On families of testing formulae for a pp formula

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Abstract

Astier and Tressl have recently proven that a pp formula fails on a finite subspace of a space of orderings if and only if a certain family of formulae is verified (V. Astier, M. Tressl, *Axiomatization of local-global principles for pp formulas in spaces of orderings,* Arch. Math. Logic 44, No. 1 (2005), 77-95). The proof given in their paper is non-constructive and uses rather advanced techniques from model theory. In this note we slightly strengthen their result by constructing another family of formulae with the same property, whose elements are given explicitly. We also illustrate the developed theory with an example of the testing family for a pp formula that is known to be a counterexample to the pp conjecture.

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Throughout this paper (X, G) denotes a space of orderings in the sense of [5, pp. 21-22]: X is a Boolean topological space, G is some group of continuous functions from X to $\{1, -1\}$, which contains the constant function -1, separates points of X, and also satisfies some additional axioms. The subbasic sets in the topology of X are of the form

$$U(a) = \{x \in X : a(x) = 1\},$$

for a given $a \in G$.

As a subspace of (X, G) we understand a pair $(Y, G|_Y)$, where $Y \neq \emptyset$ is some intersection of subbasic sets, and $G|_Y$ is the group of all restrictions $a|_Y$, $a \in G$ [5, pp. 32-33]. A subspace of a space of orderings is a space of orderings itself [5, Theorem 2.4.3]. While considering subspaces, we will usually use the same notation for elements $a \in G$ and their restrictions $a|_Y$. If (Y, H) is a subspace of (X, G) and $a, b \in H$, we define the value set

$$D_Y(a, b) = \{c \in H : \forall x \in Y[c(x) = a(x) \text{ or } c(x) = b(x)]\}.$$

In the case when $Y = X$ or when it is clear in which subspace we work, we shall write $D(a, b)$ instead of $D_Y(a, b)$.

The theory of spaces of orderings is, in a sense, equivalent to the theory of reduced special groups in the language L_{SG} of special groups (see [2] for a full list of axioms). The language L_{SG} consists of a ternary relation symbol $a \in D(b, c)$, corresponding to the relation of being an element of the above mentioned value set, a binary functional symbol $\cdot$ called multiplication, and two constants -1 and 1. The set of terms T is defined by induction as the smallest set containing individual variables and constants, which is closed under the functional symbol. For terms $a, b, c \in T$ we define atomic formulae to be of the form either $a = b$ or $a \in D(b, c)$. Since the language of special groups differs from the language of groups, we shall denote special groups by $(G, D, -1)$. An SG-morphism is a group homomorphism f between two reduced special groups $(G, D, -1)$ and $(H, D, -1)$ such that $f(-1) = -1$ and

$$\forall a, b, c \in G\{[a \in D(b, c)] \Rightarrow [f(a) \in D(f(b), f(c))]\}.$$

For a reduced special group $(G, D, -1)$ denote by X_G the set of all SG-morphisms of G into the two-element reduced special group $\mathbb{Z}_2 = \{-1, 1\}$. (X_G, G) is a space of orderings ([2, Propositon 3.10]). Moreover, for a space of orderings (X,G), $(G,D_X,-1)$ is a reduced special group ([2, Proposition 3.11]), and the two correspondences:

$$(G,D,-1) \mapsto (X_G,G) \text{ and } (X,G) \mapsto (G,D_X,-1)$$

are reciprocal to each other ([2, Proposition 3.14]).

We also note that spaces of orderings are essentially the same thing as real reduced multifields (see [8]), and we could use the language of rings with multivalued addition instead of the language of special groups.

Positive primitive (pp for short) formulae in the language L_{SG} are of considerable interest to the people working with the theory of spaces of orderings. Recall that a pp formula is a formula of the form of a finite conjunction of atomic formulae preceded by finitely many existential quantifiers. Numerous important properties of quadratic forms over spaces of orderings are expressible as pp formulae – examples include the property of two forms being isometric, the property of an element being represented by a form etc. A type of a very general local-global principle formulated by Marshall (see [6]), known as the pp conjecture, asks whether a pp formula holds in a space of orderings provided it holds in every finite subspace. Although the conjecture is known to be false in general (see [3], [4]), it is still important to have some methods of determining if a given pp formula holds true on every finite subspace of a given space of orderings, or not.

As usual, $\mathbb{N}$ will denote the set of positive integers. We fix a pp formula

$$P(\underline{y}) = \exists \underline{t} \bigwedge_{j=1}^{m} \theta_j(\underline{t},\underline{y})$$

where θ_j are atomic formulae and $\underline{y} = (y_1,\ldots,y_k)$, $\underline{t} = (t_1,\ldots,t_n)$ are tuples of individual variables in the language L_{SG}. Define the family of spaces of orderings and constants:

$$\begin{aligned}\mathbb{K}_P = \{(Y,H,\underline{b}) :& (Y,H) \text{ is a finite space of orderings,}\\ & \underline{b} \in H^k, P(\underline{b}) \text{ fails in } (Y,H),\\ & P(\underline{b}) \text{ holds in every proper subspace of } (Y,H)\}\end{aligned}$$

and the corresponding family of formulae in the language L_{SG}:

$$\mathcal{F}_P = \{Q(\underline{y}) = \exists \underline{s} \bigwedge_{j=1}^{m'} \theta'_j(\underline{s}, \underline{y}) : \theta'_j \text{ are atomic formulae in the language } L_{SG}$$
$$\underline{s} = (s_1, \dots, s_{n'}) \text{ is a tuple of individual variables, } n' \in \mathbb{N},$$
$$\forall (Y, H, \underline{b}) \in \mathbb{K}_P[Q(\underline{b}) \text{ fails in } (Y, H)]\}.$$

In other words, the family $\mathcal{F}_P$ consists of all pp formulae $Q(\underline{y})$ in the language L_{SG} such that $Q(\underline{b})$ fails in (Y, H) for all $(Y, H, \underline{b}) \in \mathbb{K}_P$.

Theorem 1. *Let (X, G) be a space of orderings, let $\underline{a} \in G^k$. The following two conditions are equivalent:*

1. *$P(\underline{a})$ fails in some finite subspace of (X, G);*
2. *for every $Q(\underline{y}) \in \mathcal{F}_P$ the formula $Q(\underline{a})$ fails in (X, G).*

This is almost exactly the same result as the one proven by Astier and Tressl in [1], although their definition of the family $\mathbb{K}_P$ slightly differs from ours. The implication 1. $\Rightarrow$ 2. is easy, whilst the implication 2. $\Rightarrow$ 1. is more complicated, and we shall prove it in two different ways. The first proof is model-theoretic in nature and similar to the proof in [1], the second one will appear as a part of the proof of Theorem 2. Both proofs make use of the following lemma, utilized also in Astier and Tressl's original proof:

Lemma 1. *Let $B(n, 0) = 1$ for $n \in \mathbb{N}$, and let*

$$B(n, k) = 2^k 2^{2^{nk B(n,k-1)}}, \text{ if } k \geq 1, n \in \mathbb{N}.$$

Then, for every space of orderings (X, G), for every $\underline{a} \in G^k$, and for every pp formula $P(\underline{y})$ with n quantifiers and k parameters, if $P(\underline{a})$ fails to hold in $(Z, G|_Z)$, where Z is a finite subspace of (X, G) (or, more generally, is a subspace Z such that $(Z, G|_Z)$ has a finite chain length), then there is a subspace Y of (X, G) such that $P(\underline{a})$ fails to hold in $(Y, G|_Y)$ and $|Y| \leq B(n, k)$.

Proof. See [1, Lemma 4]. □

We proceed to the proof of Theorem 1.

Proof. 1. $\Rightarrow$ 2. By Zorn's Lemma there is a finite subspace (Y, H) of (X, G) such that $P(\underline{a}|_Y)$ fails in (Y, H) and holds on every proper subspace of (Y, H). Then $(Y, H, \underline{a}|_Y) \in \mathbb{K}_P$. Fix a $Q(\underline{y}) \in \mathcal{F}_P$ – we thus have that $Q(\underline{a}|_Y)$ fails in (Y, H). It follows that $Q(\underline{a})$ fails in (X, G), for if for some $\underline{s} \in G^{n'}$ all $\theta'_j(\underline{s}, \underline{a})$ would hold in (X, G), $j \in \{1, \dots, m'\}$, then all $\theta'_j(\underline{s}|_Y, \underline{a}|_Y)$ would hold in (Y, H), $j \in \{1, \dots, m'\}$, that is $Q(\underline{a}|_Y)$ would hold in (Y, H).

2. $\Rightarrow$ 1. Consider the theory of $\mathbb{K}_P$ in the language L_{SG} extended by the constants $\underline{y}$:

$$\mathcal{T}(\mathbb{K}_P) = \{A(\underline{y}) : A(\underline{y}) \text{ is a formula in the free variables } \underline{y}, \\ \forall (Y, H, \underline{b}) \in \mathbb{K}_P [A(\underline{b}) \text{ holds in } (Y, H)]\}.$$

We shall construct an SG-morphism π from the reduced special group G to a reduced special group H such that, for every $A(\underline{y}) \in \mathcal{T}(\mathbb{K}_P)$, $A(\underline{b})$, holds in (Y, H), (Y, H) denoting the space of orderings induced by H, and $\underline{b} = \pi(\underline{a})$. Suppose this is already done. Note that $(Y, H, \underline{b})$ is an element of $\mathbb{K}_P$ (by Lemma 1, there is a uniform bound B on $|H|$ for $(Y, H, \underline{b}) \in \mathbb{K}_P$, and the conditions "$H$ has at most B elements", "$P(\underline{b})$ fails in (Y, H)" and "$P(\underline{b})$ holds in proper subspaces of (Y, H)" are expressible as formulae in $\mathcal{T}(\mathbb{K}_P)$). Now the set $\{y \circ \pi : y \in Y\}$ is a generating set for a finite subspace of (X, G) and, since $P(\underline{b})$ fails in (Y, H), $P(\underline{a})$ fails in this subspace, which finishes the proof. Since a first order structure provides a map (the interpretation) from the set of constants of the language into its underlying set, if we work in the language $\mathcal{L}$ of special groups extended by new constants for the elements of G it suffices to construct a model of the following $\mathcal{L}$-theory:

$$\{A(\underline{a}) : A(\underline{y}) \in \mathcal{T}(\mathbb{K}_P)\} \cup \\ \cup \{g_1 = g_2 \cdot g_3 : g_1, g_2, g_3 \in G, g_1 = g_2 \cdot g_3 \text{ holds in } (X, G)\} \cup \\ \cup \{g_1 \in D(g_2, g_3) : g_1, g_2, g_3 \in G, g_1 \in D(g_2, g_3) \text{ holds in } (X, G)\}.$$

(note that $\{A(\underline{a}) : A(\underline{y}) \in \mathcal{T}(\mathbb{K}_P)\}$ includes axioms of the theory reduced special groups). We will denote a model H of $\mathcal{T}(\mathbb{K}_P)$ by $(Y, H, \underline{b})$ to indicate the associated space of orderings (Y, H) and distinguished parameters $\underline{b}$.

In order to construct the above mentioned model, take any finite collection $S_1(\underline{g}_1), \dots, S_s(\underline{g}_s)$ of atomic formulae of the form $g_1 = g_2 \cdot g_3$ or $g_1 \in D(g_2, g_3)$ that hold true in (X, G). Some of the entries of the $\underline{g}_i$ may coincide with each other, or may be ± 1, or may coincide with entries of $\underline{a}$. Relabelling suitably, we can write each $S_i(\underline{g}_i)$ as $S_i(\underline{g}, \underline{a})$, where $\underline{g} = (g_1, \dots, g_t)$, $g_1, \dots, g_t \in G$ are distinct from each other, and from ± 1, and from the entries of $\underline{a}$. We also have the problem that some of the entries of $\underline{a}$ may be equal to each other or to ± 1. For each k, l such that $a_k = a_l$ we add the atomic formula "$a_k = a_l$" to our collection. Similarly, add "$a_k = 1$" (resp., "$a_k = -1$") if $a_k = 1$ (resp., $a_k = -1$). Define $Q(\underline{y})$ to be the pp formula $\exists \underline{u} \bigwedge_{i=1}^{s} S_i(\underline{u}, \underline{y})$. The formula $\bigwedge_{i=1}^{s} S_i(\underline{g}, \underline{a})$ holds in (X, G), which clearly implies that $Q(\underline{a})$ holds in (X, G). If $Q(\underline{b})$ fails for each $(Y, H, \underline{b})$ in the class $\mathbb{K}_P$, then $Q(\underline{y})$ belongs to the class $\mathcal{F}_P$, so $Q(\underline{a})$ fails in (X, G), by our assumptions. This is a contradiction. Thus $Q(\underline{b})$ holds for some $(Y, H, \underline{b}) \in \mathbb{K}_P$ with some $\underline{h} = (h_1, \ldots, h_t)$ verifying it. Fix such a $(Y, H, \underline{b})$. View (Y, H) as an $\mathcal{L}$-structure by interpreting $g \in G$ to be the respective entry of $\underline{h}$, if g is an entry of $\underline{g}$, or to be b_k if $g = a_k$, or to be ± 1 if $g = \pm 1$, or to be some arbitrary element of H otherwise. Obviously, $(Y, H, \underline{b})$ serves as a model of $\{A(\underline{a}) : A(\underline{y}) \in \mathcal{T}(\mathbb{K}_P)\}$ and, in particular, of every finite subset of thereof. By the compactness theorem, we have constructed the desired model. $\square$

We continue to work with the formula $P(\underline{y})$ defined before. Let x be a variable in the language L_{SG} not appearing in $P(\underline{y})$. We define a new formula $P'(\underline{y}, x)$ to be the formula obtained from $P(\underline{y})$ by replacing each atomic formula $w_1 \in D(w_2, w_3)$ in $P(\underline{y})$ with

$$\exists s_1 \exists s_2 [(s_1 \in D(1, x)) \wedge (s_2 \in D(1, x)) \wedge (w_1 \in D(s_1 w_2, s_2 w_3))]$$

(note that, since we are working in the reduced theory, any atomic formula of the form $w_1 = w_2$ can be replaced by $w_1 w_2 \in D(1, 1)$).

One sees that for each space of orderings (X, G) and for each subspace of the form $U(b)$, $b \in G$, if $\underline{a} \in G^k$ then $P'(\underline{a}, b)$ holds in (X, G) if and only if $P(\underline{a})$ holds in the subspace $U(b)$. Let $\lambda \geq 1$ be an integer. We shall construct a sequence of formulae $P_\lambda^{(i)}(\underline{y})$, $i \geq 0$, by induction. For $i = 0$ take $P_\lambda^{(0)}(\underline{y}) = P(\underline{y})$. For $i = 1$, let

$$P_\lambda^{(1)}(\underline{y}) = \exists z_0 \ldots \exists z_\lambda \bigwedge_{j=1}^{\lambda} [(z_{j-1} \in D(1, z_j)) \wedge P'(\underline{y}, z_{j-1} z_j)],$$

and for $i \geq 2$ we define $P_\lambda^{(i)}(\underline{y})$ by performing the above action on $P_\lambda^{(i-1)}(\underline{y})$ instead of $P(\underline{y})$ (note that this construction depends on λ):

$$P_\lambda^{(i)}(\underline{y}) = \exists u_0 \ldots \exists u_\lambda \bigwedge_{j=1}^{\lambda} [(u_{j-1} \in D(1, u_j)) \wedge (P_\lambda^{(i-1)})'(\underline{y}, u_{j-1} u_j)].$$

Trivially, for every space of orderings (X,G) and every $\underline{a} \in G^k$, $P(\underline{a}) \Rightarrow P_\lambda^{(1)}(\underline{a})$ (by taking $z_0 = z_1 = \ldots = z_\lambda = 1$) and, consequently, $P(\underline{a}) \Rightarrow P_\lambda^{(1)}(\underline{a}) \Rightarrow \ldots \Rightarrow P_\lambda^{(i)}(\underline{a})$, for every $i \geq 2$.

Define the number

$$c_P = \max\{\mathrm{cl}(X,G) : (X,G,\underline{a}) \in \mathbb{K}_P\},$$

where $\mathrm{cl}(X,G)$ denotes the chain length of the space (X,G). By Lemma 1, this number is well defined. Moreover, c_P is uniformly bounded from above by $B(n,k)$, although we do not claim that this bound is best possible.

We shall prove a certain extension of Theorem 1:

Theorem 2. *Let $\lambda > c_P$, let (X,G) be a space of orderings, let $\underline{a} \in G^k$. The following three conditions are equivalent:*

1. *$P(\underline{a})$ fails in some finite subspace of (X,G);*
2. *for every $Q(\underline{y}) \in \mathcal{F}_P$ the formula $Q(\underline{a})$ fails in (X,G);*
3. *for every $i \geq 0$ the formula $P_\lambda^{(i)}(\underline{a})$ fails in (X,G).*

The implication 1. ⇒ 2. is the "easy part" of Theorem 1. In the proof of Theorem 2 given below we actually show 2. ⇒ 3. ⇒ 1., and thus provide a second, rather different proof of 2. ⇒ 1. in Theorem 1.

Proof. 2. ⇒ 3. It suffices to show that for every $i \geq 0$ $P_\lambda^{(i)}(\underline{y}) \in \mathcal{F}_P$. Obviously $P(\underline{y}) \in \mathcal{F}_P$, so $P_\lambda^{(0)}(\underline{y}) \in \mathcal{F}_P$. Let $(Y,H,\underline{b}) \in \mathbb{K}_P$. Then $\mathrm{cl}(Y,H) \leq c_P < \lambda$, and hence, for every $c_0, \ldots, c_\lambda \in H$ satisfying $c_{j-1} \in D(1,c_j)$, $j \in \{1,\ldots,\lambda\}$, there exists $j_0 \in \{1,\ldots,\lambda\}$ such that $c_{j_0-1}c_{j_0} = 1$. This forces $P'(\underline{b}, c_{j_0-1}c_{j_0})$ to be logically equivalent to $P(\underline{b})$, which implies that the formula $P_\lambda^{(1)}(\underline{b})$ fails in (Y,H) and, consequently, $P_\lambda^{(1)}(\underline{y}) \in \mathcal{F}_P$. From the construction of $P_\lambda^{(i)}(\underline{y})$, the argument follows for $i \geq 2$ by repeating the same reasoning.

3. ⇒ 1. Using Zorn's Lemma, choose a subspace (Y,H) of (X,G) minimal subject to the condition that for every $i \geq 0$ the formula $P_\lambda^{(i)}(\underline{a})$ fails in (Y,H). We shall show that $\mathrm{cl}(Y,H) < \lambda$. Suppose that, for some $c_0, \ldots, c_\lambda \in G$, $c_{j-1} \in D(1,c_j)$ in (Y,H), and $c_{j-1}c_j \neq 1$ in (Y,H) for $j \in \{1,\ldots,\lambda\}$. Define $Z_j = U(c_{j-1}c_j) \cap Y$; clearly $(Z_j, H|_{Z_j})$ are proper subspaces of (Y,H), and thus for every $j \in \{1,\ldots,\lambda\}$ there is some $i \geq 0$ such that $P_\lambda^{(i)}(\underline{a})$ holds

in $(Z_j, H|_{Z_j})$. Since $P_\lambda^{(i)}(\underline{a}) \Rightarrow P_\lambda^{(i+1)}(\underline{a})$ we may assume that for $i \geq 0$ big enough $P_\lambda^{(i)}(\underline{a})$ holds in $(Z_j, H|_{Z_j})$ for $j \in \{1, \ldots, \lambda\}$. From the construction of $P_\lambda^{(i+1)}(\underline{y})$ it follows that $P_\lambda^{(i+1)}(\underline{a})$ holds in (Y, H) – a contradiction. The result now follows, by Lemma 1. □

We will discuss the formulae $P_\lambda^{(i)}(\underline{y})$ in some more detail in the case when $P(\underline{y})$ is one of the examples of pp formulae for which the pp conjecture fails. Note that if the conjecture is true for a formula $P(\underline{y})$, then, for a given space of orderings (X, G) and $\underline{a} \in G^k$, testing the formulae $P_\lambda^{(i)}(\underline{a})$ becomes trivial: since $P_\lambda^{(0)}(\underline{a}) = P(\underline{a})$, already the formula $P_\lambda^{(0)}(\underline{a})$ fails in (X, G).

Example 1. Consider

$$P(y_1, y_2, y_3) = \exists t_1 \exists t_2 [t_1 \in D(1, y_1) \wedge t_2 \in D(1, y_2) \wedge y_3 t_1 t_2 \in D(1, y_1 y_2)]$$

as an example of a pp formula for which the pp conjecture fails (see [3], [4]). We shall show that for this particular formula $\mathbb{K}_P$ consists of just two (up to isomorphism) elements, namely a singleton space and a space containing six elements described below. Consequently, the upper bound for chain length c_P is equal to 3. This also shows that, since $B(2, 3) = 2^{3+256\cdot 2^{512}}$, the above mentioned estimate $c_P < B(n, k)$ is largely overblown.

Let (Y, H) be a finite space of orderings, and let $\underline{b} \in H^3$ be such that $P(\underline{b})$ fails in (Y, H), and holds in every proper subspace of (Y, H). Readily, $P(\underline{b})$ is logically equivalent to the following formula:

$$b_3 \in D(1, b_1) D(1, b_2) D(1, b_1 b_2).$$

By the structure theorem for finite spaces of orderings [5, Theorem 4.2.2], and by the minimality of (Y, H), (Y, H) is either a singleton space, or a proper group extension of some space $(\overline{Y}, \overline{H})$. The singleton case is easy: one sees that if $Y = \{x\}$, then $b_1(x) = b_2(x) = 1$, $b_3(x) = -1$, i.e., $b_1 = b_2 = 1$ and $b_3 = -1$, and $\mathrm{cl}(Y, H) = 1$.

In the group extension case, since H is finite we may, by induction, assume that $H = \overline{H} \times \{1, c\}$. If one of $b_1, b_2, b_1 b_2$ is equal to -1, then, by the well-known description of value sets in group extensions [5, pp. 62-64], readily $D(1, b_1) D(1, b_2) D(1, b_1 b_2) = H$, so the formula $b_3 \in D(1, b_1) D(1, b_2) D(1, b_1 b_2)$ is trivially satisfied. Assume that none of $b_1, b_2, b_1 b_2$ is equal to -1.

We claim that $b_1 \in \overline{H}$. Indeed, suppose that $b_1 \in H \setminus \overline{H}$. Interchanging b_2 and $b_1 b_2$, if necessary, we may assume that $b_2 \in \overline{H}$. Similarly, interchanging

b_3 and b_1b_3, if required, we may assume that that $b_3 \in \overline{H}$. Since, clearly, $b_3 \notin D(1, b_2)$, there is $\overline{x} \in \overline{Y}$ such that $b_3(\overline{x}) = -1$ and $b_2(\overline{x}) = 1$. Now $\overline{x}$ gives rise to two elements $x_1, x_2 \in Y$ such that, since $b_1 \in H \setminus \overline{H}$, $b_1(x_1) = 1$ and $b_1(x_2) = -1$. Thus $b_1(x_1) = b_2(x_1) = 1$ and $b_3(x_1) = -1$, meaning that $P(\underline{b})$ fails already in $\{x_1\}$ – a contradiction.

We see that, by symmetry, also $b_2 \in \overline{H}$. Observe that, in turn, $b_3 \notin \overline{H}$ – for if $b_3 \in \overline{H}$ then, due to the minimality of (Y, H), and due to the fact that $(\overline{Y}, \overline{H})$ may be considered as a proper subspace of (Y, H):

$$b_3 \in D_{\overline{H}}(1, b_1)D_{\overline{H}}(1, b_2)D_{\overline{H}}(1, b_1b_2) = D(1, b_1)D(1, b_2)D(1, b_1b_2).$$

Since $b_1, b_2, b_1b_2 \neq -1$, there are elements of $\overline{Y}$ making b_1, b_2 and b_1b_2 positive. At the same time, if, for some $\overline{x} \in \overline{Y}$, $b_1(\overline{x}) = b_2(\overline{x}) = 1$ then, arguing as before, we would be able to construct $x \in Y$ such that $b_1(x) = b_2(x) = 1$ and $b_3(x) = -1$, so that $P(\underline{b})$ would already fail in $\{x\}$. Thus there is no element of $\overline{Y}$ making both b_1 and b_2 positive. To sum up, there exist $x_1, x_2, x_3 \in \overline{Y}$ such that b_1, b_2, b_1b_2 have the following signs:

	x_1	x_2	x_3
b_1	+	-	-
b_2	-	+	-
b_1b_2	-	-	+

Consider the subspace $(\tilde{Y}, \tilde{H})$ for which $\{x_1, x_2, x_3\}$ is a minimal generating set. Since no element of $\overline{Y}$ makes both b_1 and b_2 positive, $x_1x_2x_3 \notin \tilde{Y}$, and, consequently, $\tilde{Y} = \{x_1, x_2, x_3\}$. Let (Y', H') be the group extension of $(\tilde{Y}, \tilde{H})$ where $H' = \tilde{H} \times \{1, b_3\}$. It consists of 6 orderings x_1^+, x_2^+, x_3^+, x_1^-, x_2^-, x_3^-, with respect to which the signs of b_1, b_2, b_1b_2, b_3 are as follows:

	x_1^+	x_2^+	x_3^+	x_1^-	x_2^-	x_3^-
b_1	+	-	-	+	-	-
b_2	-	+	-	-	+	-
b_1b_2	-	-	+	-	-	+
b_3	+	+	+	-	-	-

$P(\underline{b})$ fails to hold on (Y', H'), so $(Y, H) = (Y', H')$. One easily verifies that in this case $\mathrm{cl}(Y, H) = 3$.

We have thus finished a detailed description of $\mathbb{K}_P$ and are in the position to build the formulae $P_\lambda^{(i)}(\underline{y})$. Take $\lambda = 4 > c_P$. We have

$$P_4^{(0)}(y_1, y_2, y_3) = \exists t_1 \exists t_1 [t_1 \in D(1, y_1) \wedge t_2 \in D(1, y_2) \wedge y_3 t_1 t_2 \in D(1, y_1 y_2)],$$

$$
\begin{aligned}
P_4^{(1)}(y_1,y_2,y_3) = \exists t_{11}\exists t_{12}\dots\exists t_{31}\exists t_{32}\exists z_0\dots\exists z_4\exists s_{11}\dots\exists s_{16}\exists s_{21}\dots\exists s_{46}\\
& z_0 \in D(1,z_1)\wedge\\
\wedge\quad & s_{11}\in D(1,z_0z_1)\wedge s_{12}\in D(1,z_0z_1)\wedge\dots\wedge s_{16}\in D(1,z_0z_1)\wedge\\
\wedge\quad & t_{11}\in D(s_{11},s_{12}y_1)\wedge t_{12}\in D(s_{13},s_{14}y_2)\wedge y_3t_{11}t_{12}\in D(s_{15},s_{16}y_1y_2)\wedge\\
\wedge\quad & \dots\wedge\\
& z_3\in D(1,z_4)\wedge\\
\wedge\quad & s_{41}\in D(1,z_3z_4)\wedge s_{42}\in D(1,z_3z_4)\wedge\dots\wedge s_{46}\in D(1,z_3z_4)\wedge\\
\wedge\quad & t_{31}\in D(s_{41},s_{42}y_1)\wedge t_{32}\in D(s_{43},s_{44}y_2)\wedge y_3t_{31}t_{32}\in D(s_{45},s_{46}y_1y_2)
\end{aligned}
$$

One sees that $P_4^{(1)}(y_1,y_2,y_3)$ contains 37 quantifiers and 40 atomic formulae. $P_4^{(2)}(y_1,y_2,y_3)$ will contain, respectively, 473 quantifiers and 484 atomic formulae. It would be desirable to find simpler formulae to which $P_4^{(i)}(y_1,y_2,y_3)$ would be logically equivalent.

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