots, f_k \in H$, $g \in D((1, f_1) \otimes \ldots \otimes (1, f_k))$ if and only if:

$$\forall \rho \in Y[(f_1\rho = 1 \wedge \ldots \wedge f_k\rho = 1) \Rightarrow g\sigma = 1]$$

([6, Theorem 2.4.1]). Thus, for some $\sigma \in Y$, $a_1\sigma = 1$, $a_2\sigma = 1$ and $d\sigma = -1$. Clearly $\sigma \in X_n$ for any fixed $n \in \mathbb{N}\setminus\{0\}$, so - by the Tarski Transfer Principle [3, Corollary 5.2.4] - there is a point $(a, b) \in A_n$ such that $a_1(a, b) > 0$, $a_2(a, b) > 0$ and $d(a, b) < 0$. But there is no such point in A_n (see Tab. 1) - a contradiction.

Now assume that $d \in D((1, a_1) \otimes (1, a_2))$ holds in Y. Since (Y, H) is finite, it is a direct sum of finitely many connected components, that is subspaces which correspond to equivalence classes of the following relation: if $\rho_1, \rho_2 \in Y$, then $\rho_1 \sim \rho_2$ if and only if either $\rho_1 = \rho_2$ or there exist $\rho_3, \rho_4 \in Y$ such that $\{\rho_1, \ldots, \rho_4\}$ is a 4-element fan in Y ([6, Theorem 4.2.1]). By [8, Corollary 3.6] there exists a connected component (Y_0, H_0) of (Y, H), which is not a fan, such that, if $(\overline{Y}, \overline{H})$ denotes the residue space of (Y_0, H_0) (that is a minimal space in the sense that if (Y_0, H_0) is a group extension of some space of orderings $(\widehat{Y}, \widehat{H})$, then $\overline{H} \subset \widehat{H}$), $a_1, a_2 \in \overline{H}$, neither a_1, a_2 nor a_1a_2 is equal to -1, $(1, a_1) \otimes (1, a_2)$ is isotropic over (Y_0, H_0) and $d \notin \overline{H}$. Clearly $P(a_1, a_2, d)$ already fails to hold in (Y_0, H_0), so - due to minimality of (Y, H) - $(Y, H) = (Y_0, H_0)$.

Since $a_1, a_2, a_1a_2 \neq -1$, there are elements of $\overline{Y}$ making a_1, a_2 and a_1a_2 positive. At the same time, since $(1, a_1) \otimes (1, a_2)$ is isotropic, there is no element of $\overline{Y}$ making both a_1 and a_2 positive. Fix $\sigma_1, \sigma_2, \sigma_3 \in \overline{Y}$ such that a_1, a_2 and a_1a_2 have the following signs:

	σ_1	σ_2	σ_3
a_1	+	-	-
a_2	-	+	-
a_1a_2	-	-	+

Tab. 2

Consider the subspace $(\tilde{Y}, \tilde{H})$ which is not a fan and for which $\{\sigma_1, \sigma_2, \sigma_3\}$ is a minimal generating set. Thus elements of $\tilde{Y}$, viewed as characters, are products $\prod_{i=1}^{3} \sigma_i^{e_i}$ such that $\sum_{i=1}^{3} e_i \equiv 1 \mod 2$ and do not contain the element $\sigma_1\sigma_2\sigma_3$ ([6, Theorem 3.1.3]) – consequently, $\tilde{Y} = \{\sigma_1, \sigma_2, \sigma_3\}$. Let (Y_1, H_1) be the group extension of $(\tilde{Y}, \tilde{H})$ by d. It consists of 6 orderings σ_1^+, σ_2^+, σ_3^+, σ_1^-, σ_2^-, σ_3^-, with respect to which the signs of a_1, a_2, a_1a_2, d are as follows:

	σ_1^+	σ_2^+	σ_3^+	σ_1^-	σ_2^-	σ_3^-
a_1	+	-	-	+	-	-
a_2	-	+	-	-	+	-
a_1a_2	-	-	+	-	-	+
d	+	+	+	-	-	-

Tab. 3

$P(a_1, a_2, d)$ fails to hold on (Y_1, H_1), so $(Y, H) = (Y_1, H_1)$.

Define the following subspaces of (X, G):

$$
\begin{aligned}
V^{11,22} &= U(-\pi_1) \cap U(-\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{22,13} &= U(-\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{13,21} &= U(-\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{21,14} &= U(\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{14,23} &= U(\pi_1) \cap U(\pi_2) \cap U(-\pi_3) \cap U(-\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{23,15} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(-\pi_4) \cap U(\pi_5) \cap U(\pi_6) \\
V^{15,24} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(-\pi_4) \cap U(-\pi_5) \cap U(\pi_6) \\
V^{24,16} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(-\pi_5) \cap U(\pi_6) \\
V^{16,25} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(-\pi_5) \cap U(-\pi_6) \\
V^{25,12} &= U(\pi_1) \cap U(\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(-\pi_6) \\
V^{12,26} &= U(\pi_1) \cap U(-\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(-\pi_6) \\
V^{26,11} &= U(\pi_1) \cap U(-\pi_2) \cap U(\pi_3) \cap U(\pi_4) \cap U(\pi_5) \cap U(\pi_6).
\end{aligned}
$$

By the Tarski Transfer Principle subspaces $V^{i_1j_1,i_2j_2}$ form a partition of (X, G) and, clearly, signs of a_1, a_2 and d on the $V^{i_1j_1,i_2j_2}$ are exactly the same as on the sector $A_n^{i_1j_1,i_2j_2}$, for respective i_1, i_2, j_1, j_2. Comparing those signs we see that $\sigma_1^- \in V^{23,15}$, $\sigma_1^+ \in V^{14,23}$ or $\sigma_1^+ \in V^{15,24}$, $\sigma_2^- \in V^{25,12}$, $\sigma_2^+ \in V^{16,25}$ or $\sigma_2^+ \in V^{12,26}$ and $\sigma_3^+ \in V^{22,13}$, $\sigma_3^- \in V^{11,22}$ or $\sigma_3^- \in V^{13,21}$.

Consider the following two 4-element fans:

$$\{\sigma_1^+, \sigma_1^-, \sigma_2^+, \sigma_2^-\} \text{ and } \{\sigma_1^+, \sigma_1^-, \sigma_3^+, \sigma_3^-\}.$$

If $\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{12,26}$, then, in particular, $\pi_3(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction, since for every 4-element fan $\{\rho_1, \ldots, \rho_4\}$ $\prod_{i=1}^4 \rho_i = 1$ (note that we can also use π_2 instead of π_3). On the other hand, if $\sigma_1^+ \in V^{14,23}$ and $\sigma_2^+ \in V^{16,25}$, then $\pi_5(\sigma_1^+\sigma_1^-\sigma_2^+\sigma_2^-) = -1$ - a contradiction. Thus $\sigma_1^+ \in V^{15,24}$.

If $\sigma_1^+ \in V^{15,24}$ and $\sigma_3^- \in V^{13,21}$, then $\pi_3(\sigma_1^+\sigma_1^-\sigma_3^+\sigma_3^-) = -1$ - a contradiction. But if $\sigma_1^+ \in V^{15,24}$ and $\sigma_3^- \in V^{11,22}$, then $\pi_2(\sigma_1^+\sigma_1^-\sigma_3^+\sigma_3^-) = -1$, which eliminates the last case and yields a final contradiction. □

To obtain a concrete counterexample in the space $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$ we use a standard trick. The formula $P(a_1, a_2, d)$ constructed in the proof can be written in the following form:

$$\exists t_1, t_2[(t_1, a_1t_1) \cong (1, a_1)] \wedge [(t_2, a_2t_2) \cong (1, a_2)] \wedge [(dt_1t_2, a_1a_2dt_1t_2) \cong (1, a_1a_2)]$$

and we know that, for suitably chosen n, it fails in the space (X_n, G_n), although it holds true in each of its finite subspaces [2, Proposition 6]. Let $p_1 = x^2 + y^2 - 1$ and $p_2 = 1 + \frac{1}{n} - x^2 - y^2$, so that $X_n = U(p_1) \cap U(p_2)$. Clearly the formula

$$\begin{aligned} &\exists t_1 \exists t_2 [(t_1, a_1 t_1) \otimes (1, p_1) \otimes (1, p_2) \cong (1, a_1) \otimes (1, p_1) \otimes (1, p_2)] \wedge \\ &\quad \wedge \quad [(t_2, a_2 t_2) \otimes (1, p_1) \otimes (1, p_2) \cong (1, a_2) \otimes (1, p_1) \otimes (1, p_2)] \wedge \\ &\quad \wedge \quad [(d t_1 t_2, a_1 a_2 d t_1 t_2) \otimes (1, p_1) \otimes (1, p_2) \cong (1, a_1 a_2) \otimes (1, p_1) \otimes (1, p_2)] \end{aligned}$$

holds true in every finite subspace of $(X_{\mathbb{R}(x,y)}, G_{\mathbb{R}(x,y)})$, but fails in general.

Remarks: (1) The case of the field $\mathbb{Q}(x, y)$ is already well-understood. Let $f(x, y) = 0$ be an equation of an irreducible conic section without rational points, for example let $f(x, y) = x^2 + y^2 - 3$. Then the space (X_f, G_f) of orderings compatible with the valuation v induced by f is a subspace of the space $(X_{\mathbb{Q}(x,y)}, G_{\mathbb{Q}(x,y)})$. Moreover, this space is also a group extension of the space of orderings of the residue field $\mathbb{Q}(x, y)_v$, that is the function field of the curve $f(x, y) = 0$. If the pp conjecture was true for the space $(X_{\mathbb{Q}(x,y)}, G_{\mathbb{Q}(x,y)})$, then it would be also true for the space (X_f, G_f) ([2, Proposition 6]) and, consequently, for the space $(X_{\mathbb{Q}(x,y)_v}, G_{\mathbb{Q}(x,y)_v})$ ([7, Proposition 2.3]), which is a contradiction ([5, Theorem 6]).

(2) One would expect the pp conjecture to fail for spaces of orderings of finitely generated algebraic extensions of the field $\mathbb{R}(x, y)$ or, more generally, $R(x, y)$, for a real closed field R.

(3) Up to date, nothing is known about the pp conjecture for spaces of orderings of fields $\mathbb{R}((x, y))$, $\mathbb{R}((x, y))_{alg}$, or $\mathbb{R}\{\{x, y\}\}$, as well as $R((x, y))$, $R((x, y))_{alg}$, or $R\{\{x, y\}\}$, for R being a real closed field.

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