THE SPACE OF $\mathbb{R}$-PLACES OF $\mathbb{R}(x, y)$ IS NOT METRIZABLE

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Abstract. For $n = 1$, the space of $\mathbb{R}$-places of the rational function field $\mathbb{R}(x_1, \dots, x_n)$ is homeomorphic to the real projective line. For $n \geq 2$, the structure is much more complicated. We prove that the space of $\mathbb{R}$-places of the rational function field $\mathbb{R}(x, y)$ is not metrizable. We explain how the proof generalizes to show that the space of $\mathbb{R}$-places of any finitely generated formally real field extension of $\mathbb{R}$ of transcendence degree ≥ 2 is not metrizable. We also consider the more general question of when the space of $\mathbb{R}$-places of a finitely generated formally real field extension of a real closed field is metrizable, and provide some partial answers.

1. Introduction and main result

For a formally real field F, let X_F denote the space of orderings of F, M_F the space of $\mathbb{R}$-places of F, i.e., places $\xi : F \to \mathbb{R} \cup \{\infty\}$, and let

$$\lambda : X_F \to M_F$$

be the natural map. We refer the reader to [2], [3], [10] or [12] for basic terminology and basic results. X_F is a Boolean space. M_F is compact and Hausdorff. λ is continuous and surjective. The topology on M_F is the quotient topology.

We are interested in function fields of real algebraic varieties, i.e., finitely generated formally real field extensions of $\mathbb{R}$. The rational function field $\mathbb{R}(x_1, \dots, x_n)$, $n \geq 0$ is an important special case. It is well-known that the complexity of the space of orderings and the space of $\mathbb{R}$-places of $\mathbb{R}(x_1, \dots, x_n)$ increases rapidly as n increases, e.g., see [1], [9] or [15]. $M_{\mathbb{R}}$ is a singleton set. $M_{\mathbb{R}(x)}$ is the real projective line. The structure is already complicated when $n = 2$.

Theorem 1.1. *For any uncountable real closed field R, the space of $\mathbb{R}$-places of the rational function field $R(x, y)$ is not metrizable.*

The object of the paper is to prove Theorem 1.1 and also to prove certain variations and extensions of this result; see Sections 2 and 3. It is conjectured that the requirement that R be uncountable is not necessary. We are interested in the case $R = \mathbb{R}$, but the proof is no easier in this case.

A version of Theorem 1.1 appears already in the e-print [11] by the first author and the third author, but there is a gap in the proof. The proof given here fills the gap and, at the same time, it is shorter and more direct.

It is known that a compact Hausdorff space is metrizable iff it is second countable. The non-trivial implication ($\Leftarrow$) is a consequence of Urysohn's Metrization Theorem

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[7, p. 125]. Here we will use only the trivial implication ($\Rightarrow$). We will also use the obvious fact that a topological space of uncountable cellularity cannot be second countable.

Orderings on a formally real field F can be viewed as order relations ($<$) or, equivalently, as cones of positivity (P). The so-called Harrison sets

$$H(a) := \{P \in X_F \mid a \in P\},\ a \in F^*,$$

are clopen (closed and open) in X_F and form a subbasis for the topology. We recall the definition of the map $\lambda : X_F \to M_F$. For $P \in X_F$,

$$A(P) := \{a \in F \mid n + a, n - a \in P \text{ for some } n \in \mathbb{N}\}$$

is a valuation ring of F with maximal ideal

$$I(P) := \{a \in F \mid \frac{1}{n} - a, \frac{1}{n} + a \in P \text{ for all } n \in \mathbb{N}\}.$$

The ordering induced by P on the residue field $A(P)/I(P)$ is Archimedean, so there is an order preserving embedding $A(P)/I(P) \hookrightarrow \mathbb{R}$. The place $\lambda(P) : F \to \mathbb{R} \cup \{\infty\}$ is defined by composition, in the obvious way.

The fibers of the map $\lambda : X_F \to M_F$ are described by the Baer-Krull Theorem; see [8, II, Sect. 7], [12, Sect. 1.3] or [13, Sect. 1.5]. If F is a finitely generated formally real field extension of a real closed field R, then each fiber $\lambda^{-1}(\xi)$, $\xi \in M_F$ satisfies

$$|\lambda^{-1}(\xi)| = 2^{k_\xi}$$

for some integer k_ξ in the interval $[0, d]$, where d is the transcendence degree of F over R. See [12, Sect. 3.5 and 3.6] for the proof and for related results on fans and the stability index due to L. Bröcker.

Suppose now that $R = \mathbb{R}$, i.e., that F is a finitely generated formally real field extension of $\mathbb{R}$. If $\operatorname{trdeg}(F : \mathbb{R}) = 1$ then there is a unique non-singular projective model of F, say V, and the center map $M_F \to V(\mathbb{R})$ is a homeomorphism. If $\operatorname{trdeg}(F : \mathbb{R}) \geq 2$ then Hironaka's results on resolution of singularities imply that M_F is expressible as an inverse limit

$$M_F = \varprojlim_V V(\mathbb{R}),$$

V running through non-singular projective models of F; see [3, Th. 3.5] or [14, Corollary, page 439]. As explained in [14], a certain relative version of this result remains valid for $R \neq \mathbb{R}$.

2. The proof

Let $F = R(x, y)$. Let $X := X_F$, $Y :=$ the closed subspace of X defined by $0 < x < r$ $\forall$ positive $r \in R$, i.e.,

$$Y := \cap_{r \in R^+} H(r - x) \cap H(x).$$

For arbitrary $r \in R$, define U_r to be the open set in Y defined by $-ax < y - r < ax$ for some $a \in \mathbb{N}$, i.e.,

$$U_r = \cup_{a \in \mathbb{N}} [H(ax - (y - r)) \cap H(ax + (y - r))] \cap Y.$$

Claim 1. $U_r \neq \emptyset$. This is clear. E.g., take the restriction to F of an ordering on the iterated formal power series field $R((x))((y - r))$ making x positive (by Baer-Krull, there are two such orderings).

Claim 2. $U_r \cap U_s = \emptyset$ if $r \neq s$. For suppose $<$ is an ordering in $U_r \cap U_s$. Then $|y-r| < ax$, $|y-s| < bx$, $a, b \in \mathbb{N}$. Then $|r-s| \leq |y-r| + |y-s| < (a+b)x$. Since $x < t$ for each positive $t \in R$, this forces $r = s$.

Claim 3. If $<$ and $<'$ are orderings in Y which have the same associated $\mathbb{R}$-place and $<$ belongs to U_r then so does $<'$. Since $<$ belongs to U_r, $-a < \frac{y-r}{x} < a$ for some $a \in \mathbb{N}$, i.e., $\frac{y-r}{x}$ lies in the valuation ring of F corresponding to $<$. Since $<$ and $<'$ have the same $\mathbb{R}$-place, this valuation ring is the same as the valuation ring of F corresponding to $<'$, i.e., $-b <' \frac{y-r}{x} <' b$ for some $b \in \mathbb{N}$, so $<'$ also belongs to U_r.

Let $M := M_F$ and let $\lambda : X \to M$ be the standard map. Take $N := \lambda(Y)$. Since Y is compact, N is Hausdorff and λ is continuous, the restriction $\lambda : Y \to N$ is closed, and the topology on N coincides with the quotient topology [7, Th. 8, p. 95], i.e., for any subset V of N, V is open in N iff $\lambda^{-1}(V) \cap Y$ is open in Y.

We show that N has cellularity $\geq |R|$, i.e., that there exists a family of non-empty pairwise disjoint open sets $\{V_r\}_{r \in R}$, in N. This is a consequence of Claims 1, 2 and 3. For $r \in R$, let $V_r := \lambda(U_r)$. By Claim 3, $\lambda^{-1}(V_r) \cap Y = U_r$ and U_r is open in Y so V_r is open in N. By Claims 1 and 2, $V_r \neq \emptyset$ for each r and $V_r \cap V_s = \emptyset$ for $r \neq s$. It follows that N is not second countable. Since N is compact, this implies in turn that N is not metrizable, so M is not metrizable. □

Remark 2.1. Let S denote the real closure of $R(x)$ at the unique ordering satisfying $0 < x < r$ for all $r \in R^+$. The restriction map $X_{S(y)} \to X_{R(x,y)}$ induced by the inclusion $R(x,y) \hookrightarrow S(y)$ is injective and identifies $X_{S(y)}$ with Y [5, Lemma 8]. The subspace N of $M_{R(x,y)}$ is the image of $M_{S(y)}$ under the restriction map $M_{S(y)} \to M_{R(x,y)}$, but this map is *not* injective. E.g., the $\mathbb{R}$-places given by $y/\sqrt{x} \mapsto \pm 1$ both restrict to the same $\mathbb{R}$-place on $R(x,y)$. Consequently, the argument in [11, Cor. 4.11] is incorrect.

3. VARIATIONS AND EXTENSIONS OF THE RESULT

The proof of Theorem 1.1 given above does not require $F = R(x,y)$, but only that $x, y \in F$, $R \subseteq F$ and $U_r \neq \emptyset$ for uncountably many $r \in R$. Here X, Y and U_r are assumed to be defined as in the proof of Theorem 1.1. Using this observation, one sees that the argument carries over to other situations as well. For example, we have the following:

Theorem 3.1. *Suppose $R(t,u) \subseteq F \subseteq S((t,u))$ where R is an uncountable real closed field and S is a real closed extension of R. Here $S((t,u))$ denotes the formal power series field in two variables t, u with coefficients in S, i.e., the field of fractions of the formal power series ring $S[[t,u]]$. Then M_F is not metrizable.*

Proof. Take $x = t$, $y = \frac{u}{t}$. Define X, Y, U_r as in the previous section. It suffices to show that $U_r \neq \emptyset$ for each $r \in R$. For this it suffices to show the existence of an ordering on $S((t,u))$ satisfying $0 < x$ and $-ax < y - r < ax$, i.e., $0 < t$ and $-at^2 < u - rt < at^2$, for some $a \in \mathbb{N}$. Since $S[[t,u]] = S[[t, u-rt]]$, it follows that $S((t,u)) \subseteq S((t))((u-rt))$. Take $<$ to be the restriction to $S((t,u))$ of one of the two orderings on $S((t))((u-rt))$ making $t > 0$. □

Corollary 3.2. *For any uncountable real closed field R, the space of $\mathbb{R}$-places of the formal power series field $R((x,y))$ is not metrizable.*

Proof. Immediate from Theorem 3.1. □

Corollary 3.3. *Suppose R is an uncountable real closed field and F is a finitely generated formally real field extension of R of transcendence degree ≥ 2. Then M_F is not metrizable.*

Proof. Suppose $\operatorname{trdeg}(F:R) = d$. Viewing F as the function field of a real algebraic variety V over R and going to the completion of the coordinate ring of V at some fixed real non-singular point of V, we see that $F \subseteq R((x_1, \dots, x_d))$ where $x_1, \dots, x_d$ are elements in the coordinate ring which form a system of uniformizing parameters at the fixed non-singular point, e.g., see [13, Th. 12.2.2 and Th. 12.6.1]. Applying Theorem 3.1, with $t = x_1, u = x_2$, and $S :=$ the real closure of $R((x_3, \dots, x_d))$ at some fixed ordering of $R((x_3, \dots, x_d))$, we see that M_F is not metrizable. □

If R is an Archimedean real closed field and F is a finitely generated formally real field extension of R of transcendence degree 1, then M_F is the disjoint union of finitely many simple closed curves [4, Th. 2.1], so M_F is metrizable in this case.

Corollary 3.4. *Suppose R is an uncountable Archimedean real closed field and F is a finitely generated formally real field extension of R. Then M_F is metrizable iff* $\operatorname{trdeg}(F:R) \leq 1$.

In sharp contrast to this we have the following result:

Theorem 3.5. *If S is a proper real closed extension of $\mathbb{R}$ then the space of $\mathbb{R}$-places of the rational function field $S(y)$ is not metrizable.*

Proof. Fix an element $x \in S \setminus \mathbb{R}$. Replacing x by $\pm\frac{1}{x}$ or $\pm(x-a)$ for suitable $a \in \mathbb{R}$, we may assume $0 < x < \frac{1}{n}$ for each $n \in \mathbb{N}$. Define X, Y, U_r as in the previous section, where $R := \mathbb{R}$, $F := S(y)$. (Note that $Y = X$.) By considering the restriction to F of any ordering of $S((y-r))$ (there are two such orderings, by Baer-Krull), we see that $U_r \neq \emptyset$ for each $r \in \mathbb{R}$. □

Remark 3.6. The map $\lambda : X_{R(x)} \to M_{R(x)}$, where R is an arbitrary real closed field, is described in [15]. Orderings of $R(x)$ are identified with cuts of R in the standard way [5], [6], [9], [15]. Cuts of R are represented by cut symbols [15, Prop. 3.2.2]. The fibers of λ are determined by applying [15, Th. 4.3.2] in the case $n = 1$. The fibers consist of either a single ordering, corresponding to a single cut symbol (Q, p), or a pair of orderings, corresponding to a pair of cut symbols $(Q, p, +)$, $(Q, p, -)$. The fibers are also described in [11], but there are mistakes in [11]. Corollary 4.5, Proposition 4.6 and Corollary 4.7 of [11] are incorrect.

Of course, this still leaves us with the following general question:

Question 3.7. Given a finitely generated formally real field extension F of a real closed field R, when is M_F metrizable?

In this degree of generality the question is still open, even in the purely transcendental transcendence degree 1 case. One might conjecture that it is the case iff either $\operatorname{trdeg}(F:R) = 0$ (i.e., $F = R$) or $\operatorname{trdeg}(F:R) = 1$ and R is Archimedean.

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