EXTENDING PIECEWISE POLYNOMIAL FUNCTIONS IN TWO VARIABLES

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Abstract. We study the extendibility of piecewise polynomial functions defined on closed subsets of $\mathbb{R}^2$. The compact subsets on which every piecewise polynomial function is extensible can be characterized in terms of quasi-convexity if they are definable in an o-minimal expansion of the real field. Even the non compact closed definable subsets can be characterized if semi-algebraic function germs at infinity are dense in the Hardy field of definable germs. We also present a piecewise polynomial function defined on a compact convex subset of $\mathbb{R}^2$ which is not extensible.

1. Introduction

A piecewise polynomial function is a continuous function $f : A \to \mathbb{R}$ for which there exist finitely many polynomials $p_1, \dots, p_k$ such that for every $a \in A$ $f(a) = p_i(a)$ for some i. In short, we say that f is a *pwp* function. If A is a semialgebraic set, then the continuity implies that f is semialgebraic, and it coincides with the standard definition of *pwp* function which a priori restricts to semialgebraic sets. Our definition, however, allows us to study *pwp* functions on more transcendental subsets of $\mathbb{R}^n$. Here, we are mainly interested in subsets of $\mathbb{R}^2$ which are definable in an o-minimal expansion $\mathcal{M}$ of the real field, as in this case we always obtain extendibility for compact convex definable sets, while this is not true anymore without the definability assumption. Here and in the following, *definable* always means *definable in $\mathcal{M}$ with parameters of* $\mathbb{R}$. See [3] for an introduction to o-minimal structures.

Our studies are closely related to the Pierce-Birkhoff conjecture which asserts that every *pwp* function on $\mathbb{R}^n$ is sup / inf definable from polynomials. Actually, if f is sup / inf definable on A, then the extendibility of f as *pwp* function is trivial.

A proof of the Pierce-Birkhoff conjecture for functions in two variables was sketched by Mahé in [9], see [10] for a more detailed demonstration. The Pierce-Birkhoff conjecture is still an open question for *pwp* functions of three and more variables. Note that *pwp* functions can be studied over real closed fields and ordered fields, see for example [1, 6, 8, 11, 13]. However, our discussions require at least an Archimedean real closed field, which can always be imbedded into $\mathbb{R}$. Here we only treat the field of real numbers.

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Mahé presented a semialgebraic *pwp* function of two variables which cannot be extended to $\mathbb{R}^2$ as *pwp* function, in [9, Remarks]. This example motivates the following question:

What are the subsets of $\mathbb{R}^2$ for which every *pwp* function is extensible?

We give an answer to this question for sets which are definable in an o-minimal expansion of the real field. The compact sets among these sets can be described as the *locally quasi-convex* definable subsets of $\mathbb{R}^2$. Let us make this notion precise.

We denote by $\|\cdot\|$ the Euclidean norm, and by $B_r(x)$ the open ball with radius r and center x.

Definition 1.1. A set A is said to be *locally quasi-convex* at a point $a \in A$ if there exists an $\varepsilon > 0$ and an $L > 0$ such that all points $\xi, \eta \in B_\varepsilon(a) \cap A$ can be joined by a rectifiable path γ in $B_\varepsilon(a) \cap A$ of length less than or equal to $L\|\xi - \eta\|$. We will say A is *locally quasi-convex* if A is locally quasi-convex at each $a \in A$.

For $L = 1$ we obtain the locally convex sets, so every locally convex set is locally quasi-convex. The first main result of the present paper gives a complete characterization of the compact definable subsets on which every *pwp* functions is extensible. We shall prove the following theorem.

Theorem 1.2. *Let $A \subset \mathbb{R}^2$ be a compact definable set. Then every pwp function on A can be extended to a pwp function on $\mathbb{R}^2$ if and only if A is locally quasi-convex.*

We do not know if the definability in an o-minimal structure of the sets is required. However, we will present a *pwp* function defined on a compact convex subset of $\mathbb{R}^2$ which is not extensible; see Example 3.8.

To obtain extendibility for non-compact definable closed sets we have to restrict ourselves to a small class of polynomially bounded o-minimal structures, that is, every unary definable germ at infinity is ultimately bounded by some polynomial. Let $\mathcal{H}(\mathbb{R})$ and $\mathcal{H}(\mathcal{M})$ denote the Hardy field of semialgebraic and definable unary function germs at $+\infty$, respectively. Let $\mathcal{H}(\mathcal{M})$ be endowed with the topology induced by its ordering. Then the characterization of o-minimal structures for which every *pwp* function on a locally quasi-convex closed definable subset of $\mathbb{R}^2$ is extensible reads as follows.

Theorem 1.3. *Let $\mathcal{H}(\mathbb{R})$ be dense in $\mathcal{H}(\mathcal{M})$. Let $A \subset \mathbb{R}^2$ be closed and definable. Then every pwp function on A can be extended to a pwp function on $\mathbb{R}^2$ if and only if A is locally quasi-convex.*

Examples of such o-minimal structures are the structure $\mathbb{R}_{\text{an}}$ of restricted analytic functions, cf. [2], and the structures generated in [12], see [5, Proposition 2.3]. Theorem 1.3 is sharp in the sense that if $\mathcal{H}(\mathbb{R})$ is not dense in $\mathcal{H}(\mathcal{M})$, there exists always a *pwp* function defined on a locally quasi-convex closed definable subset of $\mathbb{R}^2$ which is not extensible, see section 4.3.

In section 2 we note that *pwp* functions on $\mathbb{R}^2$ are locally Lipschitz continuous. In section 3 we study some properties of definable locally quasi-convex sets. In section 4 we prove Theorem 1.2 and Theorem 1.3.

2. Local Lipschitz continuity

Before we give some examples of *pwp* functions, we note that sum, product and composition of *pwp* functions are again *pwp* functions. Of course, polynomials are *pwp* functions, but also the absolute value function is *pwp*. Moreover, infimum and supremum of two *pwp* functions are also *pwp*.

We recall the definition of a *locally Lipschitz continuous* function.

Definition 2.1. A function $f : U \to \mathbb{R}$ is called *locally Lipschitz continuous* if for every $u \in U$ there exists an $\varepsilon > 0$ and an $L > 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

for all $x, y \in B_\varepsilon(u) \cap U$.

An analytic property of *pwp* functions is the fact, that they are locally Lipschitz continuous, which we state in the following proposition. Let ∇ denote the gradient operator.

Proposition 2.2. *Let $A \subset \mathbb{R}^n$ be convex. Then every pwp function $f : A \to \mathbb{R}$ is locally Lipschitz continuous.*

Proof. Let $x \in A$, $\varepsilon > 0$, and let f be defined by the polynomials $p_1, \ldots, p_r$. Then,

$$M := \sup\{\|\nabla p_i(y)\| : \ y \in B_\varepsilon(x), i = 1, \ldots, r\}$$

is a real number. Moreover, for every $y, z \in B_\varepsilon(x) \cap A$, the function f is piecewise differentiable along the segment connecting y and z. Hence, by the Mean Value Theorem,

$$\|f(z) - f(y)\| \leq M\|y - z\|.$$

□

3. locally quasi-convex sets

We fix an o-minimal expansion $\mathcal{M}$ of the real field. *Definable* always means *definable in $\mathcal{M}$ with parameters from* $\mathbb{R}$.

For a definable set A we denote by $\overline{A}$, A^o, and ∂A, the closure, interior and boundary of A, respectively.

3.1. **Stratification.** Definable sets can be partitioned into finitely many sets of a suitable form. One says that such partition is a stratification, and calls the sets strata, if for any two sets C and D of the partition, we have that $C \subset \overline{D}$ or $D \subset \overline{C}$ or $C \cap \overline{D} = \emptyset$ or $\overline{C} \cap D = \emptyset$. Here we are only interested in stratification of boundaries of definable subsets of $\mathbb{R}^2$. In this case one can take the strata to be Lipschitz cells; these are sets, which are either a single point or, after some suitable rotation, the graph $\Gamma(h)$ of a definable Lipschitz continuous $\mathcal{C}^1$ function $h : I \to \mathbb{R}$ where I is an open interval, see [4] or [7].

If, furthermore, for given definable sets $A_1, \ldots, A_k \subset \mathbb{R}^2$, we have that for each Lipschitz cell C and each $1 \leq i \leq k$ either $C \subset A_i$ or $C \cap A_i = \emptyset$, we say that the stratification is compatible with the sets $A_1, \ldots, A_k$. Altogether we have the following proposition.

Proposition 3.1 ([4] Theorem 1.4 or [7])**.** *Let $A, A_1, \ldots, A_k \subset \mathbb{R}^2$ be definable sets, and let $f : \partial A \to \mathbb{R}$ be a definable function. Then there is a stratification of ∂A into Lipschitz cells $S_1, \ldots, S_m$ which is compatible with the sets $A_1, \ldots, A_k$ such that f restricted to S_j is a $\mathcal{C}^1$ function for $j = 1, \ldots, m$.*

3.2. **Locally quasi-convex definable sets.** The local quasi-convexity of a definable subset of $\mathbb{R}^2$ can be graphically described as follows.

At each boundary point of a closed definable subset of $\mathbb{R}^2$ there are finitely many well defined outside angles. For example, the set $A = \{(x,y) : \ x \geq 0, \ |y| = x^2\}$ has two outside angles at $(0,0)$, one is equal to 0 the other is 2π. But this set is not locally quasi-convex at the origin. This is easily seen by the following lemma.

Lemma 3.2. *A closed definable set $A \subset \mathbb{R}^2$ is locally quasi-convex at $a \in \partial A$ if and only if each outside angle at a is strictly positive.*

Proof. ($\Rightarrow$) Let $a \in \partial A$ and assume that there is a vanishing outside angle at a. The dimension of ∂A is at most 1 so that the boundary at a is locally given by the union of the graphs of continuous function germs. After some rotation and translation, we may describe the part of the boundary at $a = (0,0)$ which causes the vanishing angle by two continuous definable function germs $f, g : [0,\delta) \to \mathbb{R}$ such that $f(0) = g(0)$. The half-tangents of f and g at 0 have the same slope, so

$$f(x) - g(x) \text{ is } o(x) \text{ as } x \searrow 0. \tag{3.1}$$

By choosing δ small enough, we may assume that $f(x) > g(x)$ for $0 < x < \delta$, that both functions f and g are continuously differentiable in $(0,\delta)$, and that

$$A \cap \{(x,y) : \ 0 < x < \delta, g(x) < y < f(x)\} = \emptyset.$$

Assume now that A is locally quasi-convex at 0 with constant $L > 0$. Take the points $\xi := (x, f(x))$ and $\eta := (x, g(x))$, where $0 < x < \delta/2$ is small enough. Then

$$\|\xi - \eta\| = f(x) - g(x).$$

Any path γ of minimal length in A connecting ξ and η passes through a. Hence

$$\text{length}(\gamma) \geq \|\xi - a\| + \|a - \eta\| \geq 2x.$$

Thus

$$2x \leq \text{length}(\gamma) \leq L\|\xi - \eta\| = L(f(x) - g(x)),$$

which contradicts (3.1).

($\Leftarrow$) Let $a \in \partial A$. Assume the outside angles at a are strictly positive. Fix a constant $L \geq 1$ so large that $L > \csc(\frac{\theta}{2})$ for each outside angle θ satisfying $\theta \leq \pi$. Then, for $\epsilon > 0$ sufficiently close to zero, $\xi, \eta \in B_\epsilon(a) \cap A$ can be joined by a rectifiable path γ in $B_\epsilon(a) \cap A$ with $\text{length}(\gamma) \leq L\|\xi - \eta\|$. It is easy to construct γ: If the whole line segment joining ξ and η belongs to A, take γ to be this line segment. Otherwise, take ξ' (resp., η') to be the point on the line segment joining ξ and η intersected with ∂A closest to ξ (resp., η) and take γ to be the line segment joining ξ and ξ' followed by the arc of ∂A joining ξ' and a followed by the arc of ∂A joining a and η' followed by the line segment joining η' and η. It remains to verify that $\text{length}(\gamma) \leq L\|\xi - \eta\|$, for $\epsilon > 0$ sufficiently small. It suffices to consider the case where some part of the line segment joining ξ and η is not in A. One may also reduce further to the case where $\xi' = \xi$ and $\eta' = \eta$. In this case the result follows by applying the following variant of the triangle inequality. □

Proposition 3.3. *If a, b and c are sides of a triangle then $\frac{a+b}{c} \leq \csc(\frac{\theta}{2})$ where θ denotes the angle opposite the side c.*

Remark: The standard triangle inequality asserts that $1 \leq \frac{a+b}{c}$.

Proof. We have

$$\frac{c^2}{(a+b)^2} = \frac{a^2+b^2-2ab\cos\theta}{(a+b)^2} = 1 - \frac{2ab}{(a+b)^2}(1+\cos\theta)$$
$$\geq 1 - \frac{1+\cos\theta}{2} = \frac{1-\cos\theta}{2} = \sin^2(\frac{\theta}{2}).$$

Here we are using the law of cosines $c^2 = a^2+b^2-2ab\cos\theta$, the half-angle formula $\frac{1-\cos\theta}{2} = \sin^2(\frac{\theta}{2})$, and the standard inequality $\sqrt{ab} \leq \frac{a+b}{2}$ relating the geometric mean and the arithmetic mean. □

The union of locally quasi-convex sets is not necessarily quasi-convex.

Example 3.4. Both the graph of the standard parabola and the x-axis in $\mathbb{R}^2$ are locally quasi-convex sets, but their union is not quasi-convex at the origin, as one of the outside angles at the origin vanishes.

Lemma 3.2 implies that finite intersections of locally quasi-convex definable subsets of $\mathbb{R}^2$ are again locally quasi-convex. This is false without the sets being definable in some o-minimal structure expanding $\mathbb{R}$.

Example 3.5. Let $A \subset \mathbb{R}^2$ be the union of the sets $[0,1]\times\{0\}$, $\{0\}\times[0,1]$ and the line segments connecting the points $(2^{-n},0)$ and $(0,2^{-n})$ for $n = 0,1,\ldots,\infty$. So A is compact and connected. Moreover, this set is locally quasi-convex. However, the intersection of A and the diagonal $\{x=y\}$ is the set $\{(2^{-n-1},2^{-n-1}); n \in \mathbb{N}\} \cup \{(0,0)\}$ which is not locally quasi-convex at $(0,0)$.

Contrary to locally convex sets, the intersection of locally quasi-convex definable subsets of $\mathbb{R}^n$, $n \geq 3$, is not necessarily locally quasi-convex again.

Example 3.6. Let A be the solution of $z^4 = x^2+y^2$, $z \geq 0$, then A is locally quasi-convex at every point, in particular at the origin. But the intersection of A with the zx-plane is the graph of $z = \sqrt{|x|}$ which is not locally quasi-convex at origin, because one outside angle vanishes.

For two continuous function $f,g : U \to \mathbb{R}$ with $f < g$ on U we set

$$(f,g)_U := \{(x,y); x \in U,\ f(x) < y < g(x)\}.$$

The quasi-convexity is a necessary condition to obtain extendibility.

Proposition 3.7. *Let $A \subset \mathbb{R}^2$ be a closed definable set which is not locally quasi-convex at some point $a \in A$. Then there exists a pwp function F on A that is not extensible to $\mathbb{R}^2$ as a pwp function.*

Proof. We may assume that $a = (0,0)$. The point a is obviously a non-isolated boundary point. Since A is not locally quasi-convex at a, there is a vanishing outside angle at a. Therefore, we may assume that after some rotation a part of ∂A is given by the graphs of two definable continuous functions $f,g : [0,\delta) \to \mathbb{R}$ such that

$$f(0) = g(0) = 0,$$
$$g(x) - f(x) \text{ is } o(x) \text{ for } x \searrow 0,$$
$$f < g \text{ on } (0,\delta),$$
$$(f,g)_{(0,\delta)} \cap A = \emptyset.$$

Define $F : A \to \mathbb{R}$ as follows. For $x \leq 0$ or $x \geq \delta$, or $0 < x < \delta$ and $y \leq f(x)$ let $F(x, y) = 0$. For $0 < x < \delta$ and $y \geq g(x)$, let $F(x, y) := \min(x, \delta - x)$. Then F is a *pwp* function on A. For $0 \leq x \leq \delta/2$,

$$|F(x, g(x)) - F(x, f(x))| = x$$

while

$$\|(x, g(x)) - (x, f(x))\| \text{ is } o(x) \text{ as } x \searrow 0.$$

Hence F is not locally Lipschitz continuous at a. Therefore Proposition 2.2 implies that F is not extensible to $\mathbb{R}^2$ as *pwp* function. □

3.3. **A counterexample.** It is easy to construct a *pwp* function on a locally quasi-convex closed set with infinitely many connected components that is not extensible. Such a set is never definable in any o-minimal structure and is also not compact. Take for example $A = \mathbb{N} \times \{0\}$ and $f : A \to \mathbb{R}$, $f(x, 0) = 1$ if x is odd, and $f(x, 0) = 0$ if x is even.

One can also define a *pwp* function on the compact set A of Example 3.5 which is not extensible. Let $f : A \to \mathbb{R}$ be defined as follows: $f(x, y) = 0$ if (x, y) belongs to $[0, 1] \times \{0\}$, $\{0\} \times [0, 1]$ or the line segments connecting the points $(2^{-n}, 0)$ and $(0, 2^{-n})$ for odd n, and let $f(x, y) = xy$ otherwise. However, the set A has not well-defined outside angles at the origin.

The question arises whether the restriction to o-minimal sets in Theorem 1.2 is necessary or not.

It seems that the o-minimality is almost necessary to obtain extendibility of *pwp* functions. Note that every convex set is locally quasi-convex with constant $L = 1$. Next we present an example of a *pwp* function defined on a compact convex set which is not extensible.

Example 3.8. The function $h : [0, 1] \to \mathbb{R}$,

$$h(x) = \begin{cases} x^6 \sin(x^{-1}) + x^6 + x^5 + x^4, & \text{if } x > 0, \\ 0, & \text{if } x = 0 \end{cases}$$

is strictly increasing and convex. Thus, the set

$$A = \{(x, y); 0 \leq x \leq 1, h(x) \leq y \leq h(1)\}$$

is a compact convex set, and even all outside angles are well defined. Define the *pwp* function f on A as follows: For $(x, y) \in A$ with

$$\frac{1}{4k\pi + 2\pi} \leq x \leq \frac{1}{4k\pi + \pi} \text{ for some integer } k \geq 0 \text{ and } y \leq x^6 + x^5 + x^4$$

let

$$f(x, y) = x^6 + x^5 + x^4 - y.$$

Otherwise, let $f(x, y) = 0$. Hence f is a *pwp* function on A.

Let us assume now that there is a *pwp* function $F : \mathbb{R}^2 \to \mathbb{R}$ with $F = f$ on A. Then F is semialgebraic. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) := F\left(x, x^5 + x^4\right).$$

Then g vanishes for

$$x = \left(4k\pi + \frac{7}{2}\pi\right)^{-1}, \quad k \in \mathbb{N},$$

and g is positive for

$$x = \left(4k\pi + \frac{3}{2}\pi\right)^{-1}, \quad k \in \mathbb{N}.$$

So the function g cannot be semialgebraic which contradicts the assumption.

Remark 3.9. The function f in the previous example is actually Lipschitz continuous; however, the domain is not definable in any o-minimal structure. We do not know, whether Lipschitz continuity of a *pwp* function on a compact definable set implies extendibility, and leave the answer to this question as open problem.

4. Proof of the Theorems

4.1. **Preliminary Lemma.** We agree to the following notation. For two continuous function $f, g : U \to \mathbb{R}$ with $f < g$ on U we set

$$[f, g]_U := \{(x, y); x \in U,\ f(x) \le y \le g(x)\}$$
$$(f)_U := \{(x, y); x \in U,\ y = f(x)\}$$

We prepare for the proof of Theorem 1.2 by proving the following technical lemma.

Lemma 4.1. *Let $a < b$ and let $p \in \mathbb{R}[x, y]$ be a polynomial such that*

$$p(a, 0) = 0 \text{ and } p(b, 0) = 0$$

Then for every semi-linear open neighbourhood V of $(a, b) \times \{0\}$ there is a pwp-function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

(1) *f vanishes outside of V*
(2) *$f = p$ on $[a, b] \times \{0\}$*

Proof. Since $T := [a, b] \times \{0\}$ is a line segment, there is a polynomial $q \in \mathbb{R}[x]$ which equals p on T. The set V is semi-linear. Hence there is an $\varepsilon > 0$ such that

$$V' := \{(x, y); x \in (a, b),\ |y| < \varepsilon \min(x - a, b - x)\}$$

is contained in V. We define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) := \begin{cases} q(x)\left(1 - \dfrac{|y|}{\varepsilon \min(x - a, b - x)}\right) & \text{if } (x, y) \in V', \\ 0 & \text{otherwise.} \end{cases}$$

Step 1: The function f is continuous.
The factor

$$1 - \frac{|y|}{\varepsilon \min(x - a, b - x)}$$

is bounded on V' and vanishes if $|y| = \varepsilon \min((x - a), (b - x))$ and $x \in (a, b)$. The polynomial q vanishes at a and b. Hence f is continuous.

Step 2: f is a *pwp* function.
Since q vanishes at a and b, the polynomial $(x - a)(b - x)$ divides q. The absolute value function is *pwp*, so that f is *pwp*. □

Let A be a compact definable locally quasi-convex set. Let S be a one-dimensional stratum of a stratification of ∂A into Lipschitz cells, which is compatible with $\overline{A^o}$. As explained in Section 3.1, we can assume, after making a suitable rotation, that $S := (h)_I$, $h : I \to \mathbb{R}$, I an open interval. Then S can be a one-sided or a two-sided boundary piece. Hence, there is a definable continuous function $\varepsilon : I \to (0, \infty)$ and, because of local quasi-convexity, a continuous semilinear function $\delta : I \to (0, \infty)$

such that $\delta(x) \to 0$ as $x \searrow a$ and as $x \nearrow b$, where $I = (a, b)$, such that one of the following cases holds:

(i) $(h, h+\varepsilon)_I \subset A$ and $(h-\delta, h)_I \cap A = \emptyset$.

(ii) $(h-\varepsilon, h)_I \subset A$ and $(h, h+\delta)_I \cap A = \emptyset$.

(iii) $(h-\delta, h+\delta)_I \cap A = S$.

Since δ is a continuous strictly positive semilinear function with bounded domain, there are continuous semilinear functions $h^{\pm} : I \to \mathbb{R}$ such that

$$h - \frac{\delta}{3} < h^- < h < h^+ < h + \frac{\delta}{3}$$

on I. This implies that

(i') $A \cup [h^-, h]_I$

(ii') $A \cup [h, h^+]_I$

(iii') $A \cup [h^-, h^+]_I$

is again a compact definable locally quasi-convex set, respectively, and $(h^+)_I$ and $(h^-)_I$ are semilinear sets.

4.2. **The compact case.** We complete the proof of Theorem 1.2 by proving the following proposition.

Proposition 4.2. *Let $A \subset \mathbb{R}^2$ be a locally quasi-convex compact definable set, and let $f : A \to \mathbb{R}$ be a pwp function. Then there is a pwp function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F = f$ on A.*

Proof. Let $S_1, \dots, S_r$ be a stratification of ∂A into Lipschitz cells such that $f|_{S_i}$ is the restriction of some polynomial p_i to S_i for $i = 1, \dots, r$. We may assume that $\dim(S_i) = 1$ if and only if $i = 1, \dots, s$. We inductively extend f to a *pwp* function on a compact semilinear neighbourhood B of A as follows. Select for every $i = 1, \dots, s$ a definable neighbourhood U_i of S_i according to the cases (i'), (ii') and (iii'). Then, we can extend f to the set $B := A \cup \overline{U_1} \cup \cdots \cup \overline{U_r}$ by setting $f = p_i$ on $\overline{U_i}$. Take a stratification $T_1, \dots, T_t$ of ∂B into Lipschitz cells such that $f|_{T_i}$ is the restriction of some polynomial to T_i for each i. The sets T_i are semilinear. Let C be the collection of the T_i with $\dim(T_i) = 0$. Then C is a finite set. Let p be a polynomial which equals f on C. Then $g = f - p$ is a *pwp* function on B and g restricted to each 1-dimensional T_i is the restriction of some polynomial q_i which vanishes in $\overline{T_i} \setminus T_i$. By applying Lemma 4.1 to the function q_i and $\overline{T_i}$ in place of p and $[a, b] \times \{0\}$ we obtain a semilinear closed neighbourhood D of B and a *pwp* function $G : D \to \mathbb{R}$ which coincides with g on B and which vanishes in ∂D. Thus G extends to a *pwp* function on $\mathbb{R}^2$ by setting $G = 0$ outside of D. Take $F = G + p$. $\square$

4.3. **The non-compact case.** If the Hardy field $\mathcal{H}(\mathbb{R})$ is not dense in $\mathcal{H}(\mathcal{M})$, then we cannot expect extendibility of *pwp* functions defined on all locally quasi-convex definable sets. Indeed, the extendibility would imply that disjoint definable arcs near infinity can be separated by a continuous semialgebraic function and therefore by an analytic semialgebraic function, which implies the density of $\mathcal{H}(\mathbb{R})$ in $\mathcal{H}(\mathcal{M})$ by [5].

We now assume that the Hardy field of semialgebraic germs at $+\infty$ lies dense in the Hardy field of definable germs at $+\infty$. We establish some preliminary lemmas.

Lemma 4.3. *Let $\Gamma \subset \mathbb{R}^2$ be an arc of a semialgebraic set of dimension 1 at infinity, and let p be a polynomial. Then, for every open semialgebraic neighbourhood U of Γ, there exists an $M > 0$ and a pwp function f such that*

$$f(x,y) = p(x,y) \text{ on } \Gamma \setminus B_M(0),$$

and f vanishes outside of U.

Proof. After some rotation we may assume that Γ is the graph of a continuous semialgebraic function $h : [a,\infty) \to \mathbb{R}$. We may further assume that $p(x,h(x)) > 0$ for sufficiently large x, as the case $p(x,h(x)) = 0$ is trivial. Select a positive semialgebraic continuous function δ such that for sufficiently large b,

$$[h-\delta, h+\delta]_{[b,\infty)} \subset U,$$

and $p(x,y) > 0$ for all $(x,y) \in [h-\delta, h+\delta]_{[b,\infty)}$. The graph of $h+\delta$ is a semialgebraic set of dimension 1. By Bezout's theorem there exists an irreducible polynomial $q \in \mathbb{R}[x,y]$ unique up to multiplication by a non-zero real number such that

$$q(x, h(x) + \delta(x)) = 0$$

for sufficiently large x. After modifying δ suitably (e.g., replacing δ by δ/n for suitable $n \geq 1$), we can assume that $q(x,h(x)) \neq 0$ for sufficiently large x. Scaling q by a suitable non-zero real, we can assume that

$$q(x,h(x)) > 0$$

for sufficiently large x. For $n \in \mathbb{N}$ let q_n be the polynomial

$$q_n(x,y) := \left(x^2+y^2\right)^n q(x,y)$$

Then there is an N such that

$$q_N(x,h(x)) > p(x,h(x)) \tag{4.1}$$

for sufficiently large x, say $x \geq M/2$. Let $\rho_M : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$\rho_M(x,y) := \inf\left(1, \sup\left(0, x^2+y^2-M^2\right)\right). \tag{4.2}$$

Then ρ_M is a *pwp* function which vanishes in $B_M(0)$ and which equals 1 outside of $B_{M+1}(0)$. On the set $[h, h+\delta]_{[M/2,\infty)}$ we define the *pwp* function f as

$$f := \rho_{M/2} \inf(q_N, p)$$

The function f vanishes on the graph of $h+\delta$, and f coincides with p on $\Gamma \setminus B_M(0)$ because of inequality (4.1). Similarly we extend p to the set $[h-\delta, h)_{[M/2,\infty)}$. Hence there is a *pwp* function $f : \mathbb{R}^2 \to \mathbb{R}$ which coincides with p on the arc $\Gamma \setminus B_M(0)$. □

The dimension of the boundary of a semialgebraic subset of $\mathbb{R}^2$ is at most 1. So we obtain a stronger version of the previous lemma.

Lemma 4.4. *Let $A \subset \mathbb{R}^2$ be a closed semialgebraic set and let $f : A \to \mathbb{R}$ be a pwp function. Then there is an $M > 0$ and a pwp function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F = f$ on $A \setminus B_M(0)$.*

Proof. We choose M so big that $\partial A \setminus B_M(0)$ consists of a disjoint union of closed (semialgebraic) arcs at infinity. For each of these arcs we find open pairwise disjoint semialgebraic neighbourhoods, so that Lemma 4.3 implies the existence of a *pwp* function $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}$ such that $\tilde{F} = f$ on $\partial A \setminus B_{M/2}(0)$, after some further increasing of M. Define $g : \mathbb{R}^2 \to \mathbb{R}$ as $g := f$ on A and $g := \tilde{F}$ outside of A, and let $F := \rho_{M/2} g$ where ρ_M is the function defined by (4.2) □

We also need a version of Lemma 4.4 for definable sets A.

Lemma 4.5. *Let $\mathcal{H}(\mathbb{R}) \subset \mathcal{H}(\mathcal{M})$ be dense. Let $A \subset \mathbb{R}^2$ be closed definable set and let $f : A \to \mathbb{R}$ be a pwp function. Then there is an $M > 0$ and a pwp function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $g = f$ on $A \setminus B_M(0)$.*

Proof. Note that a set is semialgebraic if its boundary is semialgebraic. The dimension of the boundary of a definable subset of $\mathbb{R}^2$ is at most 1. Let $A_1, \ldots, A_r$ denote the one-dimensional Lipschitz cells of $\partial A \setminus B_M(0)$ for some $M > 0$. Choose M so large that each A_i is unbounded, and such that $f = p_i$ on A_i for some polynomial p_i. We construct pairwise disjoint closed semialgebraic neighbourhoods V_i of A_i. In some linear orthogonal coordinate system, we may write

$$A_i = (h_i)_{(a_i,\infty)}$$

for some definable Lipschitz continuous function $h_i : (a_i, \infty) \to \mathbb{R}$. Let $\varepsilon_i : (a_i, \infty) \to \mathbb{R}$ be defined by

$$\varepsilon_i(x) = \frac{1}{3}\mathrm{dist}\left(h_i(x), \bigcup_{j\neq i} A_j\right)$$

Since $\mathcal{H}(\mathbb{R}) \subset \mathcal{H}(\mathcal{M})$ is dense there are continuous semialgebraic functions $\varphi_i, \psi_i : (c_i, \infty) \to \mathbb{R}$, $c_i > a_i$ with

$$h_i - \varepsilon_i < \varphi_i < h_i < \psi_i < h_i + \varepsilon_i$$

on (c_i, ∞). So the closed semialgebraic sets

$$V_i := \overline{(\varphi_i, \psi_i)_{(c_i,\infty)}}$$

are disjoint (in the original coordinate system). Choose M so big that

$$\partial A \setminus B_M(0) \subset \bigcup_i V_i,$$

then the set

$$B := \bigcup_i V_i \setminus B_M(0)$$

is a closed semialgebraic neighbourhood of $\partial A \setminus B_M(0)$. Define $g : B \to \mathbb{R}$ as

$$g(x) := \begin{cases} f(x), & \text{if } x \in A \\ p_i(x), & \text{if } x \in V_i \setminus A. \end{cases}$$

This function is a semialgebraic *pwp* function such that $g = f$ on $A \setminus B_M(0)$. The result follows now from Lemma 4.4. □

Theorem 1.3 is now implied by the following proposition.

Proposition 4.6. *Let $\mathcal{H}(\mathbb{R}) \subset \mathcal{H}(\mathcal{M})$ be dense. Let $A \subset \mathbb{R}^2$ be a closed definable locally quasi-convex set. Then every pwp function $f : A \to \mathbb{R}$ extends to a pwp function $F : \mathbb{R}^2 \to \mathbb{R}$.*

Proof. Let M be so big, that Lemma 4.5 provides us with a *pwp* function $g : \mathbb{R}^2 \to \mathbb{R}$ with $g = f$ on $A \setminus B_M(0)$. The set $A \cap B_{2M}(0)$ is locally quasi-convex. Hence, by Proposition 4.2, there is a *pwp* function $G : \mathbb{R}^2 \to \mathbb{R}$ such that $G = f$ on $A \cap B_{2M}(0)$. Take a *pwp* function $\rho : \mathbb{R}^2 \to [0, 1]$ that vanishes outside of $B_{2M}(0)$ and equals 1 in $B_M(0)$. Set

$$F := \rho G + (1 - \rho)g.$$

□

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