CYLINDERS WITH COMPACT CROSS-SECTION AND THE STRIP CONJECTURE

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1. Introduction

The proof of the strip conjecture given just recently in [7] (see Theorem 3.1 below for a statement of the result) is best understood as a refinement of a proof of a general result for cylinders with compact cross-section (see Theorem 2.2 below). We elaborate on this statement. We give a detailed proof of Theorem 2.2 and we sketch a proof of Theorem 3.1. For a full proof of Theorem 3.1, we refer the reader to [7]. It should be noted that our proof of Theorem 2.2 is not the original proof given in [10, Th. 3], but rather it is a modification of an argument given even earlier in [3, Th. 5.1].

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2. Cylinders with compact cross-section

In this section $x = \underline{x} = (x_1, \dots, x_n)$, an n-tuple of variables, and y is a single variable. We begin by recalling Schmüdgen's famous Positivstellensatz, which we refer to also as Schmüdgen's Theorem:

Theorem 2.1. [18, Cor. 3] *Suppose $g_1, \dots, g_s \in \mathbb{R}[x]$ and $K := \{p \in \mathbb{R}^n \mid g_i(p) \geq 0, i = 1, \dots, s\}$ is compact. Then $\forall\ f \in \mathbb{R}[x]$, $f > 0$ on $K \Rightarrow f$ belongs to the preordering of $\mathbb{R}[x]$ generated by $g_1, \dots, g_s$.*

Schmüdgen's Theorem extends as follows:

Theorem 2.2. [10, Th. 3] *Suppose $g_1, \dots, g_s \in \mathbb{R}[x]$, $K := \{p \in \mathbb{R}^n \mid g_i(p) \geq 0, i = 1, \dots, s\}$ is compact, $f = f(x, y) = \sum_{i=0}^{2d} a_i(x) y^i \in \mathbb{R}[x, y]$, $a_{2d} > 0$ on K, $f > 0$ on $K \times \mathbb{R}$. Then f belongs to the preordering of $\mathbb{R}[x, y]$ generated by $g_1, \dots, g_s$.*

We recall the terminology: A *preordering* of a ring A (commutative with 1) is a subset T of A satisfying $T + T \subseteq T$, $TT \subseteq T$ and $f^2 \in T\ \forall\ f \in A$. $\sum A^2$ denotes the set of all (finite) sums of squares of elements of A, the unique smallest preordering of A. The preordering of A generated by $g_1, \dots, g_s \in A$ consists of all sums $\sum_i \sigma_i g^i$, $\sigma_i \in \sum A^2$, $g^i := g_1^{i_1} \dots g_s^{i_s}$, $i := (i_1, \dots, i_s)$ running through the set $\{0, 1\}^s$.

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In this section we give a proof of Theorem 2.2. Although Theorem 2.2 is first stated and proved in [10, Th. 3], it can also be deduced by a modification of the argument in [3, Th. 5.1]. This is what we do here.

Proof. If $d = 0$ the result is immediate from Schmüdgen's Theorem, so we assume $d \geq 1$. Let $F(x, y, z) := \sum_{i=0}^{2d} a_i(x) y^i z^{2d-i}$.

Claim 1. If $x \in K$ and $y, z \in \mathbb{R}$, $(y, z) \neq (0, 0)$, then $F(x, y, z) > 0$.

Proof. If $z = 0$ then $y \neq 0$ and $F(x, y, z) = a_{2d}(x) y^{2d} > 0$. If $z \neq 0$ the $F(x, y, z) = z^{2d} f(x, \frac{y}{z}) > 0$. □

Claim 2. ∃ a real number $\epsilon > 0$ such that $f(x, y) \geq \epsilon(1 + y^2)^d$ on $K \times \mathbb{R}$.

Proof. Let $\epsilon > 0$ be the minimum value of $F(x, y, z)$ on the compact set

$$K \times \mathbb{S}^1 := \{(x, y, z) \mid x \in K, y, z \in \mathbb{R}, y^2 + z^2 = 1\}.$$

Then, for any $x \in K$, $y \in \mathbb{R}$, $(x, \frac{y}{\sqrt{1+y^2}}, \frac{1}{\sqrt{1+y^2}}) \in K \times \mathbb{S}^1$ and $f(x, y) = F(x, y, 1) = F(x, \frac{y}{\sqrt{1+y^2}}, \frac{1}{\sqrt{1+y^2}})(1 + y^2)^d \geq \epsilon(1 + y^2)^d$. □

Set

$$f_1(x, y) := f(x, y) - \epsilon(1 + y^2)^d, \text{ (so } f(x, y) = f_1(x, y) + \epsilon(1 + y^2)^d). \tag{2.1}$$

Replacing ϵ by $\frac{\epsilon}{N}$, $N > 1$, we can assume $f_1(x, y) > 0$ on $K \times \mathbb{R}$ and $f_1(x, y) = \sum_{i=0}^{2d} b_i(x) y^i$ with $b_{2d}(x) > 0$ on K.

Claim 3. $f_1(x, y) = p_1(x, y)^2 + p_2(x, y)^2$ where p_1, p_2 are polynomials in y of degree $\leq d$ with coefficients continuous functions from K to $\mathbb{R}$.

Here, $f_1(x, y) = p_1(x, y)^2 + p_2(x, y)^2$ is to be viewed, in the obvious way, as an equation in the polynomial ring $C(K)[y]$, where $C(K)$ denotes the ring of all continuous functions from K to $\mathbb{R}$.

Proof. For each $x \in K$, $f_1(x, y) = b_{2d}(x) \prod_{i=1}^{2d}(y - t_i(x))$ with $t_i(x) \in \mathbb{C}$. The unordered set of roots $\{t_1(x), \ldots, t_{2d}(x)\}$ varies continuously with x [5, page 3], $t_i(x) = r_i(x) + s_i(x)\sqrt{-1}$, $r_i(x), s_i(x) \in \mathbb{R}$. There are no real roots, so $s_i(x) \neq 0$. Reindexing, we can assume $s_i(x) > 0$ for $i = 1, \ldots, d$ and $s_{i+d}(x) = -s_i(x)$, $i = 1, \ldots, d$. Let $p(x, y) := \sqrt{b_{2d}(x)} \prod_{i=1}^{d}(y - t_i(x))$. The coefficients of $p(x, y)$ are complex-valued continuous functions of x, $p = p_1 + p_2\sqrt{-1}$, $\overline{p} = p_1 - p_2\sqrt{-1}$, p_1, p_2 have coefficients which are real-valued continuous functions of x, and $f_1 = p\overline{p} = p_1^2 + p_2^2$. □

We now use the Stone-Weierstrass Approximation Theorem to approximate the coefficients of p_1 and p_2 closely by polynomials to obtain

$$f_1 = q_1^2 + q_2^2 + \sum_{i=0}^{2d} c_i(x) y^i \tag{2.2}$$

where q_1, q_2 are polynomials in y of degree $\leq d$ with coefficients in $\mathbb{R}[x]$, $c_i(x) \in \mathbb{R}[x]$ and $|c_i(x)| < \frac{2}{5}\epsilon$ on K for $i = 0, \ldots, 2d$.

Claim 4. $\frac{2}{5}\epsilon \pm c_i(x)$ lies in the preordering of $\mathbb{R}[x]$ generated by $g_1, \dots, g_s$ for each $i = 0, \dots, 2d$.

Proof. Since $\frac{2}{5}\epsilon \pm c_i(x) > 0$ on K this is immediate by Schmüdgen's Theorem. □

Now let

$$s_1(x, y) := q_1(x, y)^2 + q_2(x, y)^2,$$

$$s_2(x, y) := \frac{2}{5}\epsilon(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} c_i(x)y^i,$$

$$s_3(x, y) := \epsilon[(1 + y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d})].$$

Using (2.1) and (2.2), we see that $f = s_1 + s_2 + s_3$. Let T denote the preordering of $\mathbb{R}[x, y]$ generated by $g_1, \dots, g_s$. It remains to show $s_1, s_2, s_3 \in T$. Clearly $s_1 \in T$.

Claim 5. $s_2 \in T$.

Proof. $\frac{2}{5}\epsilon y^i + c_i(x)y^i = [\frac{2}{5}\epsilon + c_i(x)]y^i$ and $\frac{2}{5}\epsilon + c_i(x) \in T$ by Claim 4. Thus

$$\frac{2}{5}\epsilon y^i + c_i(x)y^i \in T, \text{ for } i \text{ even.} \tag{2.3}$$

For i odd, say $i = 2m + 1$, use Claim 4 to deduce

$$\frac{2}{5}\epsilon(y + 1)^2 + c_i(x)(y + 1)^2 = [\frac{2}{5}\epsilon + c_i(x)](y + 1)^2 \in T,$$
$$\frac{2}{5}\epsilon y^2 - c_i(x)y^2 = [\frac{2}{5}\epsilon - c_i(x)]y^2 \in T, \text{ and}$$
$$\frac{2}{5}\epsilon - c_i(x) \in T.$$

Adding, we see that

$$\frac{2}{5}\epsilon[(y + 1)^2 + y^2 + 1] + c_i(x)[(y + 1)^2 - y^2 - 1] = \frac{2}{5}\epsilon[2y^2 + 2y + 2] + c_i(x)2y \in T.$$

Dividing by 2 and multiplying by y^{2m}, this yields

$$\frac{2}{5}\epsilon(y^{i+1} + y^i + y^{i-1}) + c_i(x)y^i \in T, \text{ for } i \text{ odd.} \tag{2.4}$$

Adding together the various terms of type (2.3) and (2.4), as $i = 0, \cdots, 2d$, we see that $s_2 \in T$. □

Claim 6. $s_3 \in T$.

Proof. This follows from the identities

$$\begin{aligned}
&(1 + y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) \\
&= (1 + y^2)^d + \frac{1}{5}(1 + y^2 + \cdots + y^{2d-2})(1 - y)^2 \\
&- \frac{8}{5}(y^2 + y^4 + \cdots + y^{2d-2}) - (1 + y^{2d}) \\
&= \frac{1}{5}(1 + y^2 + \cdots + y^{2d-2})(1 - y)^2 + \sum_{i=1}^{d-1}(\binom{d}{i} - \frac{8}{5})y^{2i},
\end{aligned}$$

using the fact that $0 < i < d \Rightarrow \binom{d}{i} > \frac{8}{5}$. □

Finally, combining Claim 5 and Claim 6, $f = s_1 + s_2 + s_3 \in T$. □

Schmüdgen's Theorem yields a solution to the moment problem in the compact case. In an analogous way, Theorem 2.2 yields a solution to the moment problem for cylinders with compact cross-section:

Corollary 2.3. [3, Th. 5.1] *Suppose $g_1, \dots, g_s \in \mathbb{R}[x]$, $K := \{p \in \mathbb{R}^n \mid g_i(p) \geq 0, i = 1, \dots, s\}$ is compact and T is the preordering of $\mathbb{R}[x, y]$ generated by $g_1, \dots, g_s$. Suppose $L : \mathbb{R}[x, y] \to \mathbb{R}$ is linear, $L \geq 0$ on T. Then $\exists$ a positive Borel measure μ on $K \times \mathbb{R}$ such that $\forall\, f \in \mathbb{R}[x, y]$ $L(f) = \int f\, d\mu$. In other words, SMP holds for T.*

Proof. By Haviland's Theorem, e.g., see [6, Th. 3.1.2], it suffices to show that $\forall$ $f \in \mathbb{R}[x, y]$, $f \geq 0$ on $K \times \mathbb{R} \Rightarrow L(f) \geq 0$. Suppose $f \in \mathbb{R}[x, y]$, $f \geq 0$ on $K \times \mathbb{R}$. Let $g(x, y) = f(x, y) + \epsilon(1 + y^2)^d$, $\epsilon > 0$, where $2d \geq$ the degree of f as a polynomial in y. Then $g > 0$ on $K \times \mathbb{R}$ and the leading coefficient of g is > 0 on K so, by Theorem 2.2, $g \in T$. Then $L(g) \geq 0$. Since $g = f + \epsilon(1 + y^2)^d$ and L is linear, $L(g) = L(f + \epsilon(1 + y^2)^d)) = L(f) + \epsilon L((1 + y^2)^d)$. Letting $\epsilon \to 0$ we see that $L(f) \geq 0$. □

The proof of Corollary 2.3 shows a bit more than what is stated: T actually satisfies the somewhat stronger 'double dagger' condition. See [3, Th. 5.1]. See [19] and [4] for other extensions of the result.

3. The strip conjecture

The object of this section is to sketch a proof of the following result, called the strip conjecture, which is a refinement of a special case of Theorem 2.2. See [7] for the complete proof.

Theorem 3.1. *Let x, y be variables, $f = f(x, y) \in \mathbb{R}[x, y]$, $f \geq 0$ on $[0, 1] \times \mathbb{R}$. Then f lies in the preordering of $\mathbb{R}[x, y]$ generated by x and $1 - x$. In other words, the preordering of $\mathbb{R}[x, y]$ generated by x and $1 - x$ is saturated.*

The strip conjecture seems to have been considered first in [3] and [12]. In [12], the authors claimed to know a proof of the conjecture, but this claim was later withdrawn. See also [10] and [11] for work related to the conjecture.

Note: Consider the identities

$$\begin{aligned} 1 &= x + (1 - x), \\ x &= x^2 + x(1 - x), \\ 1 - x &= x(1 - x) + (1 - x)^2, \text{ and} \\ x(1 - x) &= x^2(1 - x) + x(1 - x)^2. \end{aligned}$$

The 2nd and 3rd identity show that the preordering generated by x and $1 - x$ coincides with the preordering generated by $x(1-x)$. The 1st and 4th identity show that elements in this preordering are also expressible in the form $\alpha x + \beta(1 - x)$, where α and β are sums of squares.

In [2] Hilbert showed that there are polynomials $f(x, y) \in \mathbb{R}[x, y]$ (necessarily of degree ≥ 6) which are non-negative on all of $\mathbb{R}^2$, but are not expressible as a sum of squares in $\mathbb{R}[x, y]$. The best-known example is the polynomial $f(x, y) =$

$1 - 3x^2y^2 + x^4y^2 + x^2y^4$ [8]. This shows that the preordering $\sum \mathbb{R}[x,y]^2$ is not saturated.

In fact, before Theorem 3.1 was proved, the only example of a finitely generated saturated preordering in the 2-dimensional non-compact case was the rather artificial example given in [17, Rem. 3.14] (the preordering of $\mathbb{R}[x,y]$ generated by $x, 1-x, y$ and $1-xy$). It is hoped that the ideas used in the proof of Theorem 3.1 will yield additional examples of this sort in the future.

See [1] and [17] for examples of finitely generated saturated preorderings in the 2-dimensional compact case. See [3], [9], [14] and [15] for 1-dimensional examples. By [14, Prop. 6.1], a finitely generated preordering of the polynomial ring $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1,\dots,x_n]$ cannot be saturated if the dimension of the associated semialgebraic set is 3 or more. By [16, Th. 5.4], the same holds in dimension 2 if the preordering in question is stable, i.e., in dimension 2, saturation and stability are mutually exclusive.

Just as the proof of Theorem 2.2 requires Schmüdgen's Theorem, the proof of Theorem 3.1 requires the following result:

Lemma 3.2. *Suppose $f = f(x) \in \mathbb{R}[x]$, $f \geq 0$ on $[0,1]$. Then f lies in the preordering of $\mathbb{R}[x]$ generated by x and $1-x$. In other words, the preordering of $\mathbb{R}[x]$ generated by x and $1-x$ is saturated.*

In fact, Lemma 3.2 is well-known, e.g., it is a special case of [3, Th 2.2]. But, anyway, here is the proof:

Proof. Let $T :=$ the preordering of $\mathbb{R}[x]$ generated by x and $1-x$. We may assume $f \neq 0$. Induct on the degree of f. If f is constant it is clear. Suppose now that f has a real root r.

Case 1. $0 < r < 1$. Then $f = (x-r)^2 g$, $g \geq 0$ on $[0,1]$. Then $(x-r)^2 \in T$ and $g \in T$ (by induction), so $f = (x-r)^2 g \in T$.

Case 2. $r \leq 0$. Then $r = -a$, $a \geq 0$, and $f = (x-r)g = (x+a)g$, $g \geq 0$ on $[0,1]$. Then $x + a \in T$ and $g \in T$, so $f = (x+a)g \in T$.

Case 3. $r \geq 1$. Then $r = 1 + a$, $a \geq 0$, and $f = (r-x)g = [a + (1-x)]g$, $g \geq 0$ on $[0,1]$. Then $a + (1-x) \in T$ and $g \in T$, so $f \in T$.

In the remaining case f has a complex root $r = a + b\sqrt{-1}$, $a, b \in \mathbb{R}$, $b \neq 0$. Then $f = (x-r)(x-\overline{r})g = [(x-a)^2 + b^2]g$, $g \geq 0$ on $[0,1]$. Then $(x-a)^2 + b^2 \in T$ and $g \in T$, so $f \in T$. □

Proof of Theorem 3.1. Let $f \in \mathbb{R}[x,y]$, $f \geq 0$ on $[0,1] \times \mathbb{R}$. We may assume $f \neq 0$. Then f has even degree, $2d$ say, as a polynomial in y, $f = f(x,y) = \sum_{i=0}^{2d} a_i(x)y^i$, and $a_{2d}(x) \geq 0$ on $[0,1]$. If $d = 0$ the result follows from Lemma 3.2, so we assume $d \geq 1$. (Remark: The case $d = 1$ is already highly non-trivial.) Let T be the preordering of $\mathbb{R}[x,y]$ generated by x and $1-x$.

Claim 1. Suppose $b \in \mathbb{R}[x]$, $b \geq 0$ on $[0,1]$, and $b = \pm 1$ times a product of linear factors of the form $x - r$, $r \in [0,1]$. Then $bf \in T \Rightarrow f \in T$.

Proof. By induction on the degree of b (as a polynomial in x). Let $bf = \sigma + \tau x(1-x)$, σ, τ sums of squares in $\mathbb{R}[x,y]$. Suppose $x - r$, $r \in [0,1]$ is a factor of b.

Case 1: $0 < r < 1$. Then $b = \overline{b}(x-r)^2$, $\sigma = \overline{\sigma}(x-r)^2$, $\tau = \overline{\tau}(x-r)^2$ where $\overline{\sigma}, \overline{\tau}$ are sums of squares, and $\overline{b}f = \overline{\sigma} + \overline{\tau}x(1-x) \in T$.

Case 2: $r = 0$. Then $b = \overline{b}x$, $\sigma = \overline{\sigma}x^2$ where $\overline{\sigma}$ is a sum of squares, and $\overline{b}f = \overline{\sigma}x + \tau(1-x) \in T$.

Case 3: $r = 1$. Similar to Case 2. □

Claim 2. We can assume $a_{2d}(x) > 0$ on $[0, 1]$.

Proof. Factor a_{2d} as $a_{2d} = \overline{a}\tilde{a}$, $\overline{a}, \tilde{a} \in \mathbb{R}[x]$, $\overline{a} > 0$ on $[0, 1]$, $\tilde{a}$ is ± 1 times a product of linear factors $x - r$, $r \in [0, 1]$. Define $g \in \mathbb{R}[x, y]$ by $g(x, y) = \tilde{a}^{2d-1} f(x, \frac{y}{\tilde{a}})$. Then $g \geq 0$ on $[0, 1] \times \mathbb{R}$ and the leading coefficient of g is $\overline{a}$. Suppose the result is true for g, i.e., $g \in T$. Say $g(x, y) = \alpha(x, y) + \beta(x, y)x(1 - x)$, α, β sums of squares. Then $\tilde{a}^{2d-1} f(x, y) = g(x, \tilde{a}y) = \alpha(x, \tilde{a}y) + \beta(x, \tilde{a}y)x(1 - x) \in T$. Now apply Claim 1 to get $f \in T$. □

Claim 3. We can assume f is square free.

Proof. Suppose $f = g^2 h$. Use the fact that $g \neq 0 \Rightarrow$ the set $\{(a, b) \in [0, 1] \times \mathbb{R} \mid g(a, b) \neq 0\}$ is dense in the strip to get that $h \geq 0$ on the strip. To show $f \in T$ it suffices to show $h \in T$. □

Claim 4. We can assume f has only finitely many zeros in the strip.

Proof. We know $x \nmid f$ (because $x \nmid a_{2d}$) and similarly $1 - x \nmid f$, so f has only finitely many zeros on the boundary of the strip. If f has infinitely many zeros in the interior then some irreducible factor p of f has infinitely many zeros in the interior. Then, using Bezout's Theorem, e.g., see [6, Lem. 9.4.1], p has a non-singular zero in the interior which is not a zero of any other irreducible factor of f. Then f changes sign at this non-singular zero, contradicting our assumption that $f \geq 0$ on the strip. □

If f is square-free, then no irreducible factor of f can change sign in the interior of the strip, so each irreducible factor has constant sign on the strip. Replacing p by $-p$ if necessary, for each irreducible factor p, we may assume each irreducible factor of f is ≥ 0 on the strip. In this way we are reduced further, to the case where f itself is irreducible. But this does not seem to help us much in the proof.

The idea of the proof is as follows: If $a_{2d} > 0$ on $[0, 1]$ and f has no zeros in $[0, 1] \times \mathbb{R}$ then, by the proof of Theorem 2.2, $\exists$ a real number $\epsilon > 0$ such that $f(x, y) \geq \epsilon(1 + y^2)^d$ on $[0, 1] \times \mathbb{R}$. In the general case, such a real number ϵ cannot possibly exist. The idea is to replace ϵ by a suitable polynomial $\epsilon(x)$.

Lemma 3.3. *Assume $f(x, y) = \sum_{i=0}^{2d} a_i(x)y^i \in \mathbb{R}[x, y]$, $f \geq 0$ on $[0, 1] \times \mathbb{R}$, $a_{2d} > 0$ on $[0, 1]$ and $f(x, y)$ has only finitely many zeros in $[0, 1] \times \mathbb{R}$. Then $\exists$ a polynomial $\epsilon(x) \in \mathbb{R}[x]$, $\epsilon(x) \geq 0$ on $[0, 1]$, $f(x, y) \geq \epsilon(x)(1 + y^2)^d$ on $[0, 1] \times \mathbb{R}$ and, $\forall$ $x \in [0, 1]$, $\epsilon(x) = 0$ iff $\exists$ $y \in \mathbb{R}$ such that $f(x, y) = 0$.*

Proof. See [7]. The proof requires some understanding of the complex analytic branches of the curve $f = 0$; see [13, Sect. 12.3], for instance. □

We show that a suitably modified version of the argument in Theorem 2.2 carries through, with ϵ replaced by $\epsilon(x)$. Fix $\epsilon(x)$ as in Lemma 3.3 and define $f_1(x, y)$ by

$$f_1(x, y) := f(x, y) - \epsilon(x)(1 + y^2)^d \text{ (so } f(x, y) = f_1(x, y) + \epsilon(x)(1 + y^2)^d). \tag{3.1}$$

Replacing $\epsilon(x)$ by $\frac{\epsilon(x)}{N}$, $N > 1$ we can assume the leading coefficient of $f_1(x, y)$ is > 0 on $[0, 1]$. At this point, the argument becomes a bit technical: it is necessary to take pains to ensure that the continuous functions considered are analytic at the points where $\epsilon(x) = 0$.

Lemma 3.4. *$f_1(x,y) = \sum_{i=1}^{k}[p_{i0}(x,y)^2 + p_{i1}(x,y)^2 x + p_{i2}(x,y)^2(1-x)]$ for some $k \geq 1$ where each $p_{ij}(x,y)$ is a polynomial of degree $\leq d$ in y whose coefficients are real valued continuous functions of x, for $x \in [0,1]$, which are analytic at each of the finitely many zeros of $\epsilon(x)$ in $[0,1]$.*

Proof. See [7]. Again, the proof requires some understanding of the complex analytic branches of the curve $f = 0$. □

We also need a refinement of the Weierstrass Approximation Theorem:

Lemma 3.5. *Suppose $\phi, \psi : [0,1] \to \mathbb{R}$ are continuous functions, $\phi(x) \leq \psi(x)$ $\forall$ $x \in [0,1]$, $\phi(a) = \psi(a)$ for only finitely many $a \in [0,1]$, and ϕ and ψ are analytic at each $a \in [0,1]$ such that $\phi(a) = \psi(a)$. Then $\exists$ a polynomial $p \in \mathbb{R}[x]$ such that $\phi(x) \leq p(x) \leq \psi(x)$ $\forall$ $x \in [0,1]$.*

Proof. Induct on the number of points $a \in [0,1]$ satisfying $\phi(a) = \psi(a)$. If there are no such points, existence of $p(x)$ follows from the standard Weierstrass Approximation Theorem. Suppose $a \in [0,1]$ is such that $\phi(a) = \psi(a)$. Let k be the vanishing order of $\psi - \phi$ at a. If $a \in (0,1)$ then k is even. In this case, $\phi(x) = f(x) + (x-a)^k\phi_1(x)$, $\psi(x) = f(x) + (x-a)^k\psi_1(x)$, where $f(x) \in \mathbb{R}[x]$, $\phi_1(x), \psi_1(x)$ are analytic at a, and $\phi_1(a) < \psi_1(a)$. Extend ϕ_1, ψ_1 to continuous functions $\phi_1, \psi_1 : [0,1] \to \mathbb{R}$ by defining $\phi_1(x) = \frac{\phi(x)-f(x)}{(x-a)^k}$, $\psi_1(x) = \frac{\psi(x)-f(x)}{(x-a)^k}$ for $x \neq a$. Then $\phi_1(x) \leq \psi_1(x)$ for all $x \in [0,1]$, and, $\forall$ $b \in [0,1]$, $\phi_1(b) = \psi_1(b)$ iff $\phi(b) = \psi(b)$ and $b \neq a$. By induction we have $p_1(x) \in \mathbb{R}[x]$ such that $\phi_1(x) \leq p_1(x) \leq \psi_1(x)$ on $[0,1]$. Take $p(x) = f(x) + (x-a)^k p_1(x)$. The case where $a = 0$ and the case where $a = 1$ are dealt with in a similar fashion. □

We use Lemma 3.5 to approximate the coefficients of the $p_{ij}(x,y)$ closely by polynomials to obtain

$$f_1(x,y) = \sum_{i=1}^{k}[q_{i0}(x,y)^2 + q_{i1}(x,y)^2 x + q_{i2}(x,y)^2(1-x)] + \sum_{i=0}^{2d} c_i(x)y^i, \tag{3.2}$$

where the q_{ij} are polynomials in x, y of degree $\leq d$ in y, $c_i(x) \in \mathbb{R}[x]$, $|c_i(x)| \leq \frac{2}{5}\epsilon(x)$ on $[0,1]$. Combining equations (3.1) and (3.2), f decomposes as $f = s_1 + s_2 + s_3$ where

$$s_1(x,y) := \sum_{i=1}^{k}[q_{i0}(x,y)^2 + q_{11}(x,y)^2 x + q_{i2}(x,y)^2(1-x)],$$

$$s_2(x,y) := \frac{2}{5}\epsilon(x)(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} c_i(x)y^i,$$

$$s_3(x,y) := \epsilon(x)[(1+y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d})].$$

Arguing as in the proof of Theorem 2.2, but with Schmüdgen's Theorem replaced by Lemma 3.2, we see that $s_1, s_2, s_3 \in T$, so $f \in T$. □

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